18: Central Limit Theorem

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Beta random variable

An **Beta** random variable $X$ is defined as follows:

**PDF**

$$f(x) = \frac{1}{B(a, b)} x^{a-1} (1 - x)^{b-1}$$

where $B(a, b) = \int_0^1 x^{a-1} (1 - x)^{b-1} dx$, normalizing constant

**Support of $X$:** $(0, 1)$

**Expectation**

$$E[X] = \frac{a}{a + b}$$

**Variance**

$$\text{Var}(X) = \frac{ab}{(a + b)^2(a + b + 1)}$$

Beta is a distribution for probabilities.
CS109 focus: Beta where $a, b$ both positive integers

If $a, b$ are positive integers, Beta parameters $a, b$ could come from an experiment:

$a = \text{“successes”} + 1$

$b = \text{“failures”} + 1$
Back to flipping coins

- Start with a $X \sim \text{Uni}(0,1)$ over probability
- Observe $n = 7$ successes and $m = 1$ failures
- Your new belief about the probability of $X$ is:

$$f_{X|N}(x|n) = \frac{1}{c} x^7 (1-x)^1,$$

where $c = \int_0^1 x^7 (1-x)^1 \, dx$

Posterior belief, $X|N$:

- Beta($a = 8, b = 2$)

$$f_{X|N}(x|n) = \frac{1}{c} x^{8-1} (1-x)^{2-1}$$

- Beta($a = n + 1, b = m + 1$)
Understanding Beta

• Start with a $X \sim \text{Uni}(0,1)$ over probability
• Observe $n$ successes and $m$ failures
• Your new belief about the probability of $X$ is:

$$X|N \sim \text{Beta}(a = n + 1, b = m + 1)$$
Understanding Beta

• Start with a $X \sim \text{Uni}(0,1)$ over probability
• Observe $n$ successes and $m$ failures
• Your new belief about the probability of $X$ is:

$$X|N \sim \text{Beta}(a = n + 1, b = m + 1)$$

Check this out:

Beta($a = 1, b = 1$) has PDF:

$$f(x) = \frac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1} = \frac{1}{B(a, b)} x^0(1 - x)^0 = \frac{1}{\int_0^1 1dx}$$

where $0 < x < 1$.
If the prior is a Beta...

Let $X$ be our random variable for probability of success and $N$

- If our **prior belief** about $X$ is beta: $X \sim \text{Beta}(a, b)$

  ...and if we observe $n$ successes and $m$ failures:  
  $N | X \sim \text{Bin}(n + m, x)$

- ...then our **posterior belief** about $X$ is also beta. 
  $X | N \sim \text{Beta}(a + n, b + m)$

This is the main takeaway of Beta.
If the prior is a Beta...

Let $X$ be our random variable for probability of success and $N$

- If our prior belief about $X$ is beta: $X \sim \text{Beta}(a, b)$

  ...and if we observe $n$ successes and $m$ failures: $N|X \sim \text{Bin}(n + m, x)$

- ...then our posterior belief about $X$ is also beta. $X|N \sim \text{Beta}(a + n, b + m)$

Proof:

$$f_{X|N}(x|n) = \frac{p_{N|X}(n|x)f_X(x)}{p_N(n)} = \frac{\binom{n + m}{n} x^n (1 - x)^m \cdot \frac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1}}{p_N(n)}$$

$$= C \cdot x^n (1 - x)^m \cdot x^{a-1}(1 - x)^{b-1} = C \cdot x^{n+a-1}(1 - x)^{m+b-1}$$
If the prior is a Beta...

Let $X$ be our random variable for probability of success and $N$

- If our **prior belief** about $X$ is beta: $X \sim \text{Beta}(a, b)$

  ...and if we observe $n$ successes and $m$ failures: $N|X \sim \text{Bin}(n + m, x)$

  ...then our **posterior belief** about $X$ is also beta: $X|N \sim \text{Beta}(a + n, b + m)$

Beta is a **conjugate** distribution.

- Prior and posterior parametric forms are the same
- Practically, conjugate means easy update:
  Add number of “heads” and “tails” seen to Beta parameter.
If the prior is a Beta...

Let $X$ be our random variable for probability of success and $N$

- If our prior belief about $X$ is beta: $X \sim \text{Beta}(a, b)$

...and if we observe $n$ successes and $m$ failures: $N | X \sim \text{Bin}(n + m, x)$

- ...then our posterior belief about $X$ is also beta. $X | N \sim \text{Beta}(a + n, b + m)$

You can set the prior to reflect how biased you think the coin is apriori.

- This is a subjective probability!
- $X \sim \text{Beta}(a, b)$: have seen $(a + b - 2)$ imaginary trials, where $(a - 1)$ are heads, $(b - 1)$ tails
- Then $\text{Beta}(1, 1) = \text{Uni}(0, 1)$ means we haven’t seen any imaginary trials
If the prior is a Beta...

Let $X$ be our random variable for probability of success and $N$

- If our **prior belief** about $X$ is beta:
  
  $X \sim \text{Beta}(a, b)$

- ...and if we observe $n$ successes and $m$ failures:
  
  $N \mid X \sim \text{Bin}(n + m, x)$

- ...then our **posterior belief** about $X$
  
  $X \mid N \sim \text{Beta}(a + n, b + m)$

<table>
<thead>
<tr>
<th>Prior</th>
<th>Beta($a = n_{imag} + 1, b = m_{imag} + 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Posterior</td>
<td>Beta($a = n_{imag} + n + 1, b = m_{imag} + m + 1$)</td>
</tr>
</tbody>
</table>

This is the main takeaway of Beta.
The enchanted die

Let $X$ be the probability of rolling a 6 on Lisa’s die.

- Prior: Imagine 5 die rolls where only 6 showed up
- Observation: roll it a few times...

What is the updated distribution of $X$ after our observation?

Check out the demo!
Medicinal Beta

- Before being tested, a medicine is believed to “work” 80% of the time.
- The medicine is tried on 20 patients.
- It “works” for 12, “doesn’t work” for 8.

What is your new belief that the drug “works”?

Frequentist

Let $p$ be the probability your drug works.

$$p \approx \frac{12}{20} = 0.6$$

Bayesian

A frequentist view will not incorporate prior/expert belief about probability.
Before being tested, a medicine is believed to “work” 80% of the time.
- The medicine is tried on 20 patients.
- It “works” for 12, “doesn’t work” for 8.

What is your new belief that the drug “works”?

**Frequentist**

Let $p$ be the probability your drug works.

$p \approx \frac{14}{20} = 0.7$

**Bayesian**

Let $X$ be the probability your drug works.

$X$ is a random variable.
Medicinal Beta

- Before being tested, a medicine is believed to “work” 80% of the time.
- The medicine is tried on 20 patients.
- It “works” for 12, “doesn’t work” for 8.

What is your new belief that the drug “works”? (Bayesian interpretation)

What is the prior distribution of $X$? (select all that apply)

A. $X \sim \text{Beta}(1, 1) = \text{Uni}(0, 1)$
B. $X \sim \text{Beta}(81, 101)$
C. $X \sim \text{Beta}(80, 20)$
D. $X \sim \text{Beta}(81, 21)$
E. $X \sim \text{Beta}(5, 2)$
Medicinal Beta

• Before being tested, a medicine is believed to “work” 80% of the time.
• The medicine is tried on 20 patients.
• It “works” for 12, “doesn’t work” for 8.

What is your new belief that the drug “works”?  

What is the prior distribution of $X$?  (select all that apply)

A. $X \sim \text{Beta}(1, 1) = \text{Uni}(0, 1)$
B. $X \sim \text{Beta}(81, 101)$
C. $X \sim \text{Beta}(80, 20)$
D. $X \sim \text{Beta}(81, 21)$  
   Interpretation: 80 successes / 100 imaginary trials
E. $X \sim \text{Beta}(5, 2)$  
   Interpretation: 4 successes / 5 imaginary trials

(Bayesian interpretation)

Prior: $\text{Beta}(a = n_{imag} + 1, b = m_{imag} + 1)$
Posterior: $\text{Beta}(a = n_{imag} + n + 1, b = m_{imag} + m + 1)$
Medicinal Beta

• Before being tested, a medicine is believed to “work” 80% of the time.
• The medicine is tried on 20 patients.
• It “works” for 12, “doesn’t work” for 8.

What is your new belief that the drug “works”?

Prior: \( X \sim \text{Beta}(a = 5, b = 2) \)
Posterior: \( X \sim \text{Beta}(a = 5 + 12, b = 2 + 8) \)
\[ \sim \text{Beta}(a = 17, b = 10) \]
Medicinal Beta

• Before being tested, a medicine is believed to “work” 80% of the time.
• The medicine is tried on 20 patients.
• It “works” for 12, “doesn’t work” for 8.

What is your new belief that the drug “works”?

Prior: \(X \sim \text{Beta}(a = 5, b = 2)\)
Posterior: \(X \sim \text{Beta}(a = 5 + 12, b = 2 + 8) \sim \text{Beta}(a = 17, b = 10)\)

What do you report to pharmacists?
A. Expectation of posterior
B. Mode of posterior
C. Distribution of posterior
D. Nothing

(Bayesian interpretation)
Medicinal Beta

• Before being tested, a medicine is believed to “work” 80% of the time.
• The medicine is tried on 20 patients.
• It “works” for 12, “doesn’t work” for 8.

What is your new belief that the drug “works”?

Prior: \( X \sim \text{Beta}(a = 5, b = 2) \)

Posterior: \( X \sim \text{Beta}(a = 5 + 12, b = 2 + 8) \)
\( \sim \text{Beta}(a = 17, b = 10) \)

What do you report to pharmacists?

A. Expectation of posterior
\[ E[X] = \frac{a}{a + b} = \frac{17}{17 + 10} \approx 0.63 \]

B. Mode of posterior
\[ \text{mode}(X) = \frac{a - 1}{a + b - 2} = \frac{16}{16 + 9} = 0.64 \]

C. Distribution of posterior

D. Nothing
Food for thought

In this lecture:

If we don’t know the parameter $p$, Bayesian statisticians will:

- Treat the parameter as a random variable $X$ with a Beta prior distribution
- Perform an experiment
- Based on experiment outcomes, update the posterior distribution of $X$

$Y \sim \text{Ber}(p)$

Food for thought:

Any parameter for a “parameterized” random variable can be thought of as a random variable.

$Y \sim \mathcal{N}(\mu, \sigma^2)$
Today’s plan

Finish Beta

Central Limit Theorem (CLT)

CLT exercises
(silent drumroll)
Central Limit Theorem

Consider \( n \) independent and identically distributed (i.i.d.) variables \( X_1, X_2, ..., X_n \) with \( E[X_i] = \mu \) and \( \text{Var}(X_i) = \sigma^2 \).

\[
\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)
\]

As \( n \to \infty \)

The sum of \( n \) i.i.d. random variables is normally distributed with mean \( n\mu \) and variance \( n\sigma^2 \).
True happiness
Central Limit Theorem

Consider $n$ independent and identically distributed (i.i.d.) variables $X_1, X_2, \ldots, X_n$ with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$.

The sum of $n$ i.i.d. random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$. 

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

As $n \to \infty$
Quick check

What dimensions are the following random variables?

1. $X_1$

2. $(X_1, X_2, \ldots, X_n)$

3. $\sum_{i=1}^{n} X_i$

4. $\frac{1}{n} \sum_{i=1}^{n} X_i$

A. 1-D random variable
B. $n$ -D random variable (a vector)
C. not a random variable
Quick check

What dimensions are the following random variables?

1. $X_1$  
   (A)

2. $(X_1, X_2, ..., X_n)$ (B) (aka a sample)

3. $\sum_{i=1}^{n} X_i$  
   (A)

4. $\frac{1}{n} \sum_{i=1}^{n} X_i$ (A) (aka the sample mean)

A. 1-D random variable
B. $n$-D random variable (a vector)
C. not a random variable
Central Limit Theorem

Consider $n$ independent and identically distributed (i.i.d.) variables $X_1, X_2, \ldots, X_n$ with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$.

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \]

As $n \to \infty$

The sum of $n$ i.i.d. random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$. 
i.i.d. random variables

Consider \( n \) variables \( X_1, X_2, \ldots, X_n \).

\( X_1, X_2, \ldots, X_n \) are independent and identically distributed if

- \( X_1, X_2, \ldots, X_n \) are independent, and
- All have the same PMF (if discrete) or PDF (if continuous).

\[ E[X_i] = \mu \text{ for } i = 1, \ldots, n \]
\[ \text{Var}(X_i) = \sigma^2 \text{ for } i = 1, \ldots, n \]

Same thing: \( \text{i.i.d.} \) \( \text{ iid } \) \( \text{IID } \)

Side note: Multiple random variables \( X_1, X_2, \ldots, X_n \) are independent iff

\[ P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k) = \prod_{i=1}^{k} P(X_i \leq x_i) \quad \text{for all subsets } X_1, \ldots, X_k \]
Quick check

Are $X_1, X_2, \ldots, X_n$ i.i.d. with the following distributions?

1. $X_i \sim \text{Exp}(\lambda), \ X_i$ independent

2. $X_i \sim \text{Exp}(\lambda_i), \ X_i$ independent

3. $X_i \sim \text{Exp}(\lambda), \ X_1 = X_2 = \ldots = X_n$

4. $X_i \sim \text{Bin}(n_i, p), \ X_i$ independent
Quick check

Are $X_1, X_2, \ldots, X_n$ i.i.d. with the following distributions?

1. $X_i \sim \text{Exp}(\lambda)$, $X_i$ independent

2. $X_i \sim \text{Exp}(\lambda_i)$, $X_i$ independent (unless $\lambda_i$ equal)

3. $X_i \sim \text{Exp}(\lambda)$, $X_1 = X_2 = \cdots = X_n$ dependent: $X_1 = X_2 = \cdots = X_n$

4. $X_i \sim \text{Bin}(n_i, p)$, $X_i$ independent (unless $n_i$ equal) Note underlying Bernoulli RVs are i.i.d.!
Central Limit Theorem

Consider $n$ independent and identically distributed (i.i.d.) variables $X_1, X_2, ..., X_n$ with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$.

The sum of $n$ i.i.d. random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \]
Sum of dice rolls

Roll \( n \) independent dice. Let \( X_i \) be the outcome of roll \( i \). \( X_i \) are i.i.d.

\[
\sum_{i=1}^{1} X_i \quad \text{Sum of 1 die roll} \\
\sum_{i=1}^{2} X_i \quad \text{Sum of 2 die rolls} \\
\sum_{i=1}^{3} X_i \quad \text{Sum of 3 die rolls}
\]

How many ways can you roll a total of 3 vs 11?
CLT explains a lot

\[
\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)
\]

As \( n \to \infty \)

The sum of \( n \) i.i.d. random variables is normally distributed with mean \( n\mu \) and variance \( n\sigma^2 \).

Proof:

Let \( X_i \sim \text{Ber}(p) \) for \( i = 1, \ldots, n \), where \( X_i \) are i.i.d. \( E[X_i] = p \), \( \text{Var}(X_i) = p(1 - p) \)

\[
X = \sum_{i=1}^{n} X_i \quad (X \sim \text{Bin}(n, p))
\]

\( X \sim \mathcal{N}(n\mu, n\sigma^2) \) \quad (CLT, as \( n \to \infty \))

\[
X \sim \mathcal{N}(np, np(1 - p)) \quad \text{(substitute mean, variance of Bernoulli)}
\]
CLT explains a lot

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \]

As \( n \to \infty \)

The sum of \( n \) i.i.d. random variables is normally distributed with mean \( n\mu \) and variance \( n\sigma^2 \).

Sample of size 15, sum values

Distribution of \( X_i \)

Distribution of \( \sum_{i=1}^{15} X_i \)
CLT explains a lot

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \]

As \( n \to \infty \)

The sum of \( n \) i.i.d. random variables is normally distributed with mean \( n\mu \) and variance \( n\sigma^2 \).

Sample of size 15, average values

Distribution of \( X_i \)  
(sample mean)  
Distribution of \( \frac{1}{15} \sum_{i=1}^{15} X_i \)
CLT explains a lot

The sum of $n$ i.i.d. random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

Galton Board, by Sir Francis Galton (1822-1911)
Break for Friday/announcements
Announcements

Midterm exam
It’s done!
Grades: Friday 11/1
Solutions: Friday 11/1

Problem Set 4
Due: Wednesday 11/6
Covers: Up to Law of Total Expectation

Late day reminder: No late days permitted past last day of the quarter, 12/7
Proof of CLT

The sum of $n$ i.i.d. random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

Proof:

- The Fourier Transform of a PDF is called a characteristic function.
- Take the characteristic function of the probability mass of the sample distance from the mean, divided by standard deviation
- Show that this approaches an exponential function in the limit as $n \to \infty$:
  \[ f(x) = e^{-\frac{x^2}{2}} \]
- This function is in turn the characteristic function of the Standard Normal, $Z \sim \mathcal{N}(0,1)$.

(this proof is beyond the scope of CS109)
Implications of CLT

Anything that is a sum/average of independent random variables is normal ...meaning in real life, many things are normally distributed:

- Movie ratings: averages of independent viewer scores
- Polling:
  - Ask 100 people if they will vote for candidate 1
  - $p_1 = \# \text{“yes”}/100$
  - Sum of Bernoulli RVs (each person independently says “yes” w.p. $p$)
  - Repeat this process with different groups to get $p_1, \ldots, p_n$ (different sample statistics)
  - Normal distribution over sample means $p_k$
  - Confidence interval: “How likely is it that an estimate for true $p$ is close?”
What about other functions?

Let $X_1, X_2, \ldots, X_n$ i.i.d., where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

- Sum of i.i.d. RVs:
  \[ \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \]

- Average of i.i.d. RVs (sample mean):

- Max of i.i.d. RVs:

 chẳng có mẫu
What about other functions?

Let $X_1, X_2, \ldots, X_n$ i.i.d., where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \]

Sum of i.i.d. RVs

Average of i.i.d. RVs (sample mean)

Max of i.i.d. RVs
Distribution of sample mean

Let $X_1, X_2, \ldots, X_n$ i.i.d., where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

Define: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ (sample mean) \hspace{1cm} Y = \sum_{i=1}^{n} X_i$ (sum)

$Y \sim \mathcal{N}(n\mu, n\sigma^2)$ \hspace{1cm} (CLT, as $n \to \infty$)

$\bar{X} = \frac{1}{n} Y$ \hspace{1cm} $\bar{X} \sim \mathcal{N}(?, ?)$ \hspace{1cm} (Linear transform of a Normal)

A. $\bar{X} \sim \mathcal{N}(n\mu, n\sigma^2)$
B. $\bar{X} \sim \mathcal{N}\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$
C. $\bar{X} \sim \mathcal{N}(\mu, \sigma^2)$
D. $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}$)
Distribution of sample mean

Let \( X_1, X_2, ..., X_n \) i.i.d., where \( E[X_i] = \mu, \text{Var}(X_i) = \sigma^2 \). As \( n \to \infty \):

Define:\n\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{(sample mean)} \quad Y = \sum_{i=1}^{n} X_i \quad \text{(sum)}
\]

\[
Y \sim \mathcal{N}(n\mu, n\sigma^2) \quad \text{(CLT, as } n \to \infty)\]

\[
\bar{X} = \frac{1}{n} Y
\]

\[
\bar{X} \sim \mathcal{N}(?, ?) \quad \text{(Linear transform of a Normal)}
\]

A. \( \bar{X} \sim \mathcal{N}(n\mu, n\sigma^2) \)

B. \( \bar{X} \sim \mathcal{N}\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right) \)

C. \( \bar{X} \sim \mathcal{N}(\mu, \sigma^2) \)

D. \( \bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}) \)
Distribution of sample mean

Let $X_1, X_2, \ldots, X_n$ i.i.d., where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

Define:

- $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ (sample mean)
- $Y = \sum_{i=1}^{n} X_i$ (sum)

$$Y \sim \mathcal{N}(n\mu, n\sigma^2)$$

(Linear transform of a Normal)

$$\bar{X} = \frac{1}{n} Y$$

$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

(CLT, as $n \to \infty$)

The average of i.i.d. random variables (i.e., sample mean) is normally distributed with mean $\mu$ and variance $\sigma^2/n$.

What about other functions?

Let $X_1, X_2, \ldots, X_n$ i.i.d., where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \]

Sum of i.i.d. RVs

\[ \frac{1}{n} \sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}) \]

Average of i.i.d. RVs (sample mean)

Gumbel

Max of i.i.d. RVs

(see Fisher-Tippett Gnedenko Theorem)
Once upon a time...

Abraham de Moivre
CLT for $X \sim \text{Ber}(1/2)$
1733

Aubrey Drake Graham
(Drake)
A short history of the CLT

1733: CLT for $X \sim \text{Ber}(1/2)$ postulated by Abraham de Moivre

1823: Pierre-Simon Laplace extends de Moivre’s work to approximating $\text{Bin}(n, p)$ with Normal

1901: Alexandr Lyapunov provides precise definition and rigorous proof of CLT

2018: Drake releases Scorpion
- It was his 5th studio album, bringing his total # of songs to 190
- Mean quality of subsamples of songs is normally distributed (thanks to the Central Limit Theorem)
Today’s plan

Central Limit Theorem (CLT)

CLT exercises
Working with the CLT

Let $X_1, X_2, \ldots, X_n$ i.i.d., where $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$. As $n \to \infty$:

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

Sum of i.i.d. RVs

$$\frac{1}{n} \sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Average of i.i.d. RVs (sample mean)

⚠️ If $X_i$ is discrete:
Use the **continuity correction** on $Y$!
Dice game

You will roll 10 6-sided dice \((X_1, X_2, \ldots, X_{10})\).

- Let \(X = X_1 + X_2 + \cdots + X_{10}\), the total value of all 10 rolls.
- You win if \(X \leq 25\) or \(X \geq 45\).

And now the truth (according to the CLT)...

1. Define RVs and state goal.

\[
E[X_i] = 3.5, \\
\text{Var}(X_i) = \frac{35}{12} \\
X \approx Y \sim \mathcal{N}(10(3.5), 10(35/12))
\]

Want:

\[P(X \leq 25 \text{ or } X \geq 45)\]

2. Solve.

A. \(P(25 \leq Y \leq 45)\)

B. \(P(Y \leq 25.5) + P(Y \geq 44.5)\)

C. \(1 - P(25 \leq Y \leq 45)\)

D. \(1 - P(25.5 \leq Y \leq 44.5)\)
Dice game

You will roll 10 6-sided dice \((X_1, X_2, \ldots, X_{10})\).
- Let \(X = X_1 + X_2 + \cdots + X_{10}\), the total value of all 10 rolls.
- You win if \(X \leq 25\) or \(X \geq 45\).

And now the truth (according to the CLT)...

1. Define RVs and state goal.
   
   \[
   E[X_i] = 3.5, \\
   \text{Var}(X_i) = 35/12 \\
   X \approx Y \sim \mathcal{N}(10(3.5), 10(35/12))
   \]

2. Solve.
   
   \[
   P(Y \leq 25.5) + P(Y \geq 44.5) \\
   = \Phi \left( \frac{25.5 - 35}{\sqrt{10(35/12)}} \right) - \left( 1 - \Phi \left( \frac{44.5 - 35}{\sqrt{10(35/12)}} \right) \right)
   \]

   Want:
   
   \[
   P(X \leq 25 \text{ or } X \geq 45) \\
   \approx P(Y \leq 25.5) + P(Y \geq 44.5) \\
   \approx \Phi(-1.76) - (1 - \Phi(1.76)) \\
   \approx (1 - 0.9608) - (1 - 0.9608) \\
   = 0.0784
   \]
Dice game

You will roll 10 6-sided dice ($X_1, X_2, \ldots, X_{10}$).

- Let $X = X_1 + X_2 + \cdots + X_{10}$, the total value of all 10 rolls.
- You win if $X \leq 25$ or $X \geq 45$.

And now the truth (according to the CLT)...

\[
\text{As } n \to \infty: \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)
\]

(by computer)

\[
P(X \leq 25 \text{ or } X \geq 45) \approx 0.0780
\]

(by CLT)

\[
\approx P(Y \leq 25.5) + P(Y \geq 44.5)
\]

\[
\approx 0.0784
\]
Clock running time

Want to find the mean (clock) runtime of an algorithm, $\mu = t$ sec.
  • Suppose variance of runtime is $\sigma^2 = 4$ sec$^2$.

How many trials do we need s.t. estimated time $= t \pm 0.5$ with 95% certainty?

1. Define RVs and state goal.
   
   \[ \bar{X} \sim \mathcal{N}(t, \frac{4}{n}) \]  
   \( (\text{CLT}) \)  
   
   \[ \bar{X} - t \sim \mathcal{N}(0, \frac{4}{n}) \]  
   \( (\text{linear transform of a normal}) \)

2. Solve.

   Want:  
   \[ P(t - 0.5 \leq \bar{X} \leq t + 0.5) = 0.95 \]

   \[ P(0.5 \leq \bar{X} - t \leq 0.5) = 0.95 \]
Clock running time

Want to find the mean (clock) runtime of an algorithm, $\mu = t$ sec.

• Suppose variance of runtime is $\sigma^2 = 4$ sec$^2$.

How many trials do we need s.t. estimated time $= t \pm 0.5$ with 95% certainty?

1. Define RVs and state goal.

$$\bar{X} - t \sim \mathcal{N} \left( 0, \frac{4}{n} \right)$$

$$0.95 = P \left( 0.5 \leq \bar{X} - t \leq 0.5 \right)$$

2. Solve.

$$0.95 = F_{\bar{X} - t} (0.5) - F_{\bar{X} - t} (-0.5)$$

$$= \Phi \left( \frac{0.5 - 0}{\sqrt{4/n}} \right) - \Phi \left( \frac{-0.5 - 0}{\sqrt{4/n}} \right) = 2\Phi \left( \frac{\sqrt{n}}{4} \right) - 1$$

As $n \to \infty$: $\frac{1}{n} \sum_{i=1}^{n} X_i \sim \mathcal{N} (\mu, \frac{\sigma^2}{n})$
**Clock running time**

Want to find the mean (clock) runtime of an algorithm, \( \mu = t \text{ sec.} \)

- Suppose variance of runtime is \( \sigma^2 = 4 \text{ sec}^2 \).

How many trials do we need s.t. estimated time = \( t \pm 0.5 \) with 95% certainty?

1. Define RVs and state goal.

   \[ \bar{X} - t \sim \mathcal{N} \left( 0, \frac{4}{n} \right) \]

   \[ 0.95 = P \left( 0.5 \leq \bar{X} - t \leq 0.5 \right) \]

2. Solve.

   \[ 0.95 = F_{\bar{X}-t}(0.5) - F_{\bar{X}-t}(-0.5) \]

   \[ = \Phi \left( \frac{0.5 - 0}{\sqrt{4/n}} \right) - \Phi \left( \frac{-0.5 - 0}{\sqrt{4/n}} \right) = 2\Phi \left( \frac{\sqrt{n}}{4} \right) - 1 \]

   \[ 0.975 = \Phi(\sqrt{n}/4) \]

   \[ \sqrt{n}/4 = \Phi^{-1}(0.975) \approx 1.96 \quad \Rightarrow \quad n \approx 62 \]

Run algorithm repeatedly (i.i.d. trials):

- \( X_i = \text{runtime of } i\text{-th run (for } 1 \leq i \leq n) \)
- Estimate runtime to be **average** of \( n \) trials, \( \bar{X} \)

As \( n \to \infty \):
\[
\frac{1}{n} \sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})
\]
I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "[Central limit theorem]". The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.

– Sir Francis Galton
(of the Galton Board)
Extra slides

Extra CLT exercises
Crashing website

- Let $X =$ number of visitors to a website, where $X \sim \text{Poi}(100)$.
- The server crashes if there are $\geq 120$ requests/minute.

What is $P(\text{server crashes in next minute})$?

**Strategy**: Poisson (exact)

**Strategy**: CLT (approximate)

**State goal**

**Want**: $P(X \geq 120)$

**Solve**

$$P(X \geq 120) = \sum_{k=120}^{\infty} \frac{(120)^k e^{-100}}{k!} \approx 0.0282$$
Crashing website

- Let $X = \text{number of visitors to a website, where } X \sim \text{Poi}(100)$.
- The server crashes if there are $\geq 120$ requests/minute.

What is $P(\text{server crashes in next minute})$?

**Strategy: Poisson (exact)**

**State goal**

Want: $P(X \geq 120)$

**Solve**

\[
\begin{align*}
P(X \geq 120) &= \sum_{k=120}^{\infty} \frac{(120)^k e^{-100}}{k!} \\
&\approx 0.0282
\end{align*}
\]

**Strategy: CLT (approximate)**

**State approx. goal**

$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2)$

Want: $P(X \geq 120) \approx P(Y \geq 119.5)$

**Solve**

\[
\begin{align*}
P(Y \geq 119.5) &= 1 - \Phi \left( \frac{119.5 - 100}{\sqrt{100}} \right) \\
&= 1 - \Phi(1.95) \\
&\approx 0.0256
\end{align*}
\]

(sum of IID Poisson = Poisson)