CS109: Probability for Computer Scientists
Problem Set #1 is out

Auto Submission

Check your answer

Insert LaTeX
Write an Agent newish

9. Counting Cards

Counting cards refers to when a player keeps track of what cards have already been played during a card-game, in order to have a better estimate of how likely they are to win. Counting cards was successfully used by probability students from MIT to beat casinos worldwide: MIT Blackjack Team, a heist which was popularized by the movie 21. The key to counting cards in blackjack is to keep track of the probability of high cards.

In this problem we are going to consider a simpler game called High Card played on a standard 52 card deck. The game works as follows: You decide if you want to play. If you do, the casino deals you a single card. If the card is a high card, (10, Jack, Queen, King or Ace), you win $20. If it is not, you lose $20. Another player is playing as well and each game they will play (thus revealing a card). You are allowed to hire a senior defined for you, who will...
Python Review Session

Friday at 5pm PT with Ishira (online)

Find links, recordings, and setup here
Learn LaTeX

Making a handout to help you get started

\begin{aligned}
P(E) &= \sum_{i=0}^{n} e^i \\
&= 0.25 \\
\end{aligned}

Done
What Makes for a Good Answer?

A DNA-turn has 10 base pairs. Each base pair can take on one of four distinct values, (A, T, G, C). How many distinct DNA-turns of length 10 are there?

Explanation:
We can count the number of DNA-turns by thinking of the generative process as 10 steps, each of which has 4 choices. Using the product rule of counting (aka the step rule of counting), the total number of unique turns will be $4^{10}$.

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^{10} = 1048576$$

Python code:
```python
4 * 4 * 4 * 4 * 4 * 4 * 4 * 4 * 4 # 4**10 = 1048576
```
Honor Code

Always remember: You need to be able to recreate your ability on an exam. And in the real world. This is a foundation course. **Cheating in CS109 is cheating yourself.**

Talk to your friends about the **concepts**, not the solution. Words must be your own.

Practice the **art of teaching**. Three most important things to know:
1. Do not give away the answer
2. Always be respectful
3. Know what you don’t know
Late Policy

Three types of extensions:
1. Grace period (24 hours)
2. Quite Late extension (2 class days)
3. After the hard deadline (> 2 class days)

You give them to yourself. Need to talk to us if:
A. You need more than 2 long extensions / qtr
B. Extension past the hard deadline

But CS109 is a fast class. If you want a long extension I want you to be intentional about how you are going to catch up.
If you notice a bug?

It should be robust, but things can happen.

Let me know: send an email to cpiech@stanford.edu or message me on slack. I need your email and the approximate time you encountered the bug.
Want to try to hack the PsetApp? We will give you time after class :-)
Above & Beyond
Review
CS109: From Counting to Machine Learning

↓ \( \frac{1}{9} \)
Counting Theory

Core Probability

Random Variables

Probabilistic Models

Uncertainty Theory

Machine Learning
Core Counting

Counting with steps

**Definition:** Step Rule of Counting (aka Product Rule of Counting)

If an experiment has two parts, where the first part can result in one of \( m \) outcomes and the second part can result in one of \( n \) outcomes regardless of the outcome of the first part, then the total number of outcomes for the experiment is \( m \cdot n \).

Counting with “or”

**Definition:** Inclusion Exclusion Counting

If the outcome of an experiment can either be drawn from set \( A \) or set \( B \), and sets \( A \) and \( B \) may potentially overlap (i.e., it is not the case that \( A \) and \( B \) are mutually exclusive), then the number of outcomes of the experiment is \( |A \text{ or } B| = |A| + |B| - |A \text{ and } B| \).
How Many Bit Stings?

**Problem:** A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either start with "01" or end with "10". How many such strings are there?

<table>
<thead>
<tr>
<th>Start with 01</th>
<th>End with 10</th>
</tr>
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<tbody>
<tr>
<td>010000</td>
<td>000010</td>
</tr>
<tr>
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<tr>
<td><strong>011110</strong></td>
<td>111010</td>
</tr>
<tr>
<td>011111</td>
<td>111110</td>
</tr>
</tbody>
</table>

Set $A$  

Set $B$

**Answer**

$N = |A| + |B| - |A \text{ and } B|$

$= 16 + 16 - 4$

$= 28$
Challenge Problem

1. Strings
   - How many \textit{different} orderings of letters are possible for the string BOBA?

BOBA, ABOB, OBBA...
End Review
Permutations I
Orderings of Letters

How many letter orderings are possible for the following strings?

1. CHRIS

This is Jerry’s dog, Doris. She puts her little Doris paw up to her chin when she’s thinking.
### Orderings of Letters

<table>
<thead>
<tr>
<th>chirs</th>
<th>crish</th>
<th>hicrs</th>
<th>hsirc</th>
<th>irchs</th>
<th>rcish</th>
<th>rschi</th>
<th>shirc</th>
</tr>
</thead>
<tbody>
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<td>ihrsc</td>
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<td>shcire</td>
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<td>ihsrc</td>
<td>rcihs</td>
<td>rishc</td>
<td>shicr</td>
<td>srhnc</td>
</tr>
</tbody>
</table>
Orderings of letters

Step 1: Chose first letter
Step 2: Chose 2nd letter
Step 3: Chose 3rd letter
Step 4: Chose 4th letter
Step 5: Chose 5th letter
Orderings of letters

Step 1: Chose first letter (5 options)
Step 2: Chose 2nd letter (4 options)
Step 3: Chose 3rd letter (3 options)
Step 4: Chose 4th letter (2 options)
Step 5: Chose 5th letter (1 option)
A permutation is an ordered arrangement of objects.

The number of unique orderings (permutations) of \( n \) distinct objects is
\[
n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1.
\]
Unique 6-digit passcodes with \textbf{six} smudges

How many unique 6-digit passcodes are possible if a phone password uses each of \textbf{six} distinct numbers?
Unique 6-digit passcodes with six smudges

How many unique 6-digit passcodes are possible if a phone password uses each of six distinct numbers?

Total = 6! = 720 passcodes

How many unique passcodes are possible if a phone password is some ordered subset of any 6 digits?

Total = \(10 \times 9 \times 8 \times 7 \times 6 \times 5\)

\[= \frac{10!}{4!} = 151200 \text{ passcodes}\]
Unique Bit Strings

1, 0, 1, 0, 0
Sort $n$ distinct objects
Sort $n$ distinct objects

Ayesha  Tim  Irina  Joey  Waddie
Sort $n$ distinct objects

Steps:
1. Choose 1\textsuperscript{st} can 5 options
2. Choose 2\textsuperscript{nd} can 4 options
   ... 
5. Choose 5\textsuperscript{th} can 1 option

Total $= 5 \times 4 \times 3 \times 2 \times 1$
$= 120$
Permutations II
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
- Choose $k$ objects (combinations)
- Put objects in $r$ buckets

Distinct (distinguishable)
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$

- Choose $k$ objects (combinations)
  - Some distinct

- Put objects in $r$ buckets
How many ways can we sort coke cans!

Coke  Coke0  Coke  Coke0  Coke0
Sort $n$ distinct objects

Ayesha  Tim  Irina  Joey  Waddie

# of permutations =
Sort semi-distinct objects

All distinct

Ayesha  Tim  Irina  Joey  Waddie

Some indistinct

Coke  Tim  Coke  Joey  Waddie

Order $n$
distinct objects $n!$
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{permutations of distinct objects} = \text{permutations considering some objects are indistinct} \times \text{Permutations of just the indistinct objects}
\]
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{permutations of distinct objects} = \text{permutations of just the indistinct objects}
\]
General approach to counting permutations

When there are $n$ objects such that

- $n_1$ are the same (indistinguishable or indistinct), and
- $n_2$ are the same, and

... $n_r$ are the same,

The number of unique orderings (permutations) is

\[
\frac{n!}{n_1!n_2!\cdots n_r!}.
\]

For each group of indistinct objects, divide by the overcounted permutations.
Sort semi-distinct objects

How many permutations?

Order $n$ semi-distinct objects $\frac{n!}{n_1! n_2! \cdots n_r!}$
How many letter orderings are possible for the following strings?

1. BOBA

2. MISSISSIPPI
Strings

How many letter orderings are possible for the following strings?

1. BOBA

\[
\frac{4!}{2!} = 12
\]

2. MISSISSIPPI

\[
\frac{11!}{1!4!4!2!} = 34,650
\]
To the Code!

```python
import itertools

def main):
    letters = ['b','o','b','a']
    perms = set(itertools.permutations(letters))
    for perm in perms:
        pretty_perm = ''.join(perm)
        print(pretty_perm)

import math

def main():
    n = math.factorial(4)
    d = math.factorial(2)
    print(n / d)
```
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1! n_2! \cdots n_r!}$
- Choose $k$ objects (combinations)
- Put objects in $r$ buckets
Unique 6-digit passcodes with six smudges

How many unique 6-digit passcodes are possible if a phone password uses each of six distinct numbers?

Total = 6!

= 720 passcodes
Unique 6-digit passcodes with **five** smudges

How many unique 6-digit passcodes are possible if a phone password uses each of **five** distinct numbers?

**Steps:**

1. Choose digit to repeat  
   - 5 outcomes
2. Create passcode  
   - (sort 6 digits: 4 distinct, 2 indistinct)

\[
\text{Total} = 5 \times \frac{6!}{2!} = 1,800 \text{ passcodes}
\]
Combinations I
Summary of Combinatorics

Counting tasks on $n$ objects

Sort objects (permutations)

- Distinct (distinguishable)
  - $n!$

Choose $k$ objects (combinations)

- Some distinct
  - $\frac{n!}{n_1!n_2!\cdots n_r!}$

Put objects in $r$ buckets

- Distinct
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...
Combinations with cake

There are $n = 20$ people. How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   
   $n!$ ways
Combinations with cake

There are \( n = 20 \) people. How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line

   \( n! \) ways

2. Put first \( k \) in cake room

   1 way
Combinations with cake

There are \( n = 20 \) people.
How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line

\( n! \) ways

2. Put first \( k \) in cake room

1 way
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   - $n!$ ways

2. Put first $k$ in cake room
   - 1 way

3. Allow cake group to mingle
   - $k!$ different permutations lead to the same mingle

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Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   $n!$ ways

2. Put first $k$ in cake room
   1 way

3. Allow cake group to mingle
   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle
Combinations with cake

There are $n = 20$ people. How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   - $n!$ ways

2. Put first $k$ in cake room
   - 1 way

3. Allow cake group to mingle
   - $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle
   - $(n - k)!$ different permutations lead to the same mingle
Combinations

A combination is an unordered selection of $k$ objects from a set of $n$ distinct objects.

The number of ways of making this selection is

\[
\frac{n!}{k! (n-k)!} = n! \times \frac{1}{k!} \times \frac{1}{(n-k)!}
\]

1. Order $n$ distinct objects
2. Take first $k$ as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice
Combinations

A combination is an unordered selection of $k$ objects from a set of $n$ distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k! (n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

Fun Fact: $\binom{n}{k} = \binom{n}{n-k}$
Probability textbooks

How many ways are there to choose 3 books from a set of 6 distinct books?

\[
\binom{6}{3} = \frac{6!}{3!3!} = 20 \text{ ways}
\]
To the code!

How many unique hands of 5 cards are there in a 52 card deck?

```python
def main():
    cards = make_deck()
    all_hands = itertools.combinations(cards, 5)
    for hand in all_hands:
        print(hand)

def main():
    total = math.comb(52, 5)
    print(total)
```
Buckets and The Divider Method
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1! n_2! \cdots n_r!}$

- Choose $k$ objects (combinations)
  - Distinct
    - $\binom{n}{k}$

- Put objects in $r$ buckets
  - Distinct
  - Indistinct
Balls and urns  Hash tables and distinct strings

How many ways are there to hash $n$ distinct strings to $r$ buckets?

Steps:

1. Bucket 1\textsuperscript{st} string
2. Bucket 2\textsuperscript{nd} string
   ...
$n$. Bucket $n$\textsuperscript{th} string

$r^n$ outcomes
Summary of Combinatorics

Counting tasks on $n$ objects

Sort objects (permutations)
- Distinct (distinguishable)
  - $n!$

Choose $k$ objects (combinations)
- Some distinct
  - \( \binom{n}{k} \)
- Distinct
  - $\frac{n!}{n_1!n_2!\ldots n_r!}$

Put objects in $r$ buckets
- Distinct
  - $r^n$
- Indistinct
Servers and **indistinct** requests

How many ways are there to distribute $n$ **indistinct** web requests to $r$ servers?

**Goal**

Server 1 has $x_1$ requests,
Server 2 has $x_2$ requests,
...
Server $r$ has $x_r$ requests

constraint: $\sum_{i=1}^{r} x_i = n$
How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?
Bicycle helmet sales

1 possible assignment outcome:

**Goal**  Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

Consider the following generative process...
The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

**Goal** Order $n$ indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct

![Image of children and dividers]
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

Goal Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers

\((n + r - 1)!\)
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers

\[
(n + r - 1)!
\]

2. Make \( n \) objects indistinct

\[
\frac{1}{n!}
\]
The divider method: A generative proof

How many ways can we assign \( n = 5 \) indistinct children to \( r = 4 \) distinct bicycle helmet styles?

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers

\[
(n + r - 1)! 
\]

2. Make \( n \) objects indistinct

\[
\frac{1}{n!} 
\]

3. Make \( r - 1 \) dividers indistinct

\[
\frac{1}{(r - 1)!} 
\]
The divider method

The number of ways to distribute $n$ indistinct objects into $r$ buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that $n$ are indistinct objects, and $r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$
Integer solutions to equations

How many integer solutions are there to the following equation:

\[ x_1 + x_2 + \cdots + x_r = n, \]

where for all \( i \), \( x_i \) is an integer such that \( 0 \leq x_i \leq n \)?

Positive integer equations can be solved with the divider method.

Divider method (\( n \) indistinct objects, \( r \) buckets) \( \binom{n + r - 1}{r - 1} \)
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - Some distinct
- Choose $k$ objects (combinations)
  - Distinct
  - 1 group
- Put objects in $r$ buckets
  - Distinct
  - Indistinct
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Ready for the first 4 problems (and the rest on Friday)

Now, more than ever, get started early
Want to go deep?

Enigma Machine

One of the very first computers was built to break the Nazi “enigma” codes in WW2. It was a hard problem because the “enigma” machine, used to make secret codes, had so many unique configurations. Every day the Nazis would choose a new configuration and if the Allies could figure out the daily configuration, they could read all enemy messages. One solution was to try all configurations until one produced legible German. This begs the question: How many configurations are there?

The WW2 machine built to search different enigma configurations.

The enigma machine has three rotors. Each rotor can be set to one of 26 different positions. How
Challenge: Bucketing Distincts into Many (Fixed Sized) Containers

You run an experiment that has three types of outcomes \{A, B, C\}. Among 10 runs you observe:

- 7 outcomes of type A
- 2 outcomes of type B
- 1 outcome of type C

There are $3^{10}$ unique orderings of outcomes. How many of those have exactly 7 As, 2 Bs and 3 Cs?