16: Continuous Joint Distributions

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Continuous joint distributions
Remember target?

Good times...

\[
\frac{59}{309} = 0.191
\]
Quick check: What is the probability that a dart hits at (456.2344132343, 532.1865739012)?

The CS109 logo was created by throwing 500,000 darts according to a joint distribution.

If we throw another dart according to the same distribution, what is $P(\text{dart hits within } r \text{ pixels of center})$?
CS109 logo with darts

P(dart hits within $r$ pixels of center)?

1 pixel = 1 dart thrown at screen

Possible dart counts (in 100x100 boxes)
CS109 logo with darts

P(dart hits within $r$ pixels of center)?
Possible dart counts
(in infinitesimally small boxes)
Continuous joint probability density functions

If two random variables $X$ and $Y$ are jointly continuous, then there exists a joint probability density function $f_{X,Y}$ defined over $-\infty < x, y < \infty$ such that:

$$P(a_1 \leq X \leq a_2, \ b_1 \leq Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y) dy \, dx$$
From one continuous RV to jointly continuous RVs

Single continuous RV $X$
- PDF $f_X$ such that $\int_{-\infty}^{\infty} f_X(x) \, dx = 1$
- Integrate to get probabilities

Jointly continuous RVs $X$ and $Y$
- PDF $f_{X,Y}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1$
- Double integrate to get probabilities

Probability for jointly continuous RVs is **volume** under a surface.
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

Write down the definite double integral that must integrate to 1:
Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

Write down the definite double integral that must integrate to 1:

$$
\int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy = 1 \quad \text{or} \quad \int_{y=0}^{1} \int_{x=0}^{2} xy \, dy \, dx = 1
$$

(used in next slide)
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

**0. Set up integral:**

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, dx \, dy = \int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy$$

**1. Evaluate inside integral by treating $y$ as a constant:**

\[
\int_{y=0}^{2} \left( \int_{x=0}^{1} xy \, dx \right) \, dy = \int_{y=0}^{2} y \left( \int_{x=0}^{1} x \, dx \right) \, dy = \int_{y=0}^{2} y \left[ \frac{x^2}{2} \right]_{0}^{1} \, dy = \int_{y=0}^{2} y \frac{1}{2} \, dy
\]

**2. Evaluate remaining (single) integral:**

\[
\int_{y=0}^{2} y \frac{1}{2} \, dy = \left[ \frac{y^2}{4} \right]_{y=0}^{2} = 1 - 0 = 1
\]

Yes, $g(x, y)$ is a valid joint PDF because it integrates to 1.
Marginal distributions

Suppose $X$ and $Y$ are continuous random variables with joint PDF:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \, dx = 1$$

The marginal density functions (marginal PDFs) are therefore:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) dy \quad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b) dx$$
Back to darts!

Match $X$ and $Y$ to their respective marginal PDFs:
Back to darts!

Match \( X \) and \( Y \) to their respective marginal PDFs:
Extra slides

If you want more practice with double integrals, I’ve included two exercises at the end of this lecture.
Joint CDFs
An observation: Connecting CDF to PDF

For a continuous random variable \( X \) with PDF \( f \), the CDF (cumulative distribution function) is

\[
F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx
\]

The density \( f \) is therefore the derivative of the CDF, \( F \):

\[
f(a) = \frac{d}{da} F(a)
\]

(Fundamental Theorem of Calculus)
Joint cumulative distribution function

For two random variables $X$ and $Y$, there can be a joint cumulative distribution function $F_{X,Y}$:

$$F_{X,Y}(a, b) = P(X \leq a, Y \leq b)$$

For discrete $X$ and $Y$:

$$F_{X,Y}(a, b) = \sum_{x \leq a} \sum_{y \leq b} p_{X,Y}(x, y)$$

For continuous $X$ and $Y$:

$$F_{X,Y}(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y) \, dy \, dx$$

$$f_{X,Y}(a, b) = \frac{\partial^2}{\partial a \partial b} F_{X,Y}(a, b)$$
Single variable CDF, graphically

\[ \lim_{x \to +\infty} F_X(x) = 1 \]

\[ \lim_{x \to -\infty} F_X(x) = 0 \]

\[ F_X(x) = P(X \leq x) \]

\[ f_X(x) \]
Joint CDF, graphically

\[
\lim_{x,y \to -\infty} F_{X,Y}(x, y) = 0
\]

\[
f_{X,Y}(x, y)
\]

\[
F_{X,Y}(x, y) = P(X \leq x, Y \leq y)
\]

\[
\lim_{x,y \to +\infty} F_{X,Y}(x, y) = 1
\]
Independent Continuous RVs
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$ P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y) \quad \forall x, y $$

Equivalently:

$$ F_{X,Y}(x, y) = F_X(x)F_Y(y) \quad \forall x, y $$
$$ f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \forall x, y $$

Proof of PDF:

$$ f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y) $$
$$ = \frac{\partial}{\partial x} \frac{\partial}{\partial y} F_X(x)F_Y(y) = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y) $$
$$ = f_X(x)f_Y(y) $$
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

More generally, $X$ and $Y$ are independent if joint density factors separately:

$$f_{X,Y}(x, y) = g(x)h(y), \text{ where } -\infty < x, y < \infty$$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$

2. $f_{X,Y}(x, y) = 4xy$
   where $0 < x, y < 1$

3. $f_{X,Y}(x, y) = 4xy$
   where $0 < x + y < 1$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$
   Separable functions: $g(x) = 3e^{-3x}$
   $h(y) = 2e^{-2y}$

2. $f_{X,Y}(x, y) = 4xy$
   where $0 < x, y < 1$
   Separable functions: $g(x) = 2x$
   $h(y) = 2y$

3. $f_{X,Y}(x, y) = 4xy$
   where $0 < x + y < 1$
   Cannot capture constraint on $x + y$
   into factorization!

If you can factor densities over all of the support, you have independence.
Bivariate Normal Distribution
Bivariate Normal Distribution

$X_1$ and $X_2$ follow a bivariate normal distribution if their joint PDF $f$ is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

Can show that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

(Ross chapter 6, example 5d)

Often written as: $X \sim \mathcal{N}(\mu, \Sigma)$

- Vector $X = (X_1, X_2)$
- Mean vector $\mu = (\mu_1, \mu_2)$, Covariance matrix: $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

Recall correlation: $\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$

We will focus on understanding the shape of a bivariate Normal RV.
These darts were actually thrown according to a bivariate normal distribution:

$$(X, Y) \sim \mathcal{N} (\mu, \Sigma)$$

$$\mu = (450, 600)$$

$$\Sigma = \begin{bmatrix} 900^2/4 & 0 \\ 0 & 900^2/25 \end{bmatrix}$$

**Marginal PDFs:**

$$X \sim \mathcal{N} \left( 450, \frac{900^2}{4} \right)$$

$$Y \sim \mathcal{N} \left( 600, \frac{900^2}{25} \right)$$
A diagonal covariance matrix

Let $X = (X_1, X_2)$ follow a bivariate normal distribution $X \sim \mathcal{N}(\mu, \Sigma)$, where

$$\mu = (\mu_1, \mu_2), \quad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Are $X_1$ and $X_2$ independent?

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}} \left( \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left( \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right)}$$

(Note covariance: $\rho\sigma_1\sigma_2 = 0$)

$X_1$ and $X_2$ are independent with marginal distributions $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$
16: Continuous Joint Distributions (I)

Lisa Yan
May 11, 2020
Jointly continuous RVs

\(X\) and \(Y\) are jointly continuous if they have a joint PDF:

\[
f_{X,Y}(x, y) \text{ such that } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1
\]

Most things we’ve learned about discrete joint distributions translate:

- **Marginal distributions**
  \[
p_X(a) = \sum_y p_{X,Y}(a, y) \quad \text{and} \quad f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) \, dy
\]

- **Independent RVs**
  \[
p_{X,Y}(x, y) = p_X(x)p_Y(y) \quad \text{and} \quad f_{X,Y}(x, y) = f_X(x)f_Y(y)
\]

- **Expectation (e.g., LOTUS)**
  \[
  E[g(X,Y)] = \sum_x \sum_y g(x, y)p_{X,Y}(x, y) \quad \text{and} \quad E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f_{X,Y}(x, y) \, dy \, dx
\]

...etc.
Think by yourself: 2 min
Warmup exercise

$X$ and $Y$ have the following joint PDF:

$$f_{X,Y}(x, y) = 3e^{-3x}$$

where $0 < x < \infty$, $1 < y < 2$

1. Are $X$ and $Y$ independent?

2. What is the marginal PDF of $X$? Of $Y$?

3. What is $E[X + Y]$?
Warmup exercise

$X$ and $Y$ have the following joint PDF:

$$f_{X,Y}(x, y) = 3e^{-3x}$$
where $0 < x < \infty$, $1 < y < 2$

1. Are $X$ and $Y$ independent?

2. What is the marginal PDF of $X$? Of $Y$?

3. What is $E[X + Y]$?

$\begin{align*}
g(x) &= 3Ce^{-3x}, \quad 0 < x < \infty \quad C \text{ is a constant} \\
h(y) &= \frac{1}{C}, \quad 1 < y < 2
\end{align*}$
Warmup exercise

\( X \) and \( Y \) have the following joint PDF:

\[
f_{X,Y}(x, y) = 3e^{-3x}
\]

where \( 0 < x < \infty, 1 < y < 2 \)

1. Are \( X \) and \( Y \) independent?

2. What is the marginal PDF of \( X \)? Of \( Y \)?

\[
g(x) = 3C e^{-3x}, 0 < x < \infty \quad C \text{ is a constant}
\]

\[
h(y) = 1/C, \quad 1 < y < 2
\]

3. What is \( E[X + Y] \)?
Check out the question on the next slide (Slide 39). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Breakout rooms: 4 min. Introduce yourself!
The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define \( X = \# \text{ minutes past 12pm that person 1 arrives. } \) \( X \sim \text{Unif}(0, 30) \)

\( Y = \# \text{ minutes past 12pm that person 2 arrives. } \) \( Y \sim \text{Unif}(0, 30) \)

What is the probability that the first to arrive waits >10 mins for the other?

Compute: \( P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y) \) (by symmetry)

1. What is “symmetry” here?
2. How do we integrate to compute this probability?
The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define \( X = \# \text{ minutes past 12pm that person 1 arrives.} \quad X \sim \text{Unif}(0, 30) \)

\( Y = \# \text{ minutes past 12pm that person 2 arrives.} \quad Y \sim \text{Unif}(0, 30) \)

What is the probability that the first to arrive waits >10 mins for the other?

**Compute:**

\[
P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y) \quad \text{(by symmetry)}
\]

\[
= 2 \cdot \int_{x+10<y} \int f_{X,Y}(x,y) dx \, dy = 2 \cdot \int_{x+10<y, \ 0\leq x,y,\leq 30} (1/30)^2 dx \, dy \quad \text{(independence)}
\]

Errata 5/12: corrected lower bound on \( y \) to 10

\[
= \frac{2}{30^2} \int_{10}^{30} \int_{0}^{y-10} dx \, dy = \frac{2}{30^2} \int_{10}^{30} (y-10) dy = \cdots = \frac{4}{9}
\]
Interlude for jokes/announcements
Announcements

Mid-quarter feedback form
link
Open until: this Friday

Problem Set 4
Due: Monday 5/18 10am
Covers: Up to and including today
Announcements: CS109 contest

Do something cool and creative with probability

Replaces one “passing” work requirement

Optional Proposal: Sat. 5/23 11:59pm
Due: Monday 6/8, 11:59pm

Interesting probability news

What That Election Probability Means

Even when you shift the probability far left or far right, the opposing candidate still gets some wins. That doesn’t mean a forecast was wrong. That’s just randomness and uncertainty at play. The probability estimates the percentage of times you get an outcome if you were to do something multiple times.

https://flowingdata.com/2016/07/28/what-that-election-probability-means/

CS109 Current Events Spreadsheet
Bivariate Normal Distribution

The bivariate normal distribution of $X = (X_1, X_2)$:

$$X \sim \mathcal{N}(\mu, \Sigma)$$

- Mean vector $\mu = (\mu_1, \mu_2)$
- Covariance matrix: $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$
  \[\text{Cov}(X_1, X_2) = \text{Cov}(X_2, X_1) = \rho \sigma_1 \sigma_2\]

- Marginal distributions: $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$
- For bivariate normals in particular, $\text{Cov}(X_1, X_2) = 0$ implies $X_1, X_2$ independent.

We will focus on understanding the shape of a bivariate Normal RV.
Check out the question on the next slide (Slide 47). Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Breakout rooms: 3 min. Introduce yourself!


\((X, Y)\) Matching (all have \(\mu = (0, 0)\))

1. PDF

2. PDF

3. PDF

4. PDF

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & 2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0.5 \\
0.5 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -0.5 \\
-0.5 & 1 \\
\end{bmatrix}
\]
(X, Y) Matching (all have $\mu = (0, 0)$)

1. PDF

2. PDF

3. PDF

4. PDF

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$
Probabilities from joint CDFs

Recall for a single RV $X$ with CDF $F_X$:

\[ P(a < X \leq b) = F_X(b) - F(a) \]

For two RVs $X$ and $Y$ with joint CDF $F_{X,Y}$:

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]

Note strict inequalities; these properties hold for both discrete and continuous RVs.
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[ F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[ F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \\
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[ F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probability with Instagram!

(for next time)

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
In a Gaussian blur, for every pixel:
- Weight each pixel by the probability that $X$ and $Y$ are both within the pixel bounds
- The weighting function is a Bivariate Gaussian (Normal) standard deviation parameter $\sigma$

Gaussian blurring with $\sigma = 3$:

$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-\left(\frac{x^2 + y^2}{2 \cdot 3^2}\right)}$$

What is the weight of the center pixel?

$$P(-0.5 < X \leq 0.5, -0.5 < Y \leq 0.5) =$$

$$= 0.206$$

Weight matrix:

Center pixel: (0, 0)
Pixel bounds:
- $-0.5 < x \leq 0.5$
- $-0.5 < y \leq 0.5$

$\rightarrow$ Independent $X \sim \mathcal{N}(0, 3^2), Y \sim \mathcal{N}(0, 3^2)$

$\rightarrow$ Joint CDF: $F_{X,Y}(x, y) = \Phi\left(\frac{x}{3}\right) \Phi\left(\frac{y}{3}\right)$
Extra
1. Integral practice

Let $X$ and $Y$ be two continuous random variables with joint PDF:
$$f(x, y) = \begin{cases} 4xy & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X \leq Y)$?

$$P(X \leq Y) = \int_{0 \leq x, y \leq 1} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x=y}^{1} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{y} 4xy \, dx \, dy$$

$$= \int_{y=0}^{1} 4y \left[ \frac{x^2}{2} \right]_{0}^{y} \, dy = \int_{y=0}^{1} 2y^3 \, dy = \left[ \frac{2}{4} y^4 \right]_{0}^{1} = \frac{1}{2}$$
2. How do you integrate over a circle?

\[ \int \int_{x^2 + y^2 \leq 10^2} f_{X,Y}(x, y) \, dy \, dx \]

Let’s try an example that doesn’t involve integrating a Normal RV.
2. Imperfection on Disk

You have a disk surface, a circle of radius $R$. Suppose you have a single point imperfection uniformly distributed on the disk.

What are the marginal distributions of $X$ and $Y$? Are $X$ and $Y$ independent?

\[
\begin{align*}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy = \frac{1}{\pi R^2} \int_{x^2+y^2 \leq R^2} dy \\
&= \frac{1}{\pi R^2} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}
\end{align*}
\]

\[
\begin{align*}
f_Y(y) &= \frac{2\sqrt{R^2-y}}{\pi R^2} \quad \text{where } -R \leq y \leq R, \text{ by symmetry}
\end{align*}
\]

\[
f_{X,Y}(x, y) = \begin{cases} 
\frac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\
0 & \text{otherwise}
\end{cases}
\]

No, $X$ and $Y$ are dependent. 

\[
f_{X,Y}(x, y) \neq f_X(x)f_Y(y)
\]