Conditional Probability and Bayes

Review

What is a Probability?

A number [0, 1] to which we ascribe meaning

The event we care about

How many times does it occur

$$P(E) = \lim_{n \to \infty} \frac{\text{count}(E)}{n}$$

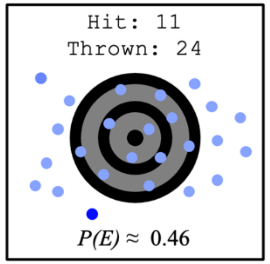
Out of (close to) infinite trials

Sources of Probability

Infinite Trials

$$P(E) = \lim_{n \to \infty} \frac{\text{count}(E)}{n}$$

n is the number of trails



The "event" *E* is that you hit the target

Datasets

Dataset of weather

Trial	Value
1	Rainy
2	Sunny
3	Rainy
4	Cloudy
5	Rainy
6	Sunny
	...
10000	Cloudy

Let E be the event that it is **Sunny**

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

$$\approx \frac{\text{Count}(E)}{10000}$$

$$\approx \frac{3332}{10000} \approx 0.3332$$

Equally Likely Outcomes

Sum of Two Die = 7?

Roll two 6-sidex dice. What is probability the sum = 7?
Let E be the event that the sum is 7

Each outcome

Value dice 1

Value dice 2

S = { [1,1] [1,2] [1,3] [1,4] [1,5] **[1,6]**
[2,1] [2,2] [2,3] [2,4] **[2,5]** [2,6]
[3,1] [3,2] [3,3] **[3,4]** [3,5] [3,6]
[4,1] [4,2] **[4,3]** [4,4] [4,5] [4,6]
[5,1] **[5,2]** [5,3] [5,4] [5,5] [5,6]
[6,1] [6,2] [6,3] [6,4] [6,5] [6,6] }

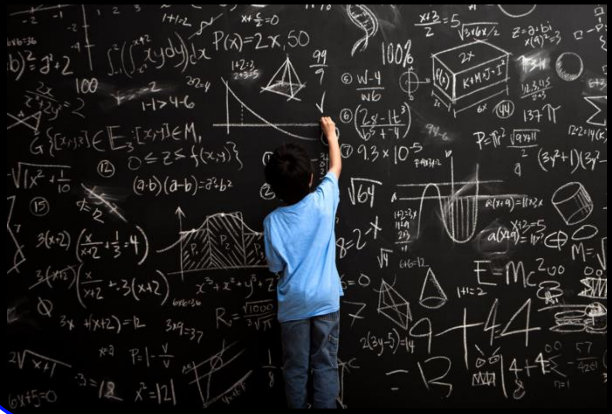
E = *in blue*

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = 0.16\bar{6}$$

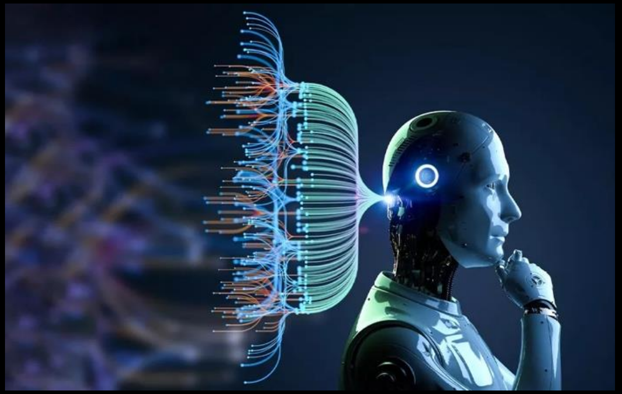
Piech, CS109, 2021

Stanford University 184

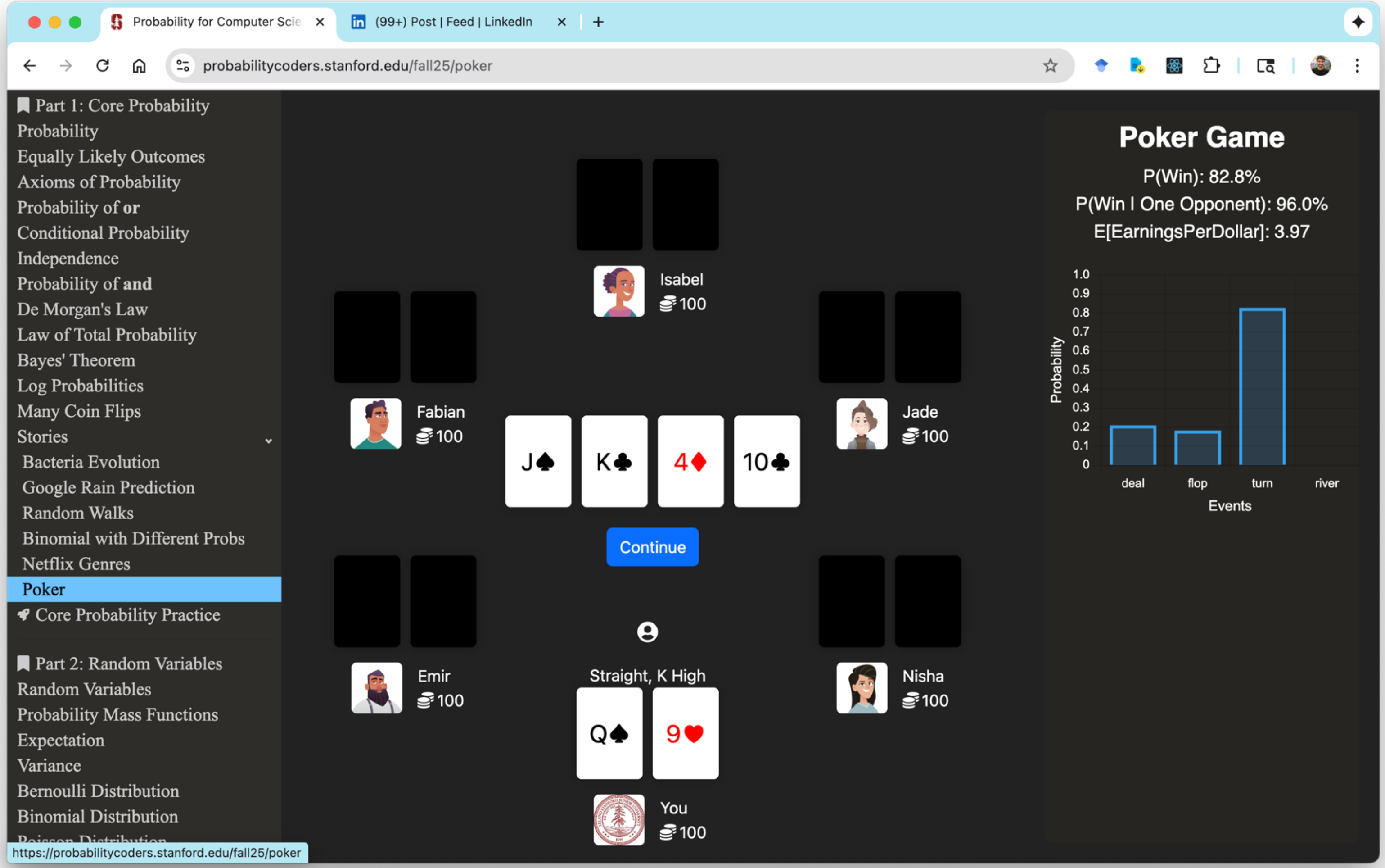
Analytics

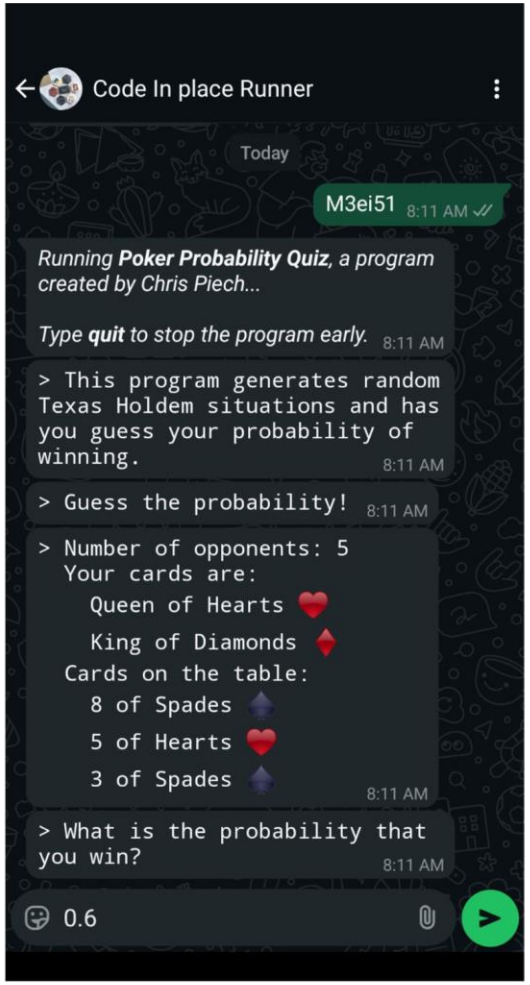


AI Models



How Does This Work?





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MLB 040

Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$

$P(E^c) = 1 - P(E)$?

$$P(E \cup E^c) = P(E) + P(E^c)$$

Axiom 3. Since E and E^c are mutually exclusive

$$P(S) = P(E) + P(E^c)$$

Since everything must either be in E or E^c

$$1 = P(E) + P(E^c)$$

Axiom 2

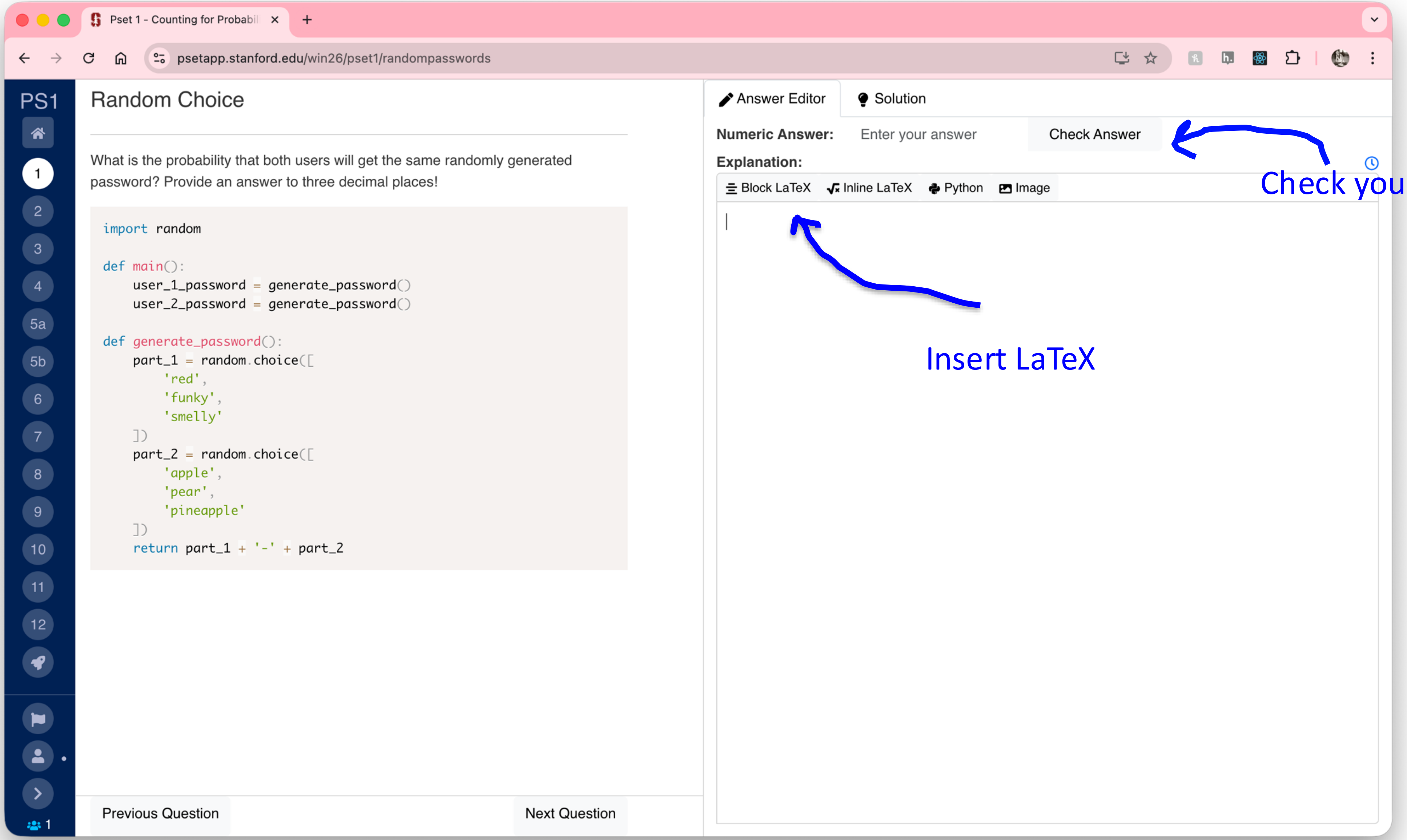
$$P(E^c) = 1 - P(E)$$

Rearrange

End Review

Announcements

Problem Set #1 is out newish

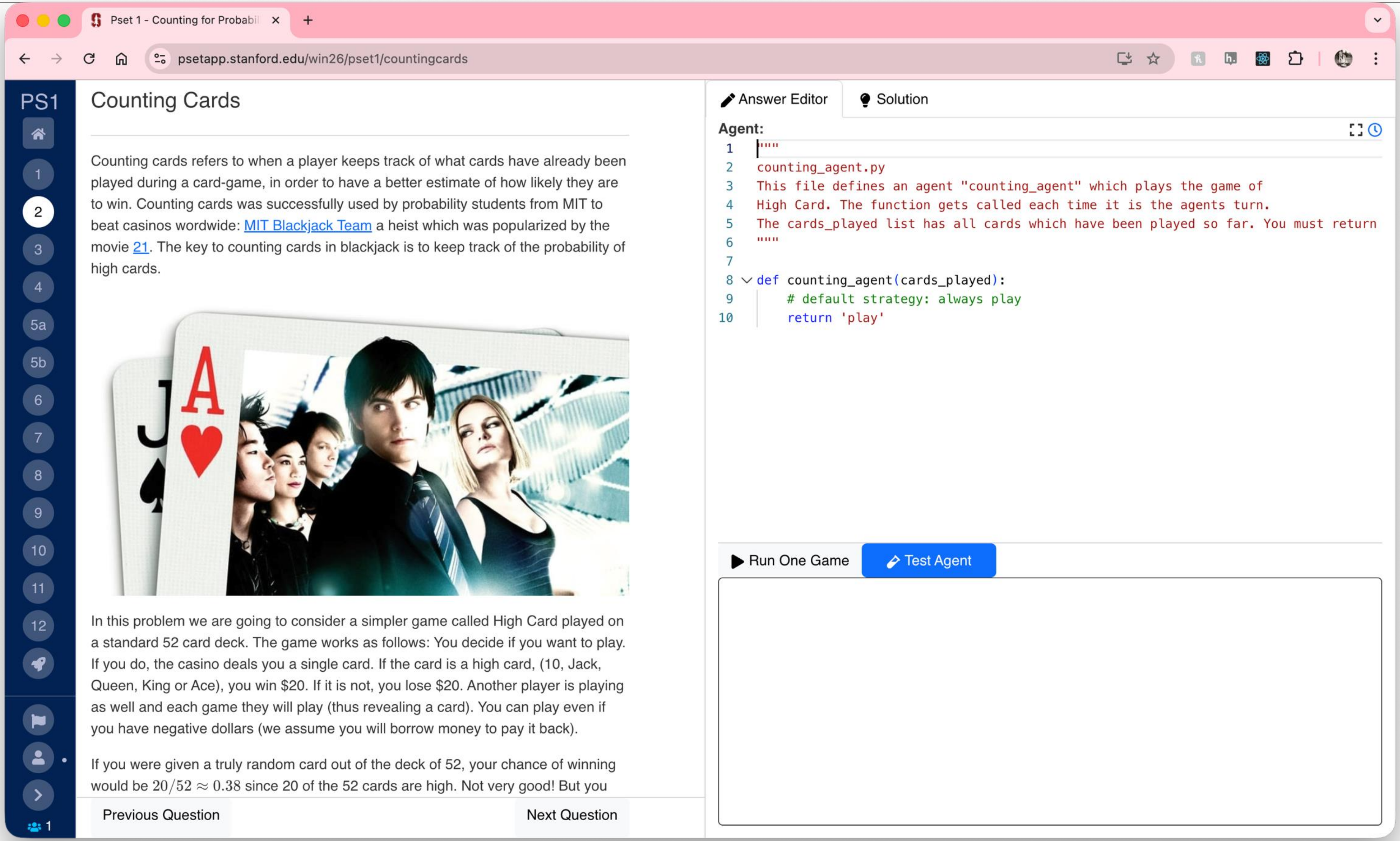


Check your answer

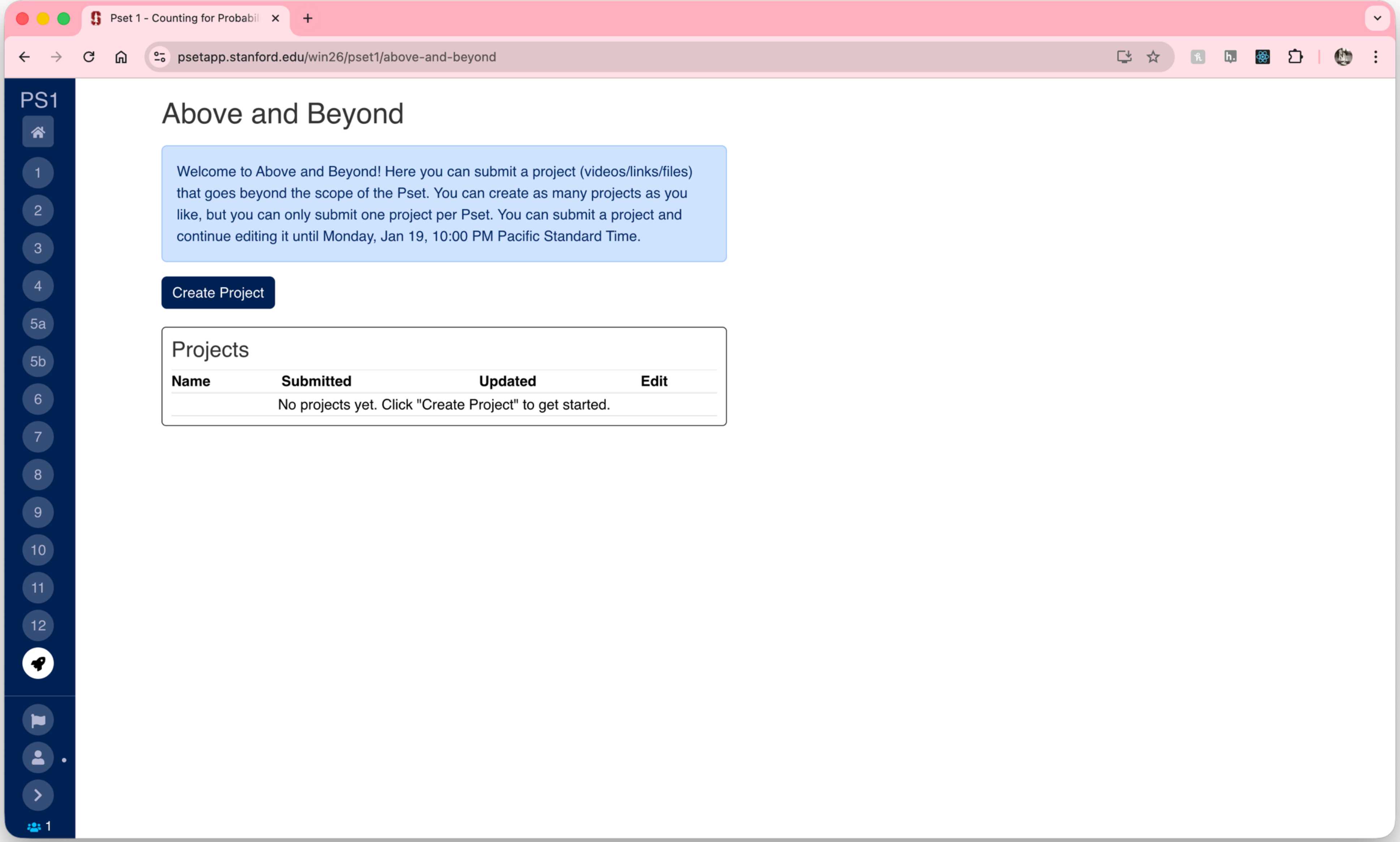
Auto Submission

Write Agents

newish



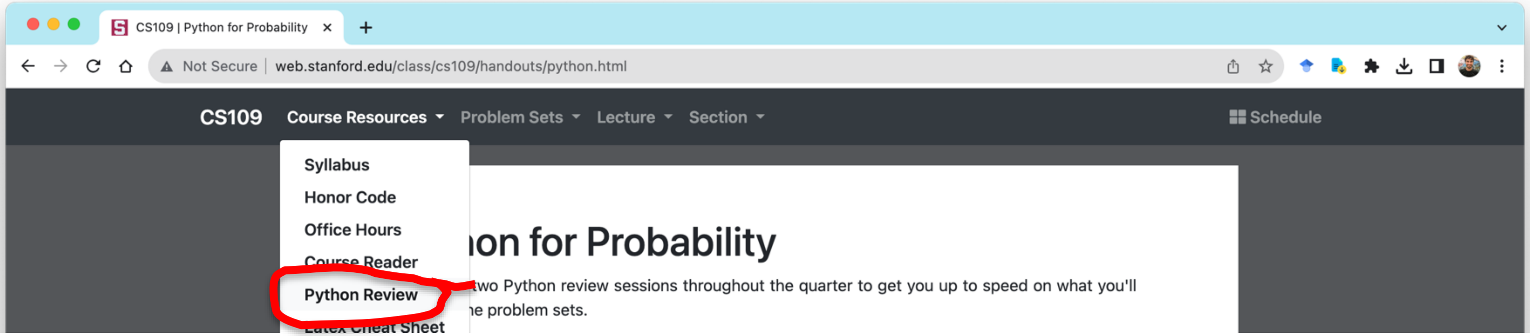
Above and Beyond new



Python Review Session

Friday at 4:30-5:30pm PT, recorded

Find links, recordings, and setup here

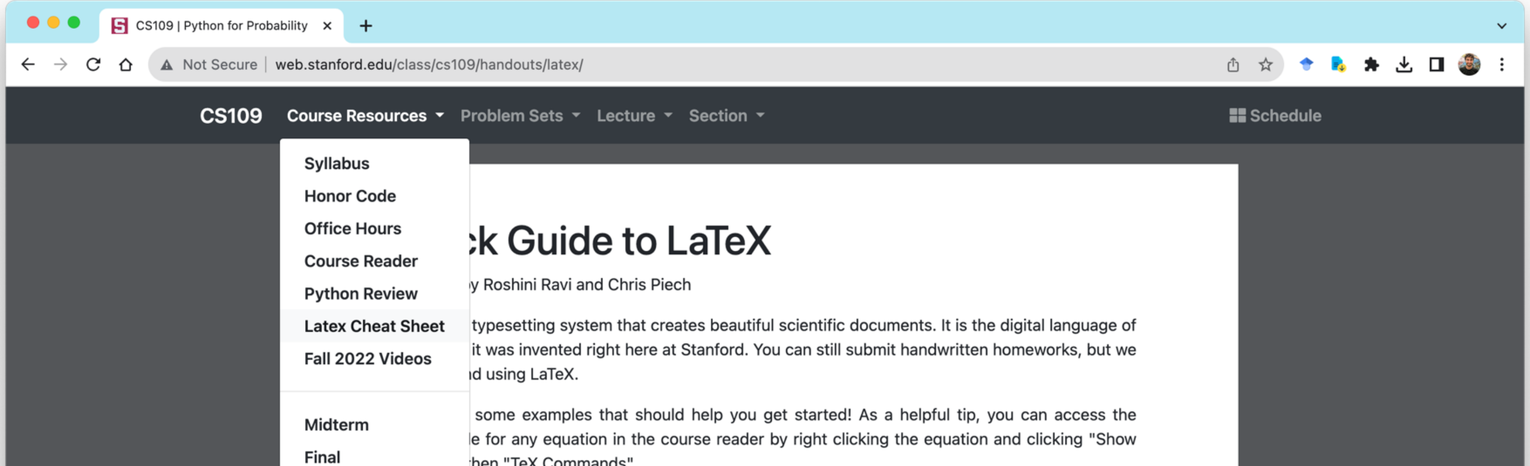


Learn LaTex

```
\begin{aligned}
P(E) 
&= \sum_{i=0}^n e^{i}\\
&= 0.25
\end{aligned}
```

$$\begin{aligned} P(E) &= \sum_{i=0}^n e^{i}\\ &= 0.25 \end{aligned}$$

Done



Inline LaTex new

Explanation:

Block LaTeX | Inline LaTeX | Python | Image

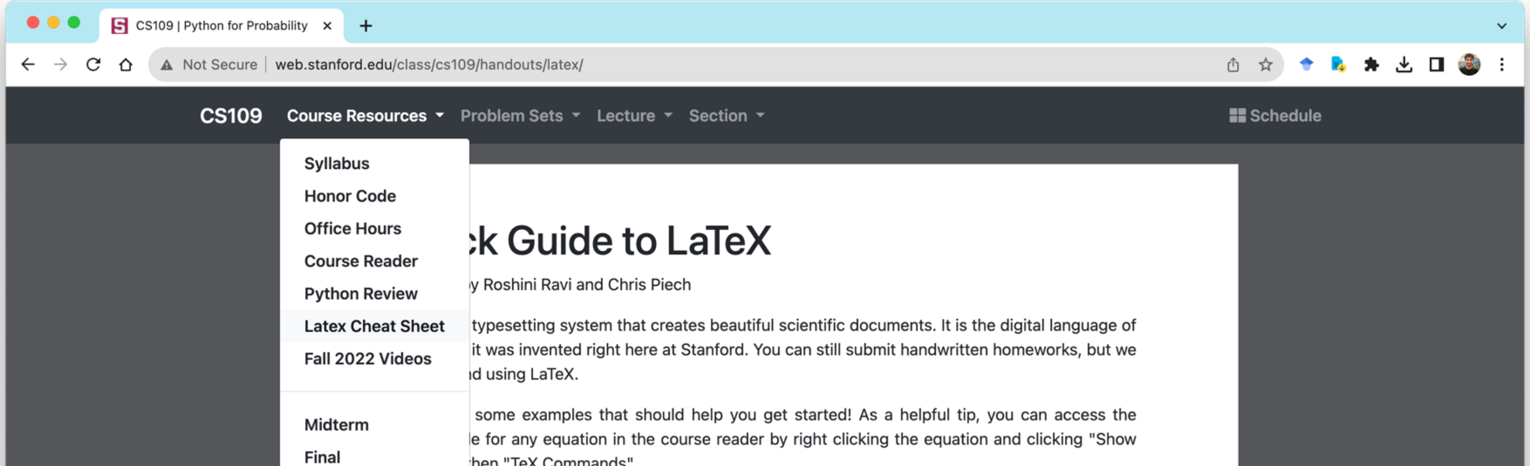
This is an example of inline latex. Let $$Y$

When you type the closing $

Explanation:

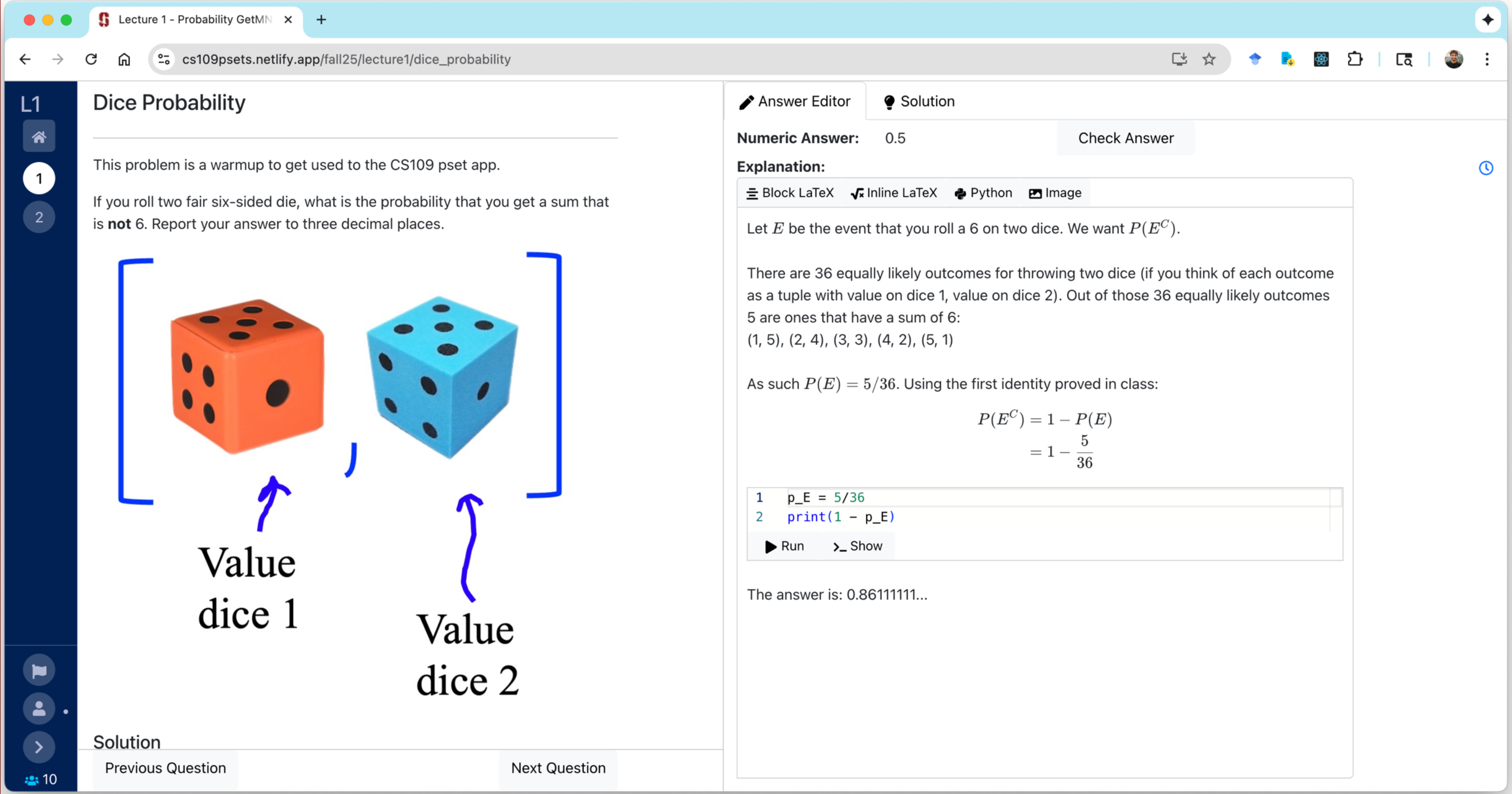
Block LaTeX | Inline LaTeX | Python | Image

This is an example of inline latex. Let Y



What Makes for a Good Answer

https://psetapp.stanford.edu/win26/lecture2/dice_probability



If you notice a bug?

It should be robust, but things can happen.

Let me know: send an email to jwoodrow@stanford.edu. I need your email and the approximate time you encountered the bug.

Honor Code

Always remember: You need to be able to recreate your ability on an exam. And in the real world. This is a foundation course. **Cheating in CS109 is cheating yourself and your friends**.

Talk to your friends about the **concepts**, not the solution. Words must be your own.

Practice the **art of teaching**. Three most important things to know:

1. Do not give away the answer
2. Always be respectful
3. Know what you don’t know

Section Signups Open Tomorrow

Sunday	Monday	Tuesday	Wed	Thursday	Friday	Saturday
			Jan 7 You are here	Jan 8 Section signup opens	Jan 9	Jan 10
Jan 11 Section signups close at 5pm	Jan 12	Jan 13 Sections announced	Jan 14	Jan 15, Jan 16 First section!		Jan 17

End Announcements

Learning Goal for Today: Conditional Probability

$P(E \text{ and } F)$

Chain rule

Definition of conditional probability

$P(E|F)$

Law of Total Probability

Bayes' Theorem

$P(E)$

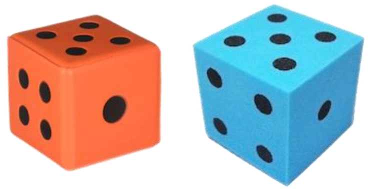
$P(F|E)$

Conditional Probability

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Roll two 6-sided fair dice. What is P(sum = 7)?



S = {(1,1) (1,2) (1,3) (1,4) (1,5) **(1,6)**
(2,1) (2,2) (2,3) (2,4) **(2,5)** (2,6)
(3,1) (3,2) (3,3) **(3,4)** (3,5) (3,6)
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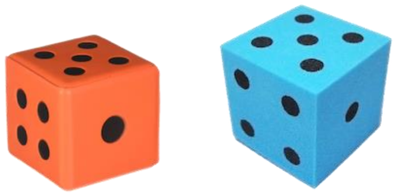
(5,1) **(5,2)** (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) }

$E =$ *In blue*

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.
You want them to sum to 4.



What is the best outcome for $P(D_1)$?

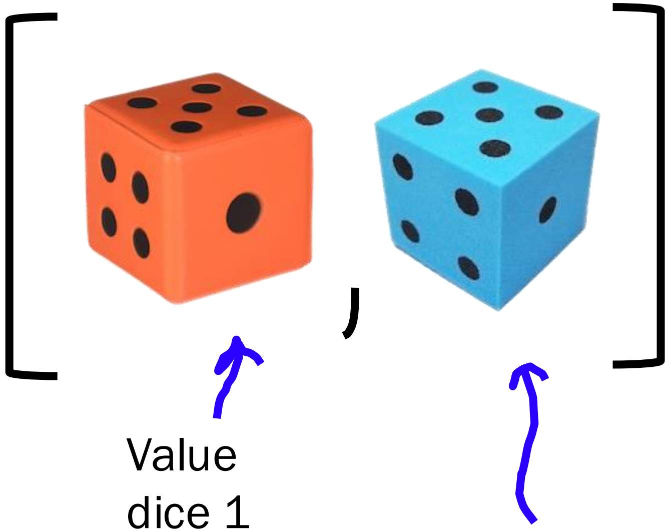
Your Choices:

- A. 1 and 3 tie for best
- B. 1, 2 and 3 tie for best
- C. 2 is the best
- D. Other/none/more than one

Sum of Two Die = 4?

Roll two 6-sidex dice. What is probability the sum = 4?
Let E be the event that the sum is 4

Each outcome



Value dice 1

Value dice 2

S = { [1,1] [1,2] [1,3] [1,4] [1,5] [1,6]
[2,1] [2,2] [2,3] [2,4] [2,5] [2,6]
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[6,1] [6,2] [6,3] [6,4] [6,5] [6,6] }

E = *In red*

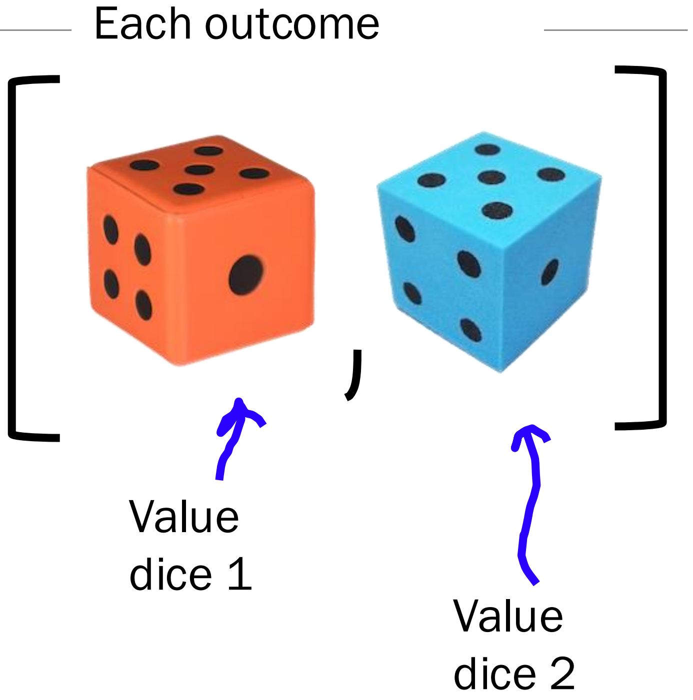
$$P(E) = \frac{|E|}{|S|} = \frac{3}{36} = 0.08\bar{3}$$

Sum of Two Die = 4? Condition on F: $D_1 = 2$

Roll two 6-sidex dice. What is probability the sum = 4?
Let E be the event that the sum is 4

S = {

[1,1]	[1,2]	[1,3]	[1,4]	[1,5]	[1,6]
[2,1]	[2,2]	[2,3]	[2,4]	[2,5]	[2,6]
[3,1]	[3,2]	[3,3]	[3,4]	[3,5]	[3,6]
[4,1]	[4,2]	[4,3]	[4,4]	[4,5]	[4,6]
[5,1]	[5,2]	[5,3]	[5,4]	[5,5]	[5,6]
[6,1]	[6,2]	[6,3]	[6,4]	[6,5]	[6,6] }



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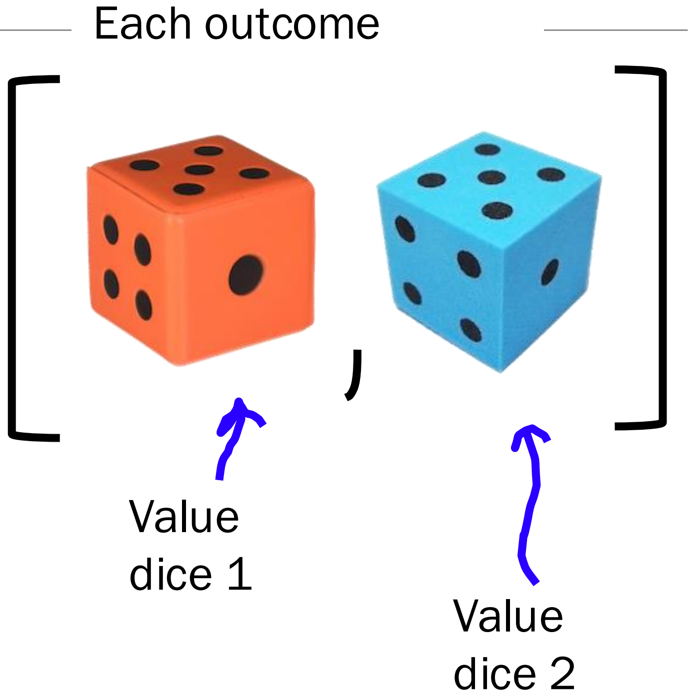

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Dice, our misunderstood friends

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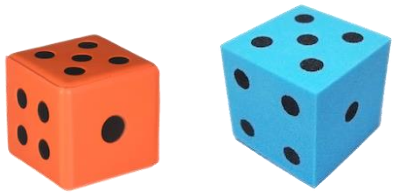


Let E be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$

$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let F be event: $D_1 = 2$.

What is $P(E$, given F already observed$)$?

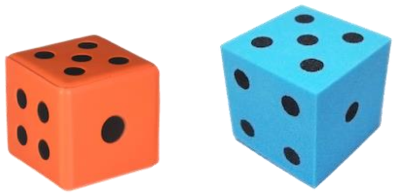
$S = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)\}$

$E = \{(2,2)\}$

$P(E) = 1/6$

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.



Let E be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$

$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let F be event: $D_1 = \mathbf{3}$.

What is $P(E$, given F already observed$)$?

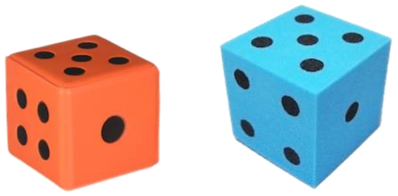
$S = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$

$E = \{(3,1)\}$

$P(E) = 1/6$

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.



Let E be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$

$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let F be event: $D_1 = \mathbf{5}$.

What is $P(E$, given F already observed$)$?

$S = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

$E = \{\ \ \}$

$P(E) = 0/6$

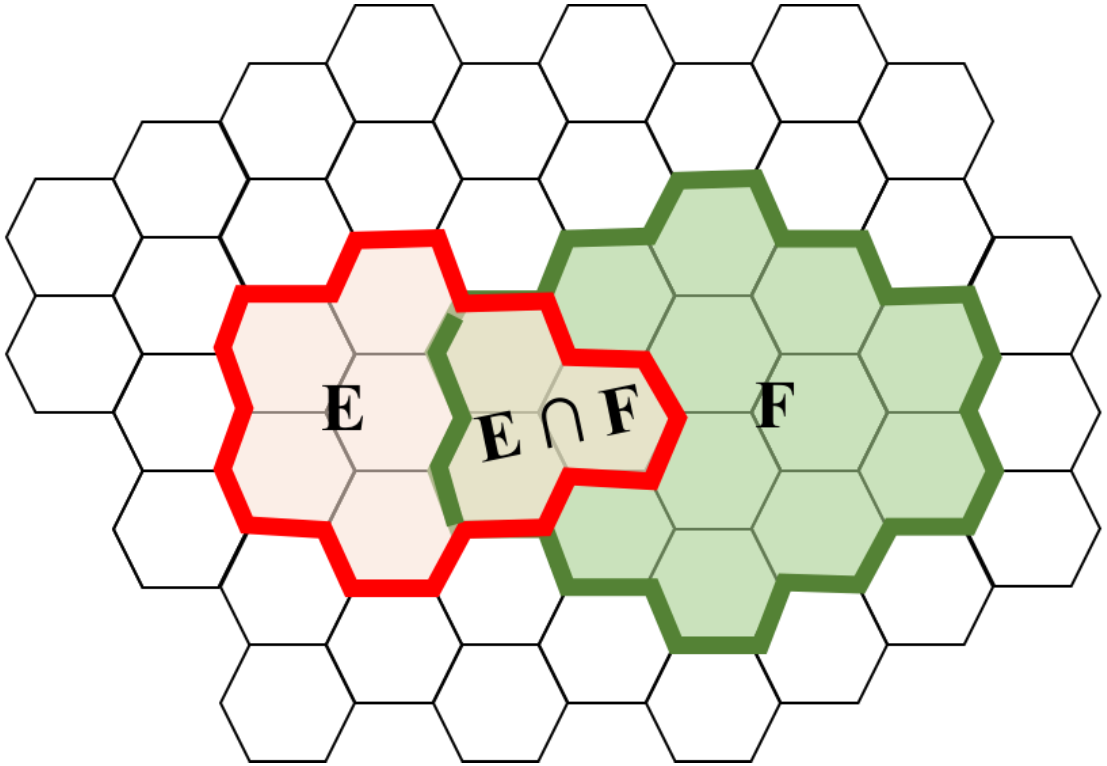
Conditional Probability

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Written as:	$P(E\|F)$
Means:	"$P(E,$ given F already observed)"
Sample space →	all possible outcomes consistent with F (i.e. S and F)
Event →	all outcomes in E consistent with F (i.e. E and F)

Conditional Probability, visual intuition

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.



$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) =$$

Conditional Probability, visual intuition

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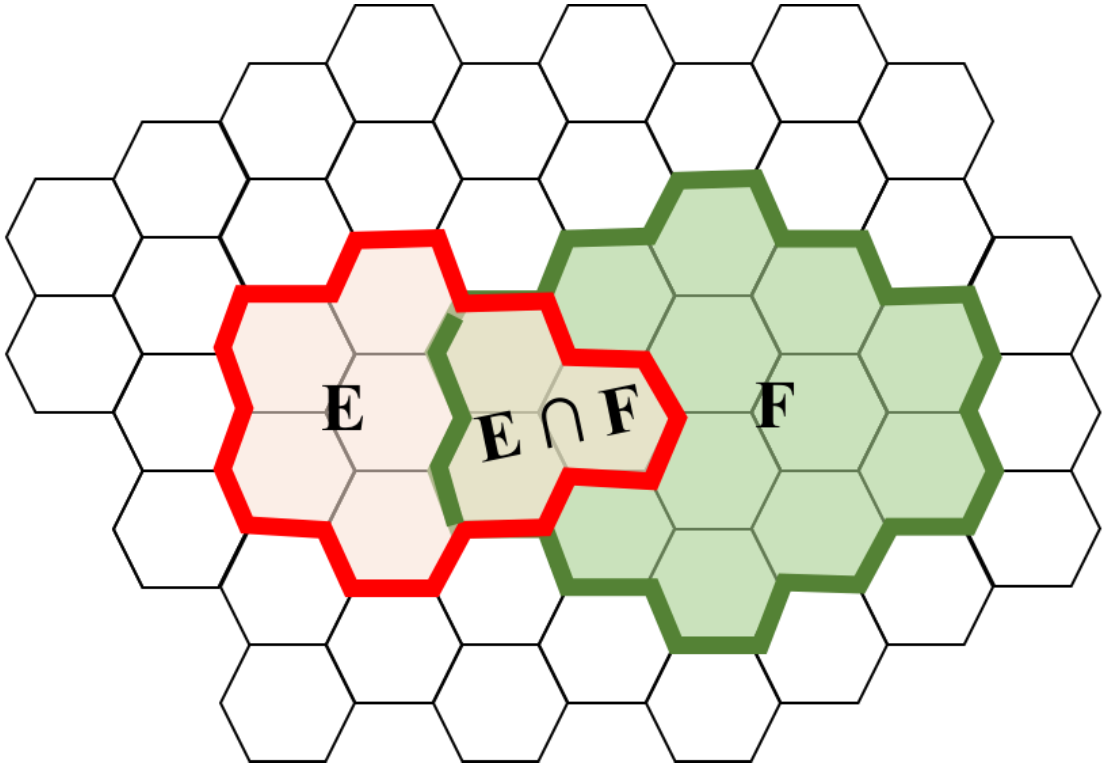

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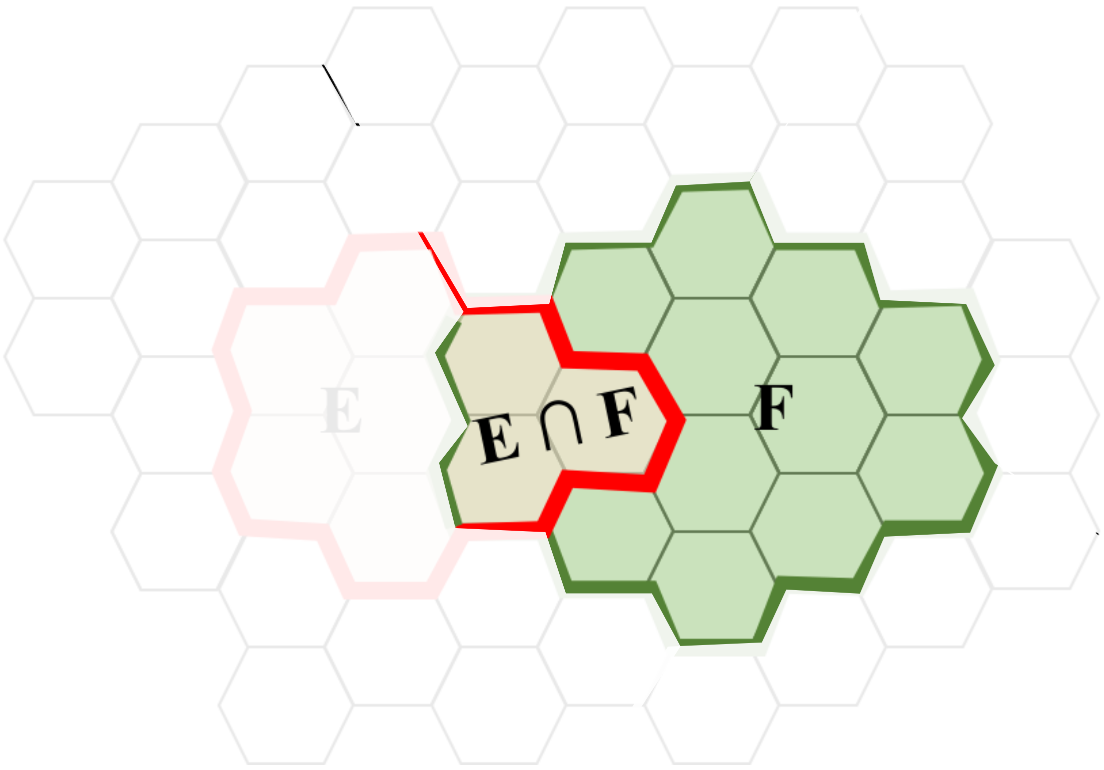

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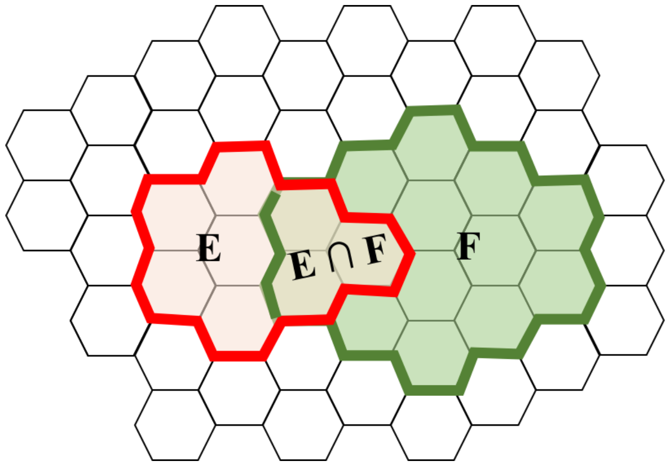
Conditional Probability, visual intuition

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Shorthand notation for set intersection (aka set "and")

$$\Pr(E|F) = \frac{\text{\# of outcomes in } E \text{ consistent with } F}{\text{\# of outcomes in } S \text{ consistent with } F} = \frac{|EF|}{|SF|} = \frac{|EF|}{|F|}$$



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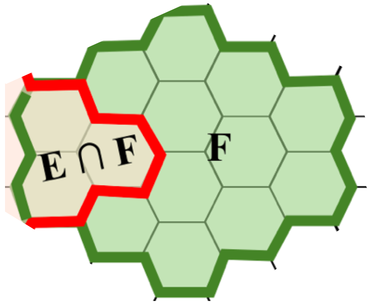

$$P(E) = \frac{8}{50} \approx 0.16$$

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Conditional probability in general

These properties hold even when outcomes are not equally likely.

General definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Shorthand for E and F

The Chain Rule (aka Product rule):

$$P(EF) = P(F)P(E|F)$$

What if P(F) = 0?

- P(E | F) undefined
- *Congratulations! Observed impossible*

Bye, land of equally likely outcomes

NETFLIX

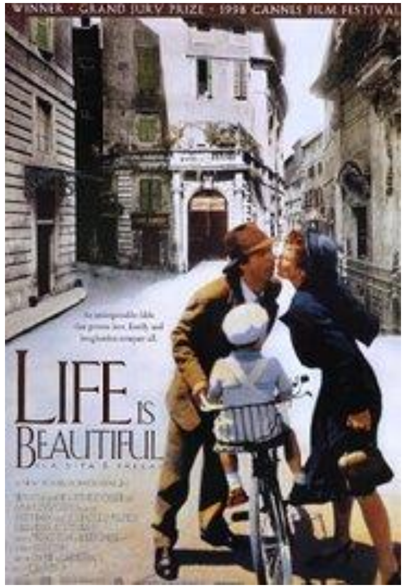
and Learn

Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

What is the probability that a user will watch Life is Beautiful?

$$P(E)$$



S = {Watch, Not Watch}

E = {Watch}

P(E) = ½ ?

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

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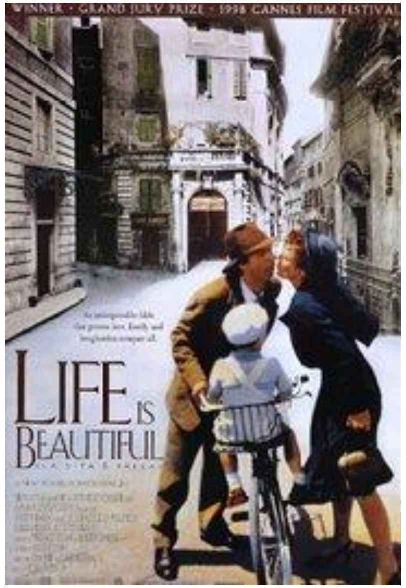

Netflix and Learn

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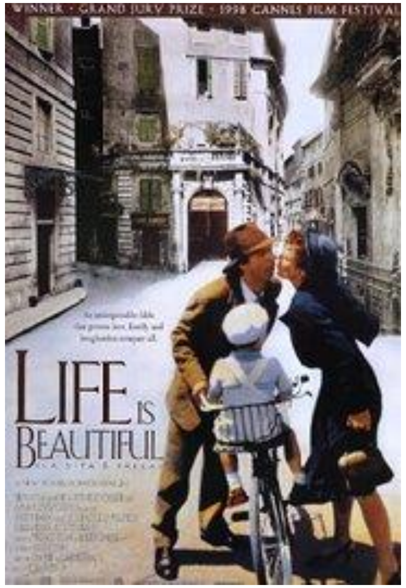


$$P(E)$$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\#\text{people who watched movie}}{\#\text{people on Netflix}}$$

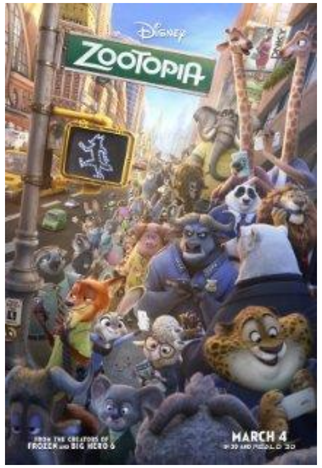
$$P(E) = 10{,}234{,}231 \,/\, 50{,}923{,}123 = 0.20$$

Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

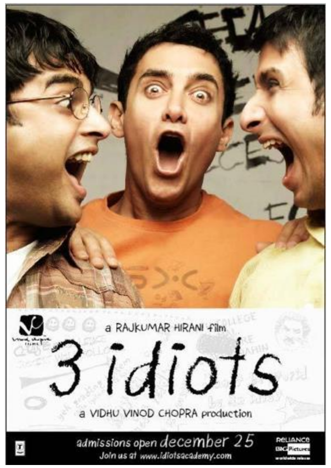
Let E be the event that a user watches the given movie.

$P(E) = 0.19$

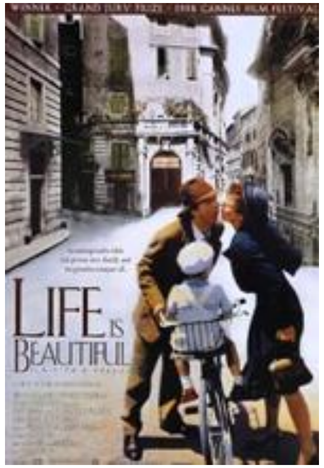


$P(E) = 0.32$

$P(E) = 0.20$



$P(E) = 0.09$



$P(E) = 0.20$

Netflix and Learn

$P(E|F) = \frac{P(EF)}{P(F)}$ Definition of Cond. Probability

Let E = a user watches Life is Beautiful.

Let F = a user watches CODA.

What is the probability that a user watches Life is Beautiful, given they watched CODA?

$$P(E|F)$$

$$P(E|F) = \frac{P(EF)}{P(F)} \approx \frac{\frac{\text{\# people who have watched both}}{\text{\# people on Netflix}}}{\frac{\text{\# people who have watched CODA}}{\text{\# people on Netflix}}}$$

Netflix and Learn

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$$\approx \frac{\text{\# people who have watched both}}{\text{\# people who have watched CODA}}$$

$$\approx 0.42$$

Netflix and Learn

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$$P(E|F) = \frac{P(EF)}{P(F)} \approx \frac{\frac{\text{\# people who have watched both}}{\text{\# people on Netflix}}}{\frac{\text{\# people who have watched CODA}}{\text{\# people on Netflix}}}$$

$$\approx \frac{\text{\# people who have watched both}}{\text{\# people who have watched CODA}}$$

$$\approx 0.42$$

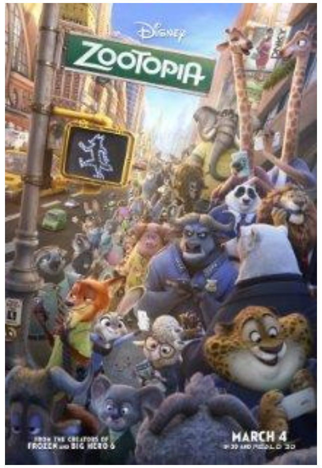
Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches CODA (2021).

$P(E) = 0.19$

$P(E|F) = 0.14$

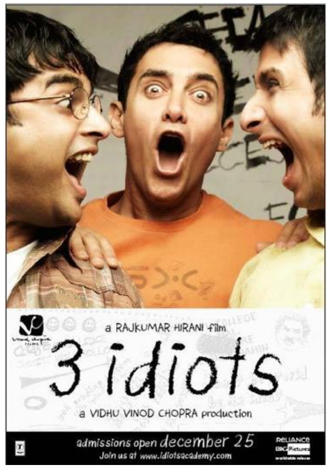


$P(E) = 0.32$

$P(E|F) = 0.35$

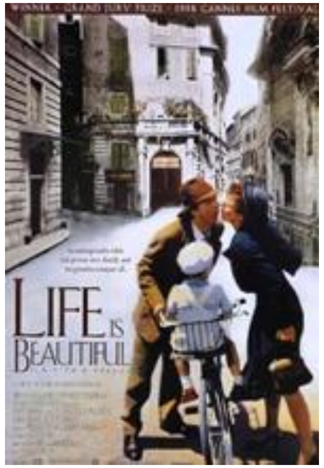
$P(E) = 0.20$

$P(E|F) = 0.20$



$P(E) = 0.09$

$P(E|F) = 0.72$



$P(E) = 0.20$

$P(E|F) = 0.42$

Machine Learning

Machine Learning is:
Probability + Data + Computers

Notation

And	Or	Given
$P(E \text{ and } F)$	$P(E \text{ or } F)$	$P(E \mid F)$
$P(E, F)$	$P(E \cup F)$	$P(E \mid F, G)$
$P(EF)$		
$P(E \cap F)$		

Probability of E given F and G

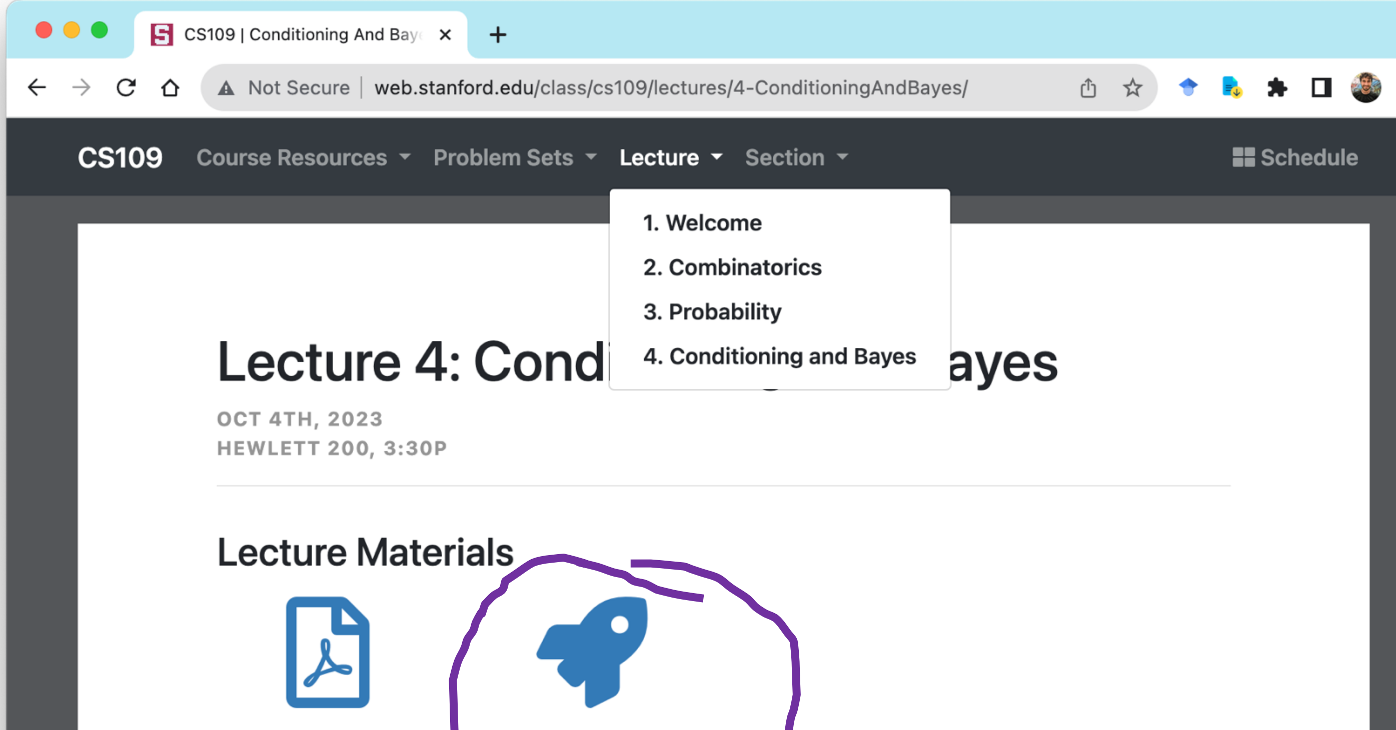
Chain Rule via Baby Poop

https://psetapp.stanford.edu/win26/lecture2/poop

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$P(EF) = P(F)P(E|F)$$

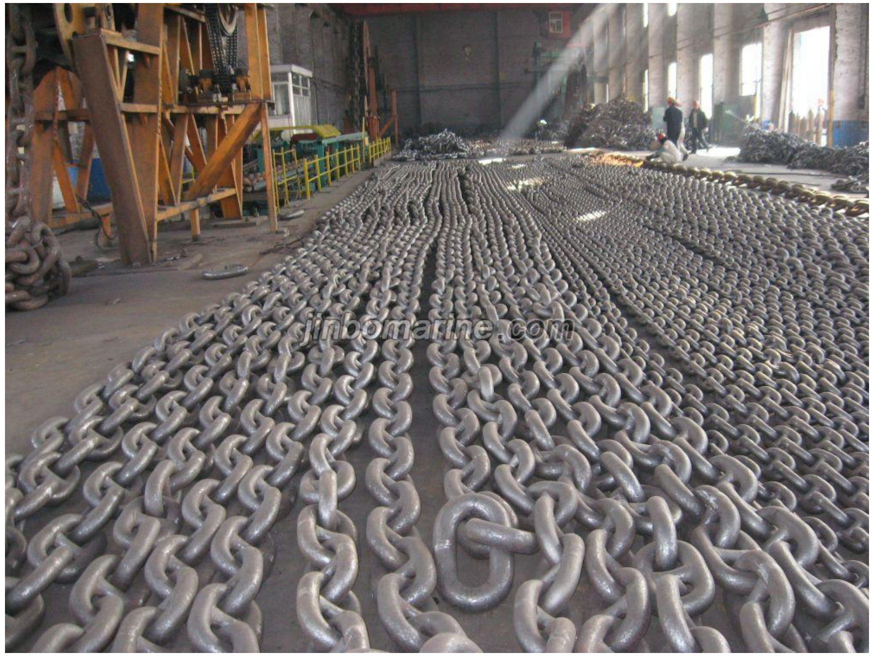
In the morning when she wakes up, a baby has a 50% chance of having pooped. The chance that a baby cries **given** that she has pooped is 50%. What is the probability that a baby **has pooped, and cries.**



Generalized Chain Rule

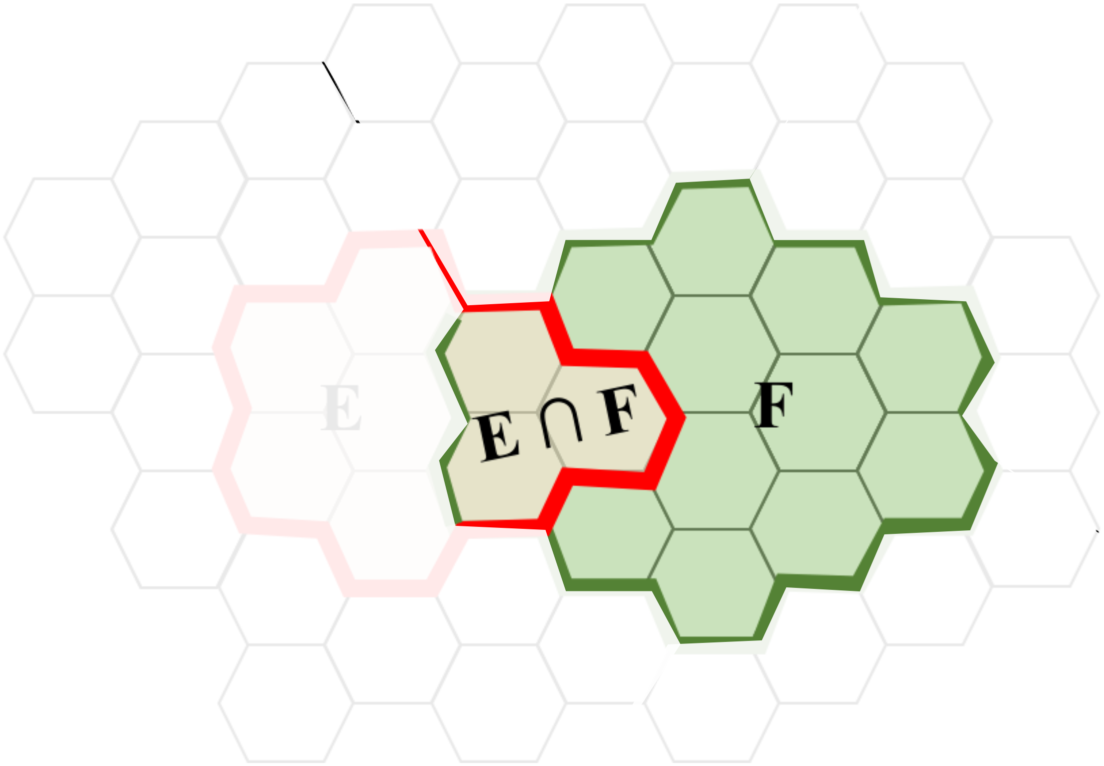
$$\Pr(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots E_n)$$

$$= \Pr(E_1) \cdot \Pr(E_2|E_1) \cdot \Pr(E_3|E_1, E_2) \cdots \Pr(E_n|E_1, E_2 \ldots E_{n-1})$$



Conditional Paradigm

When you condition on an event (or multiple events), you enter a world where all the rules of probability still hold.

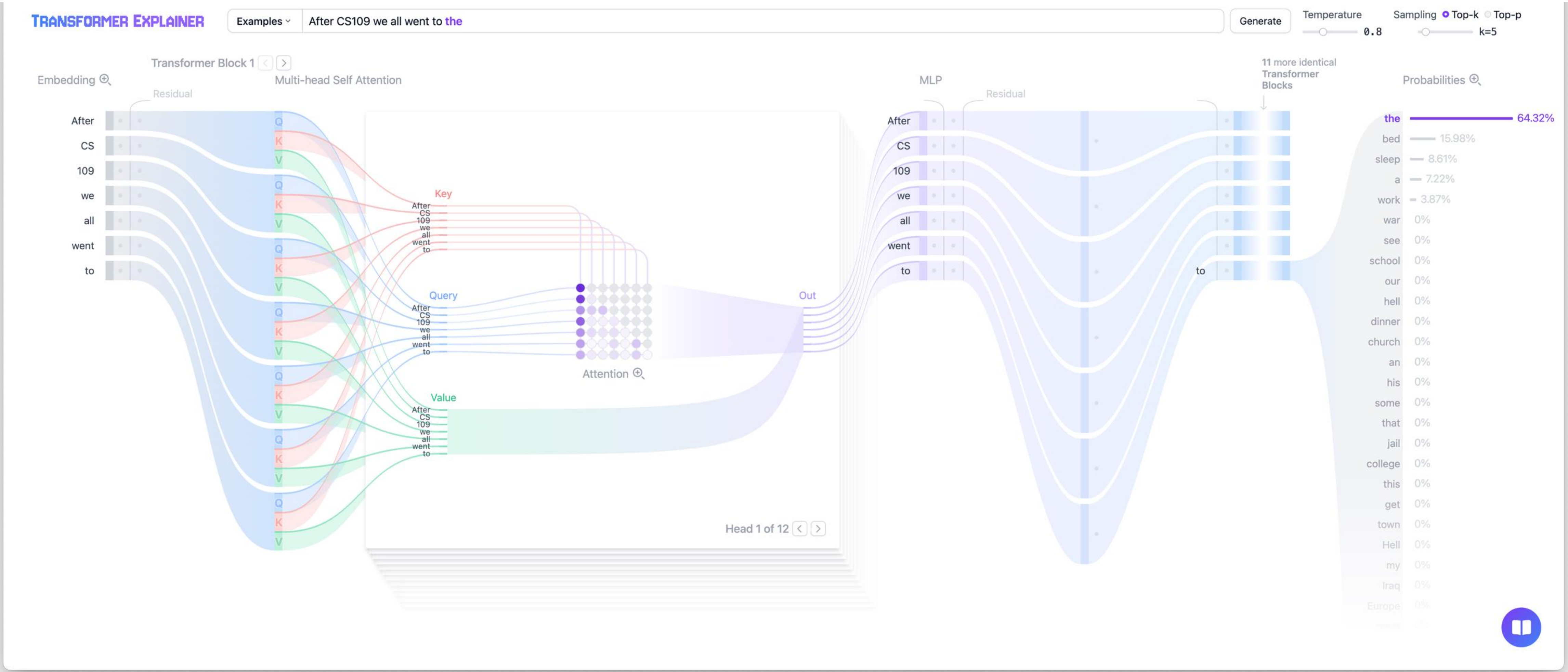


For example:

$$P(E^C|F) = 1 - P(E|F)$$

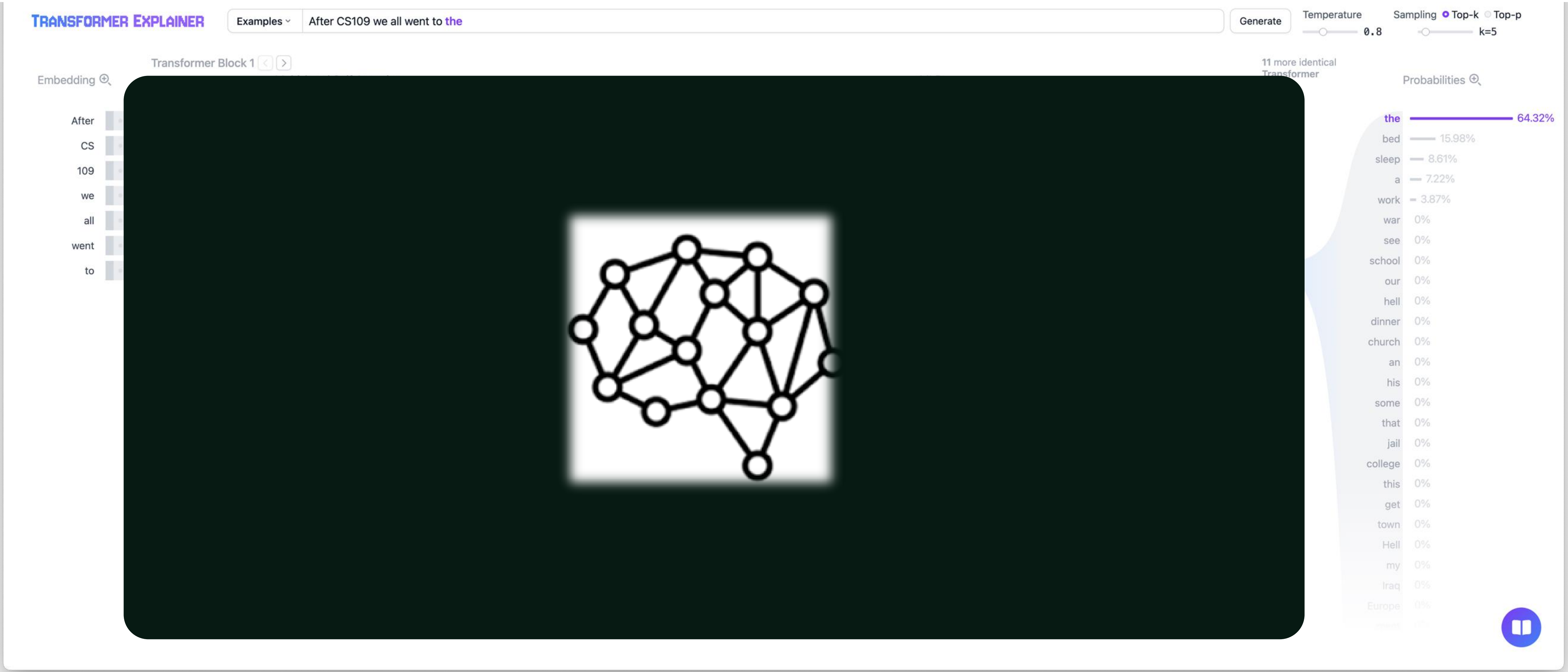
and Learn

Under the hood of a Large Language Model



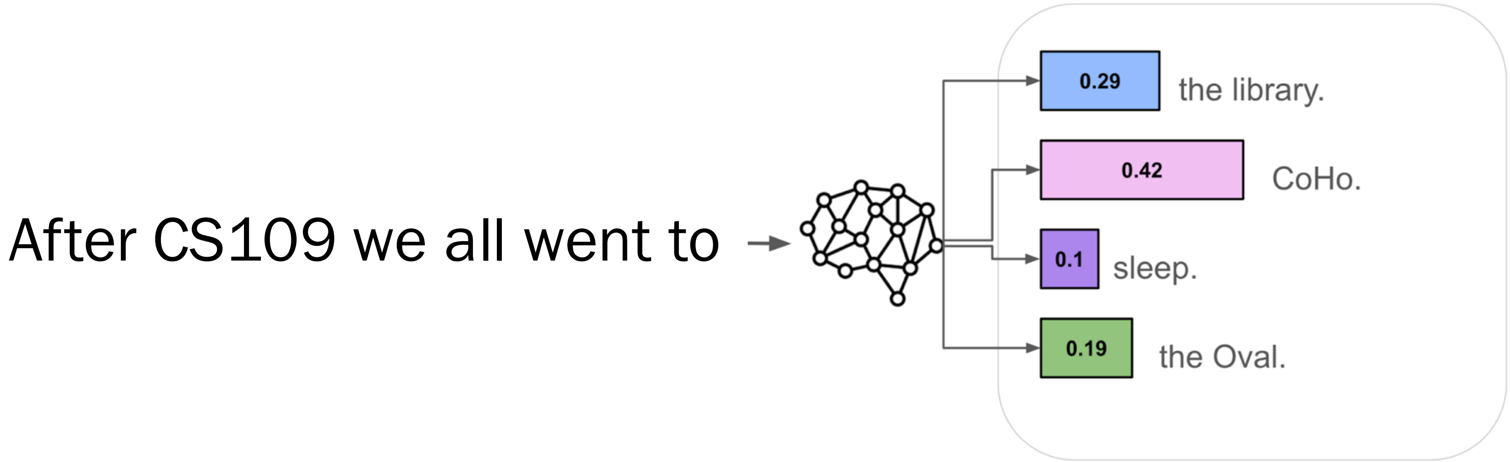
https://poloclub.github.io/transformer-explainer/

Under the hood of a Large Language Model



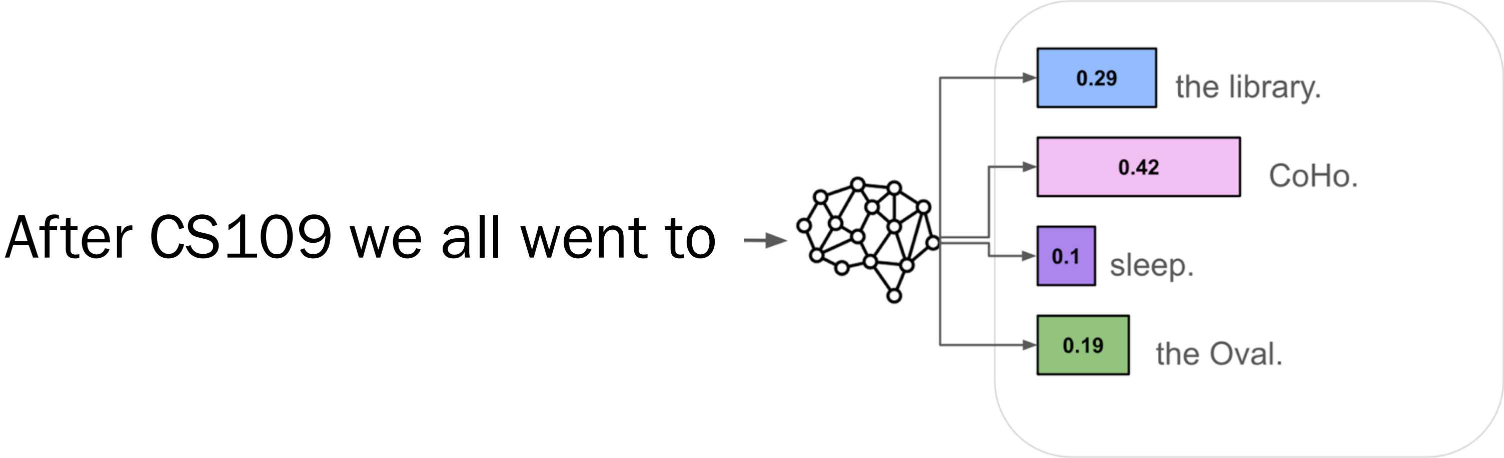
https://poloclub.github.io/transformer-explainer/

LLM is a Conditional Probability Machine



LLM is a Conditional Probability Machine

Let T_i be the ith token in a prompt.
An LLM is built to compute:
$P(T_i|T_1, \ldots, T_{i-1})$



Q1: What is the probability of the string **"After CS109 we"**

Q2: the string **"went dancing"** coming after the string **"After CS109 we"**

Law of Total Probability

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule

Definition of conditional probability

$$P(E|F)$$

Law of Total Probability

$$P(E)$$

Baby Poop Revisited

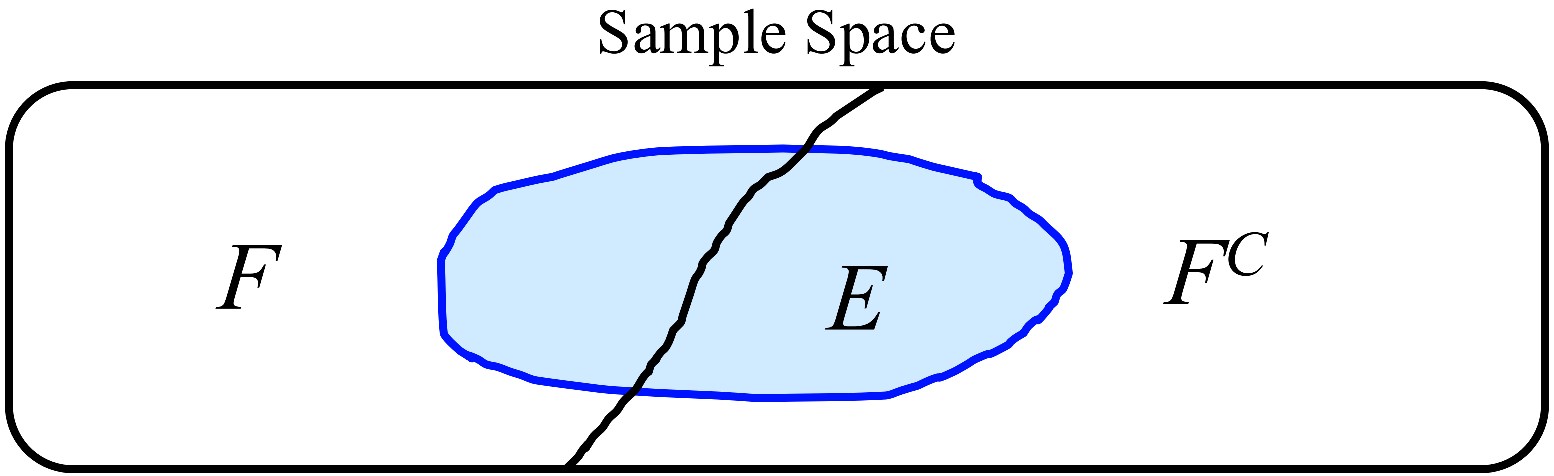
In the morning when she wakes up, a baby has a 50% chance of having pooped.
The chance that a baby cries given that she has pooped is 50%.

What is the probability of crying, unconditioned?

What information do you need?

Law of Total Probability

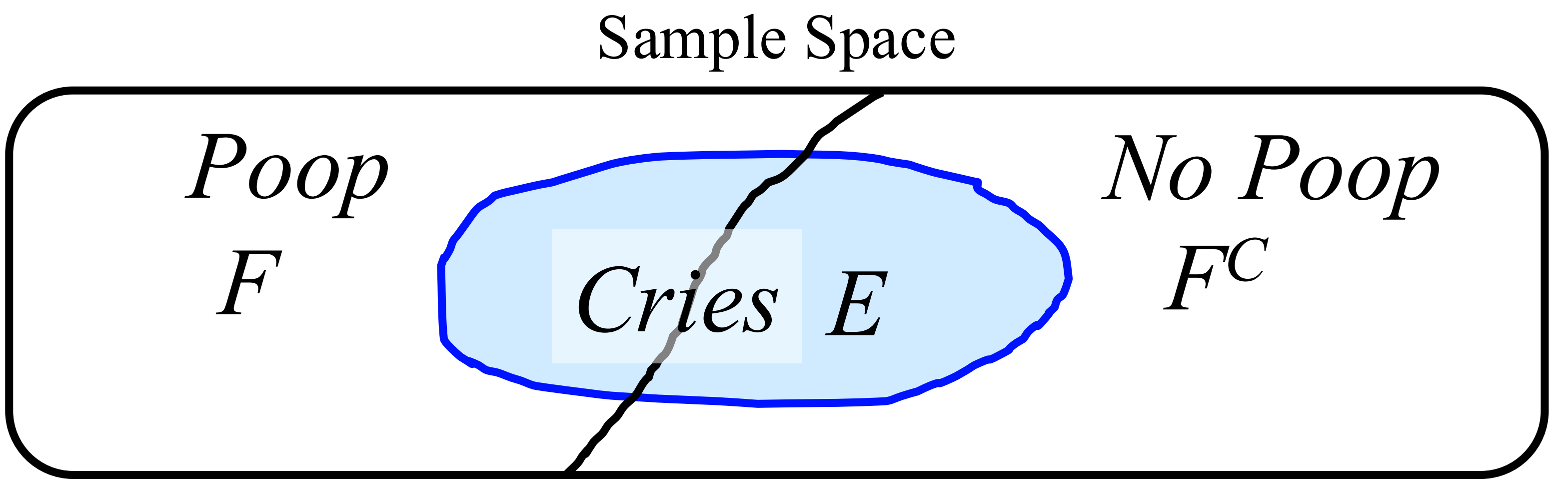
Say E and F are events in S



$$
\begin{aligned}
P(E) &= P(EF) + P(EF^C) \\
&= P(E|F)P(F) + P(E|F^C)P(F^C)
\end{aligned}
$$

Law of Total Probability

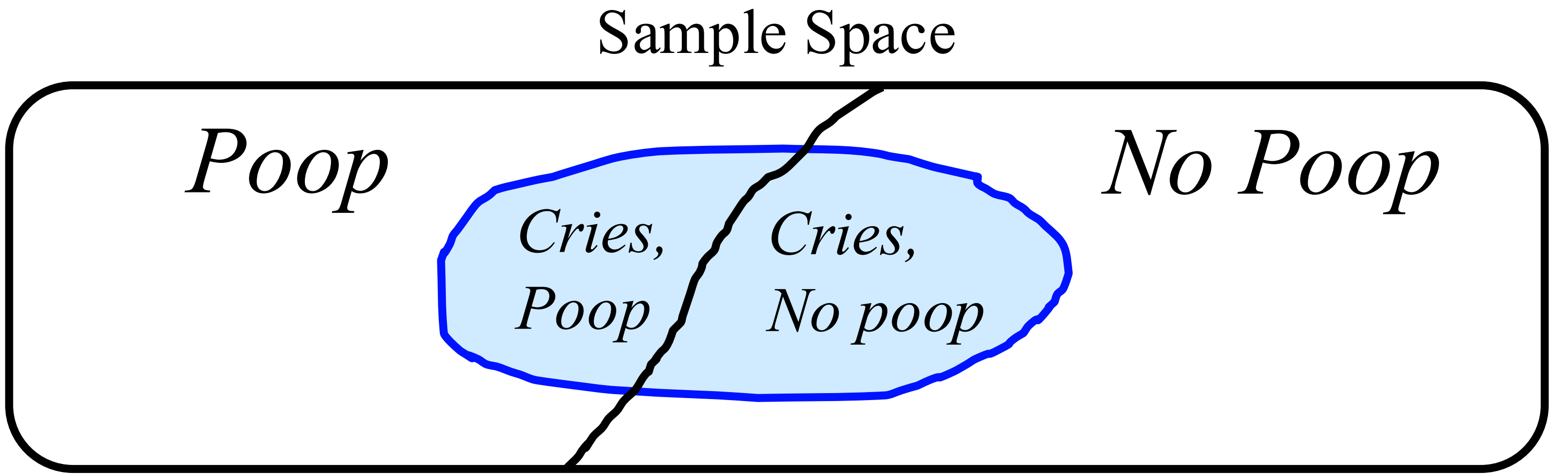
Say E and F are events in S



$$P(E) = P(EF) + P(EF^C)$$
$$= P(E|F)P(F) + P(E|F^C)P(F^C)$$

Law of Total Probability

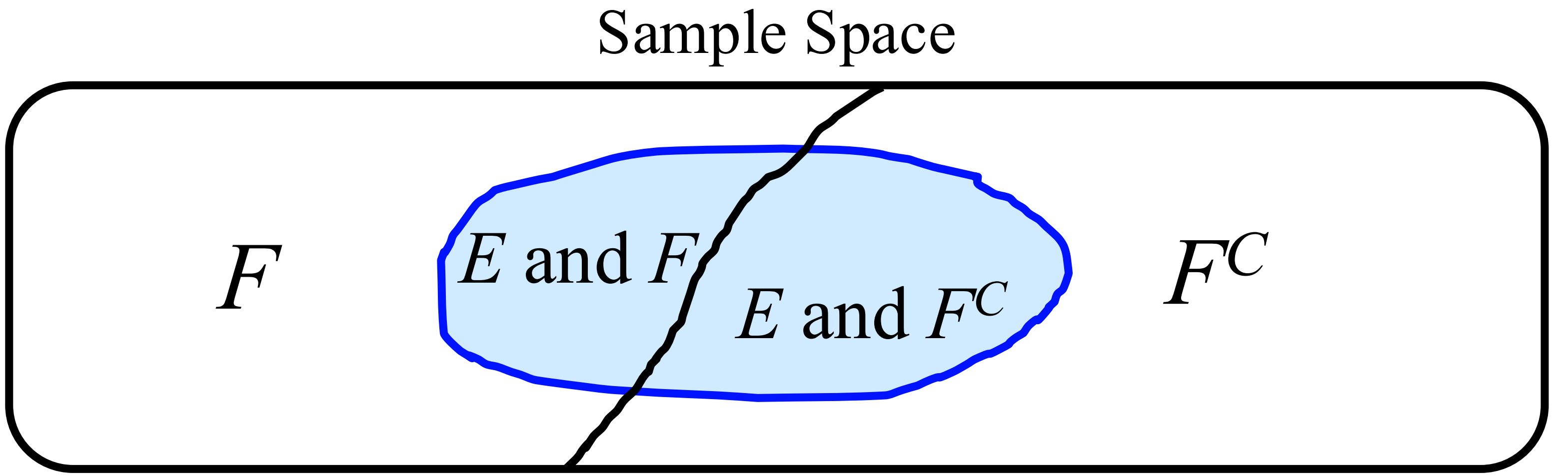
Say E and F are events in S



$$P(E) = P(EF) + P(EF^C)$$
$$= P(E|F)P(F) + P(E|F^C)P(F^C)$$

Law of Total Probability

Say E and F are events in S



$$P(E) = P(EF) + P(EF^C)$$
$$= P(E|F)P(F) + P(E|F^C)P(F^C)$$

Law of Total Probability

Thm Let F be an event where $P(F) > 0$. For any event E,

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

Proof

1. $E = (EF)$ or (EF^C) — Since F and F^C are disjoint
2. $P(E) = P(EF) + P(EF^C)$ — Probability of **or** for disjoint
3. $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ — Chain rule (product rule)

Baby Poop

In the morning when she wakes up, a baby has a 50% chance of having pooped.
The chance that a baby cries given that she has pooped is 50%.

Probability of crying (E)?

What information do you need?

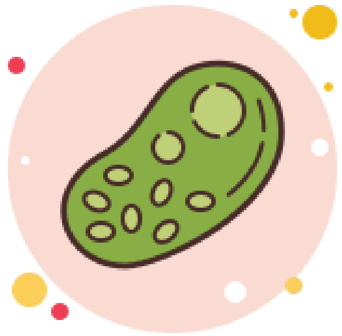
Probability of crying given no poop.

Recall that E is crying and F is poop

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

Evolution of Bacteria

$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ Law of Total Probability



You have bacteria in your gut which is causing a disease.
10% have a mutation which makes them resistant to anti-biotics
You take half a course of anti-biotics...

Probability a bacteria survives given it has the mutation: 20%
Probability a bacteria survives given it doesn't have the mutation: 1%
What is the probability that a randomly chosen bacteria survives?

Let E be the event that a bacterium survives. Let M be the event that a bacteria has the mutation. By the Law of Total Probability (LOTP):

$$
\begin{aligned}
\Pr(E) &= \Pr(E \text{ and } M) + \Pr(E \text{ and } M^C) && \text{LOTP} \\
&= \Pr(E|M)\Pr(M) + \Pr(E|M^C)\Pr(M^C) && \text{Chain Rule} \\
&= 0.20 \cdot 0.10 + 0.01 \cdot 0.90 && \text{Substituting} \\
&= 0.029
\end{aligned}
$$

Know:

P(Survive | Mutation), P(Survive), P(Mutation)

Real question. What is the probability of mutation given the bacteria survived?

P (Mutation | Survive)

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule (Product rule) ↑ ↓ Definition of conditional probability

$$P(E|F)$$

Law of Total Probability ↙

$$P(E)$$

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule
(Product rule)

Definition of
conditional probability

$$P(E|F)$$

Law of Total
Probability

Bayes'
Theorem

$$P(E)$$

$$P(F|E)$$

Bayes' Theorem

Thomas Bayes

Rev. Thomas Bayes (~1701-1761):
British mathematician and Presbyterian minister

He looked remarkably similar to Sean Astin

(but that's not important right now)

Thomas Bayes

$P(F \mid E)$

I want to calculate

P(State of the world F | Observation E)

It seems so tricky!…

The other way around is easy

P(Observation E | State of the world F)

$P(E \mid F)$

Thomas Bayes

Want $P(F \mid E)$. Know $P(E \mid F)$

$$P(F|E) = \frac{P(EF)}{P(E)}$$ Def. of Conditional Prob.

A little while later...

$$= \frac{P(E|F)P(F)}{P(E)}$$ Chain Rule

A little while later...

$$= \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ LOTP

(silent drumroll)

Bayes' Theorem

$P(E|F) \Rightarrow P(F|E)$

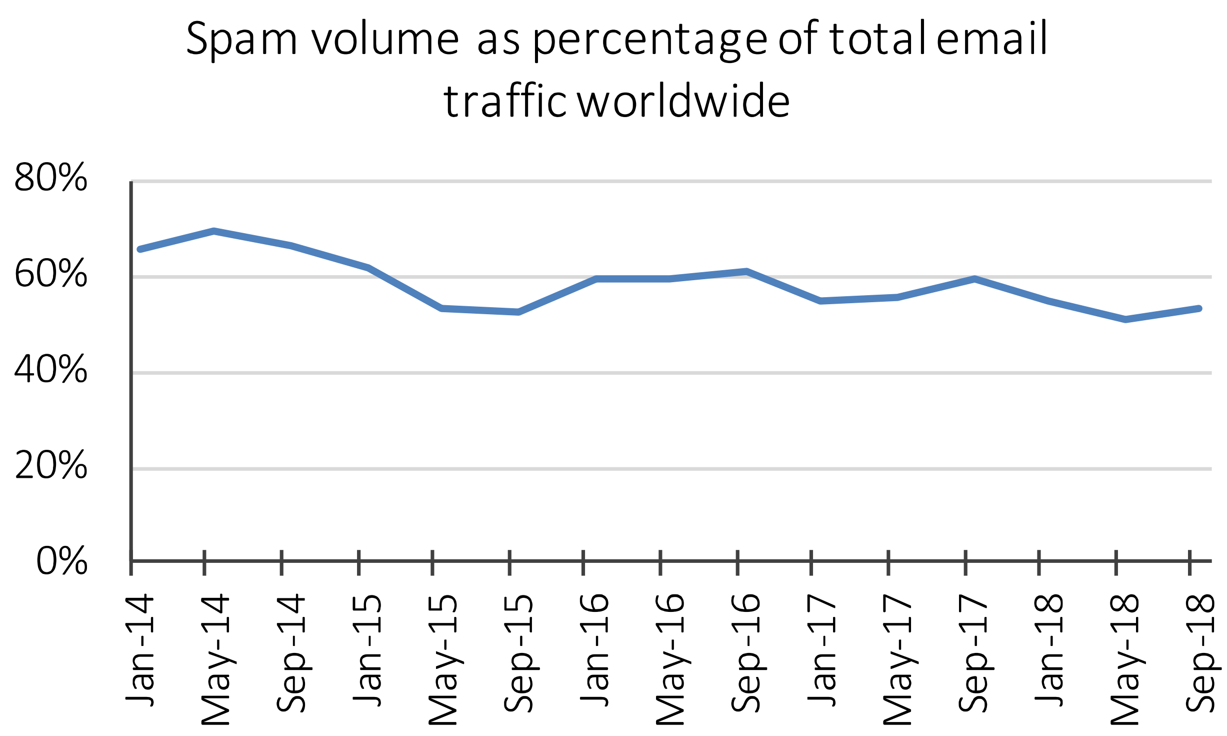
Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

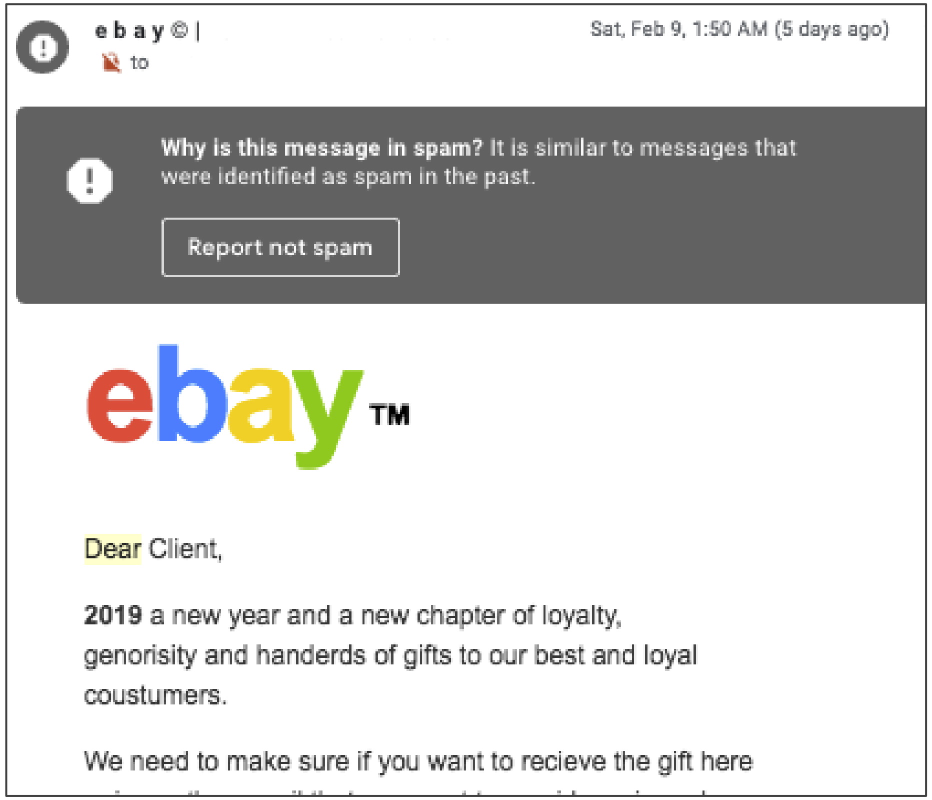
$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

Detecting spam email





We can easily calculate how many spam emails contain "Dear":

$$P(E|F) = P\left(\text{"Dear"} \,\middle|\, \begin{matrix}\text{Spam}\\\text{email}\end{matrix}\right)$$

But what is the probability that an email containing "Dear" is spam?

$$P(F|E) = P\left(\begin{matrix}\text{Spam}\\\text{email}\end{matrix} \,\middle|\, \text{"Dear"}\right)$$

Detecting spam email

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- 60% of all email in 2016 is spam.
- 20% of spam has the word "Dear"
- 1% of non-spam (aka ham) has the word "Dear"

You get an email with the word "Dear" in it.

What is the probability that the email is spam?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Let: E: "Dear", F: spam

Want: $P(\text{spam}|\text{"Dear"})$

$= P(F|E)$

Bayes' Theorem terminology

Let: E: "Dear", F: spam
Want: P(F | E)

- 60% of all email in 2016 is spam. $P(F)$
- 20% of spam has the word "Dear" $P(E|F)$
- 1% of non-spam (aka ham) has the word "Dear" $P(E|F^C)$

You get an email with the word "Dear" in it.

What is the probability that the email is spam? Want: $P(F|E)$

posterior, likelihood, prior, normalization constant

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

SARS Virus Testing

A test is 98% effective at detecting SARS

- However, test has a "false positive" rate of 1%
- 0.5% of US population has SARS
- Let E = you test positive for SARS with this test
- Let F = you actually have SARS
- What is P(F | E)?

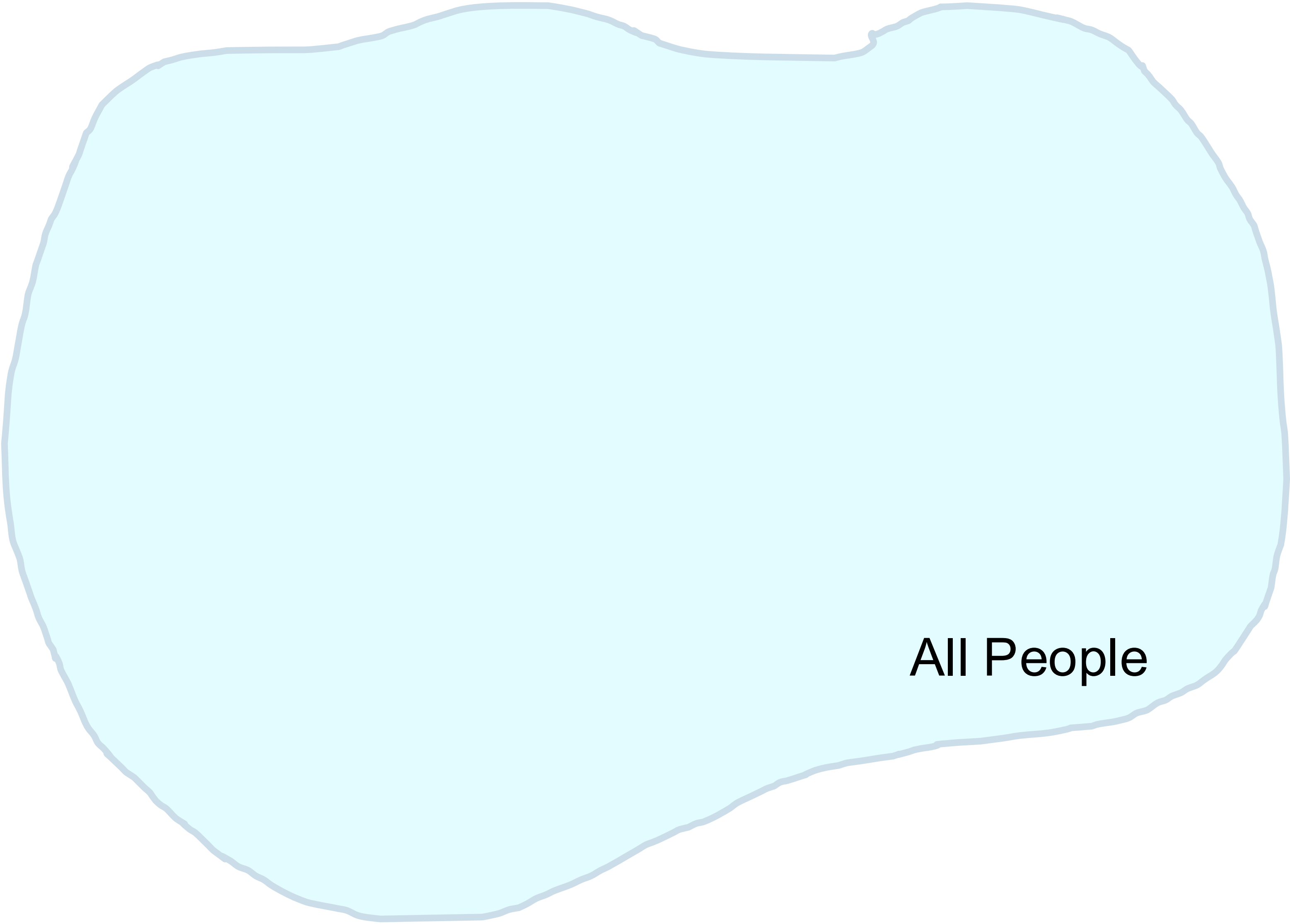
Solution:

$$P(F \mid E) = \frac{P(E \mid F)\, P(F)}{P(E \mid F)\, P(F) + P(E \mid F^c)\, P(F^c)}$$

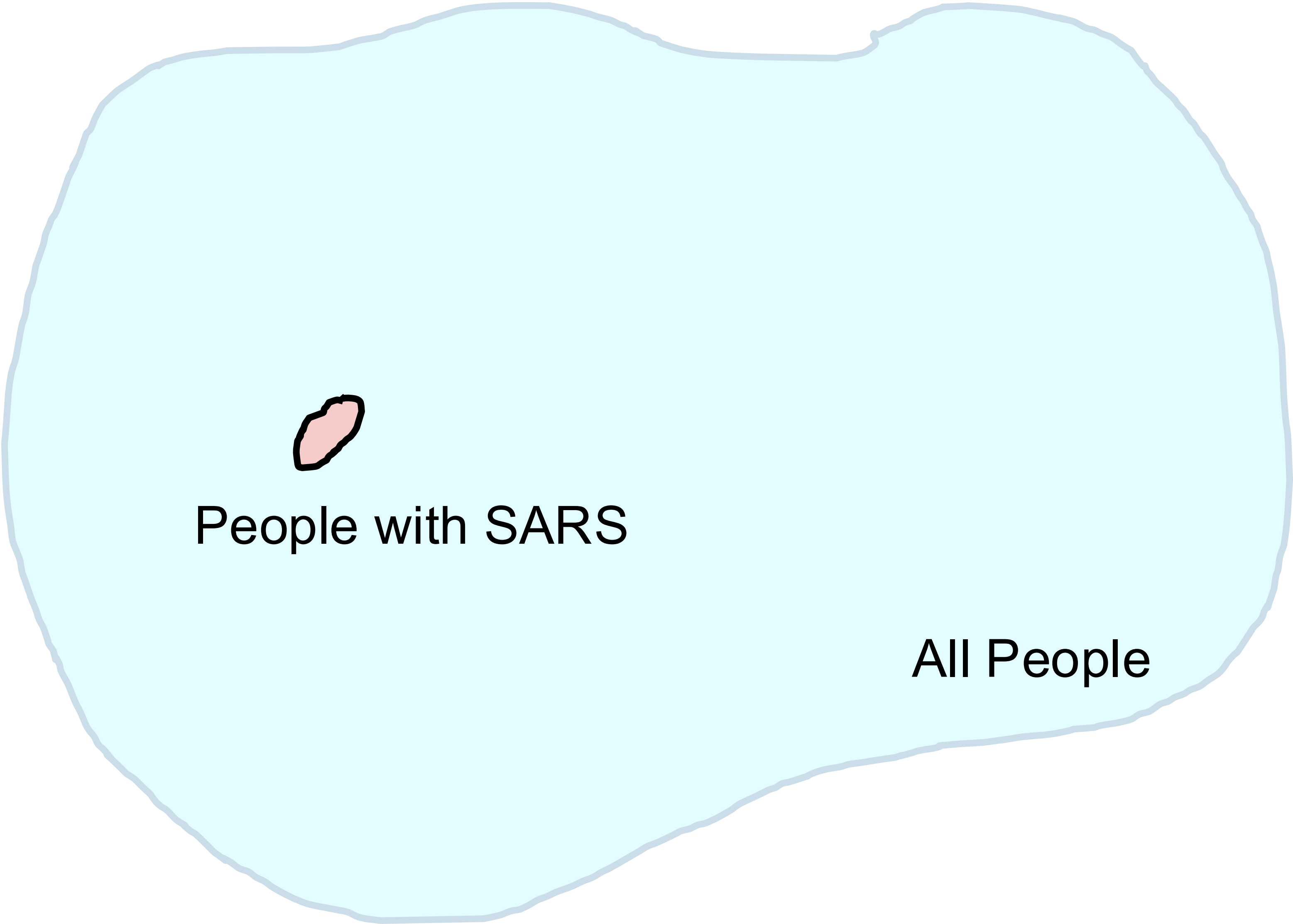
$$P(F \mid E) = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \approx 0.330$$

Intuition Time

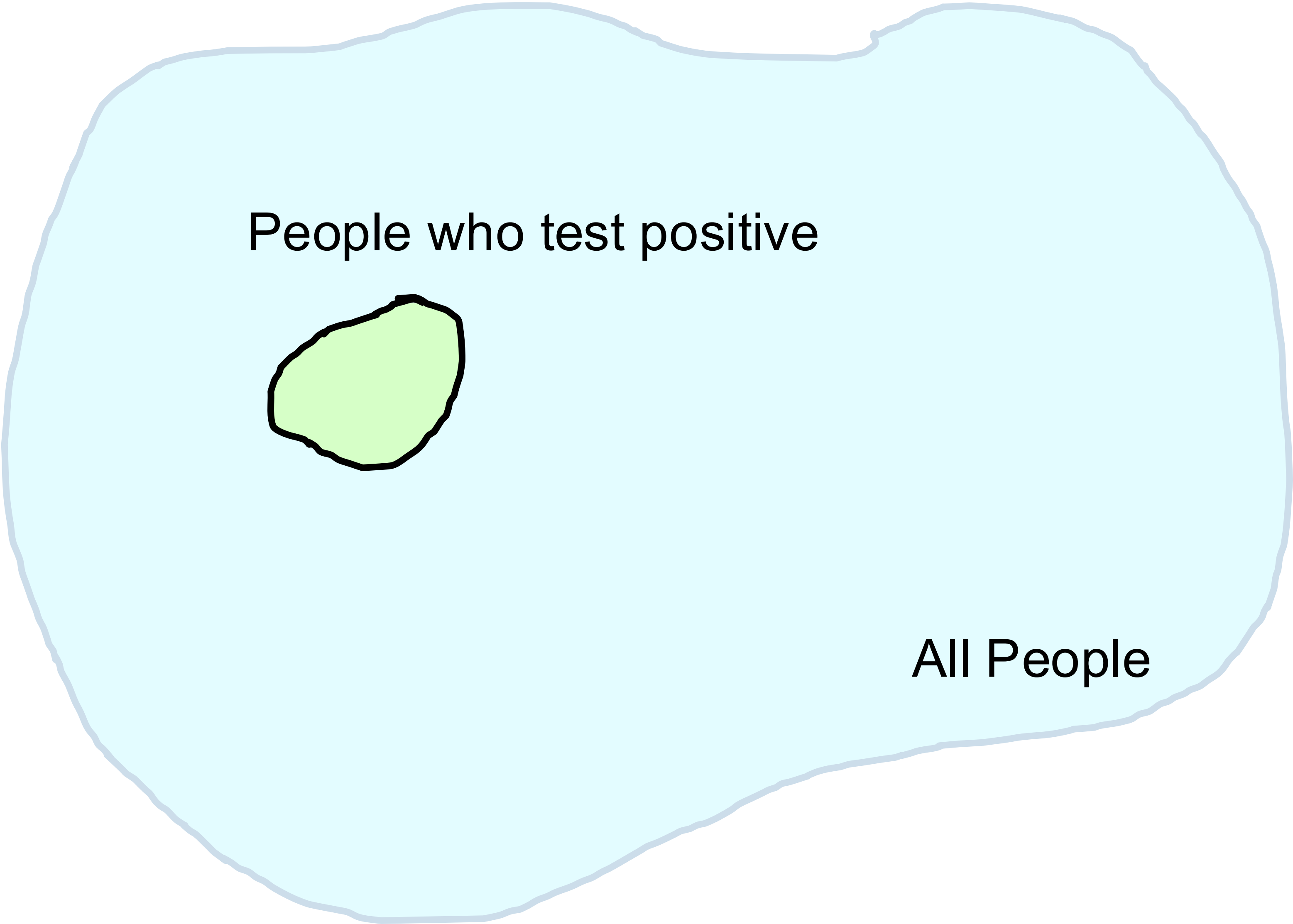
Bayes Theorem Intuition



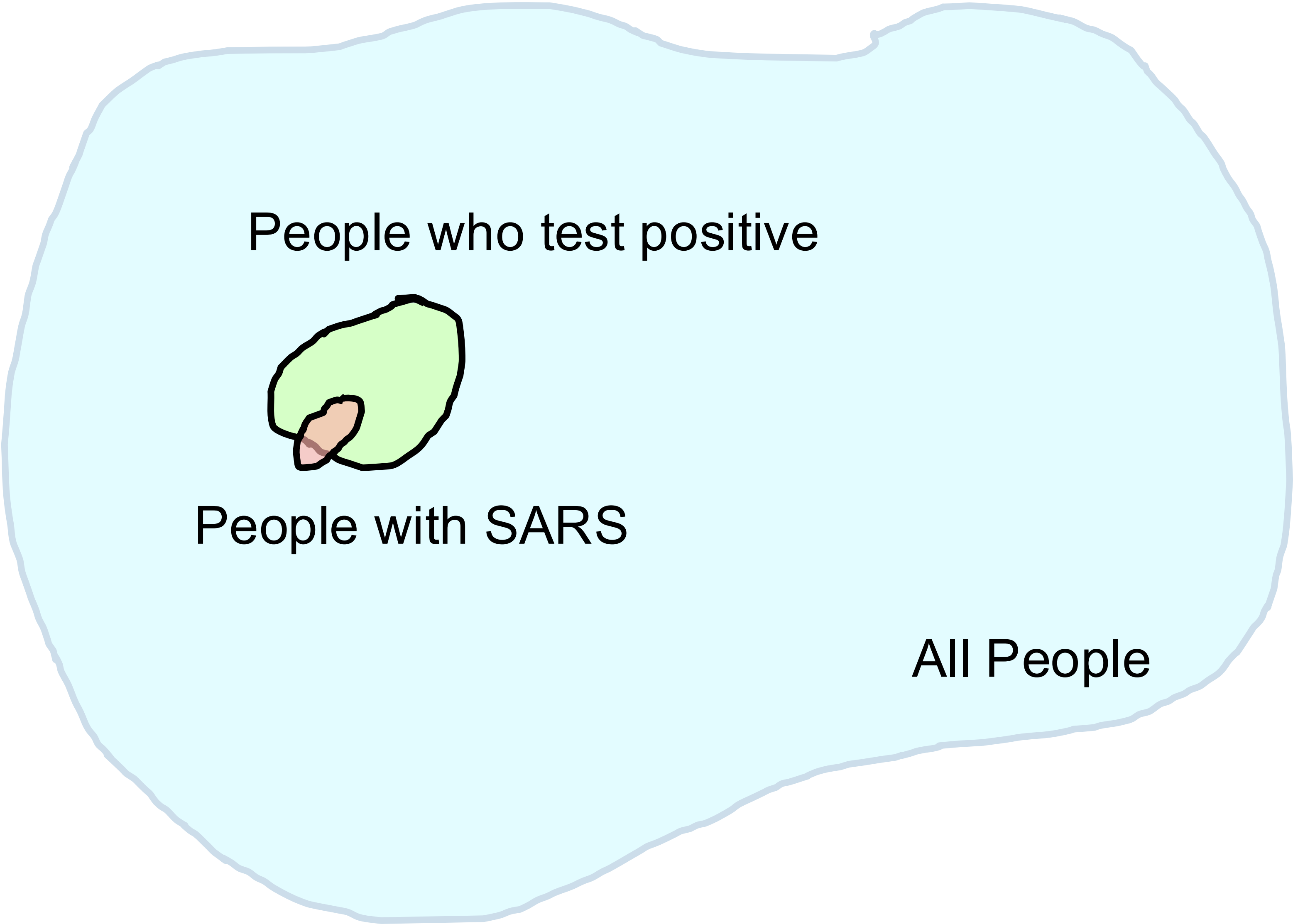
Bayes Theorem Intuition



Bayes Theorem Intuition

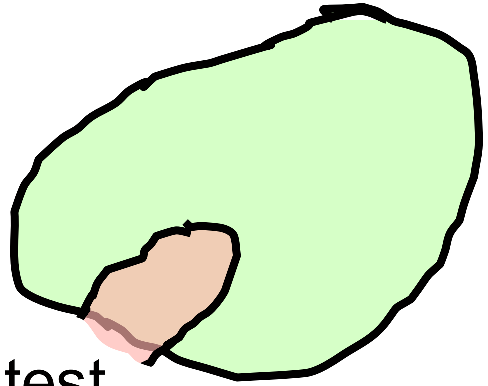


Bayes Theorem Intuition



Bayes Theorem Intuition

Conditioning on a positive result changes the sample space to this:



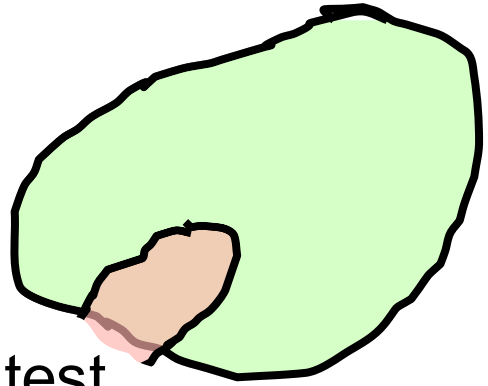
≈ 0.330

Bayes Theorem Intuition

Conditioning on a positive result changes the sample space to this:

People who test positive

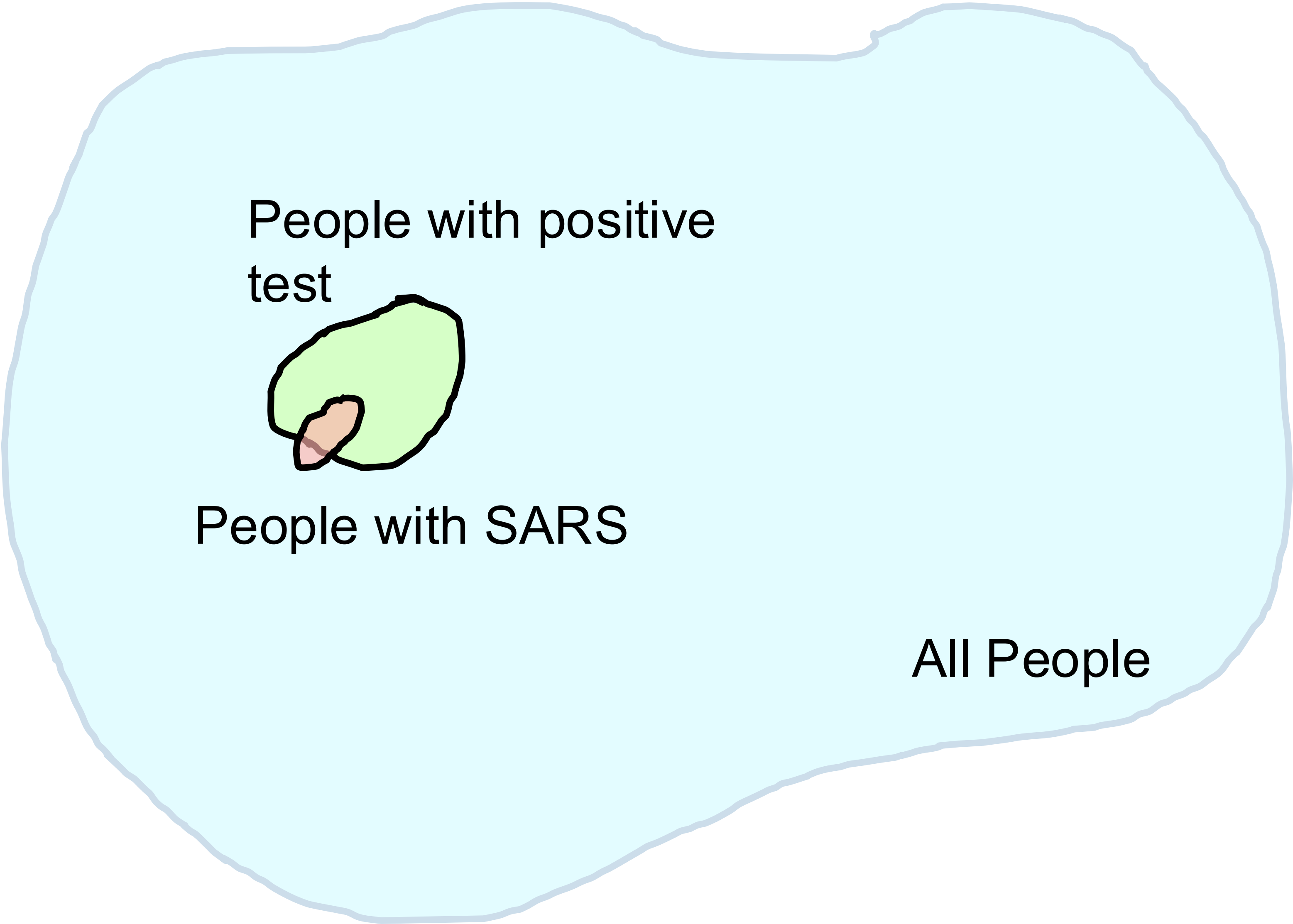
$$P(F)P(E|F) + P(F^c)P(E|F^c)$$



People who test positive and have SARS $P(F)P(E|F)$

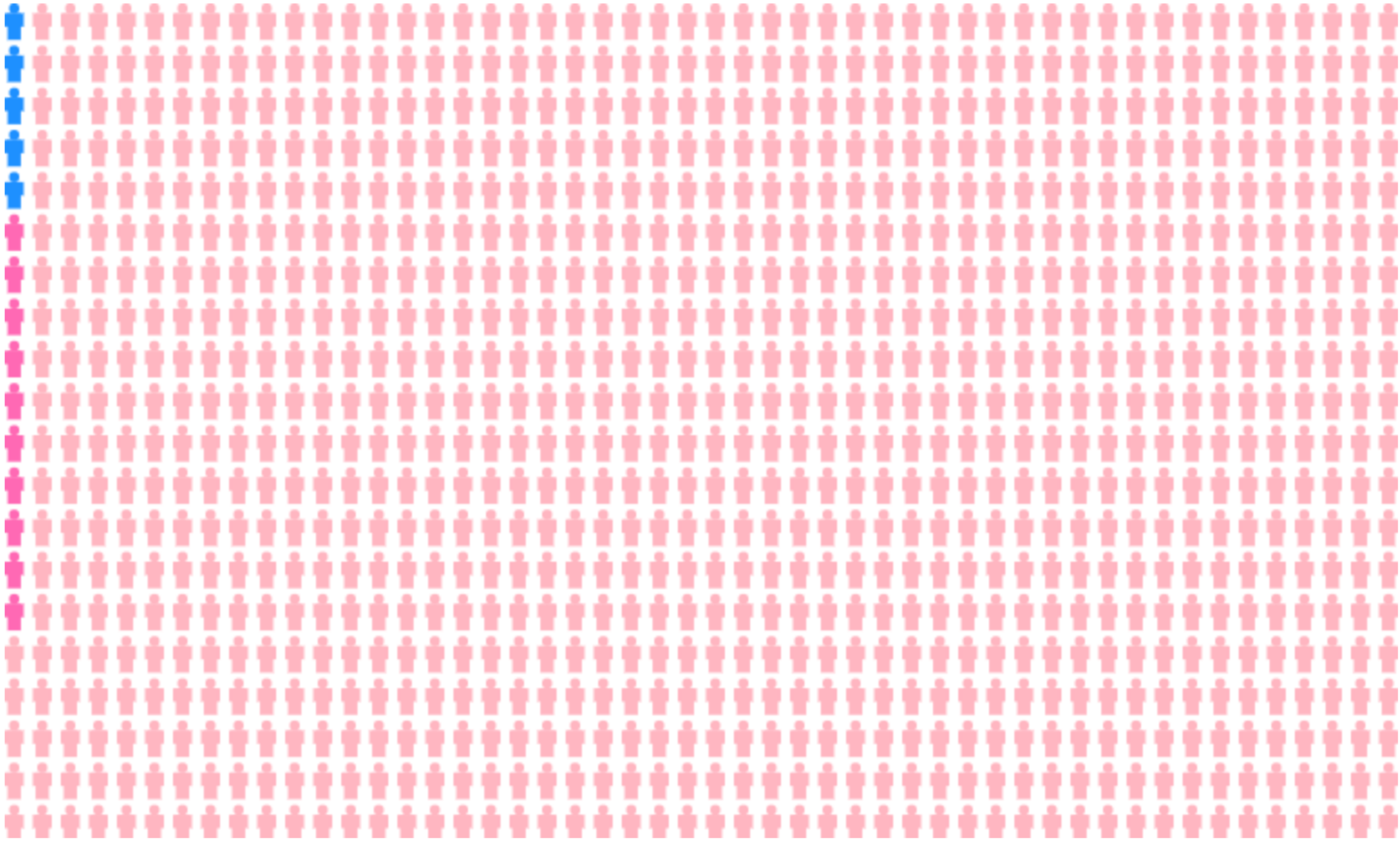
≈ 0.330

Bayes Theorem Intuition



Bayes Theorem Intuition

Say we have 1000 people:



5 have SARS and test positive, 985 **do not** have SARS and test negative.
10 **do not** have SARS and test positive.

≈ 0.333

Bayes Theorem Intuition

Conditioned on just those that test positive:

Notice that all the people with SARS are here,
but the group is still mainly folks without SARS

5 have SARS and test positive, 985 **do not** have SARS and test negative.
10 **do not** have SARS and test positive.

≈ 0.333

Why it is still good to get tested

	SARS +	**SARS –**
Test +	$0.98 = P(E \mid F)$	$0.01 = P(E \mid F^c)$
Test –	$0.02 = P(E^c \mid F)$	$0.99 = P(E^c \mid F^c)$

- Let E^c = you test <u>negative</u> for SARS with this test
- Let F = you actually have SARS
- 0.5% of population has SARS
- What is $P(F \mid E^c)$?

$$P(F \mid E^c) = \frac{P(E^c \mid F)\,P(F)}{P(E^c \mid F)\,P(F) + P(E^c \mid F^c)\,P(F^c)}$$

$$P(F \mid E^c) = \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(1 - 0.005)} \approx 0.0001$$

The End for Wednesday

Spam Revisited

How would you detect Spam using an LLM?

Real spam email:
"Pay for Viagra with a credit-card. Viagra is great. Risk free Viagra. Click for free."

1 Let E be email text. Let F be event the email is Spam.

2

60%

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

60% 40%

How would you detect Spam using an LLM?

1. Let E be email text. Let F be event the email is Spam.

2. 60%

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

60% 40%

How would you detect Spam using an LLM?

1. Let E be email text. Let F be event the email is Spam.

2. 60%

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$

60% 40%

3. Make assumption that LLM understands probability

```
prompt = "This email is spam: "
p_email_given_spam = string_pr(prompt + email) / string_pr(prompt)

prompt = ("This email is NOT spam: ")
p_email_given_ham = string_pr(prompt + email) / string_pr(prompt)
```

Mysteries

Whats an LLMs (real) belief?

Llama-3.3-70B

```python
from llm import string_pr
pr = string_pr("Hello, world")
print(pr)
```

If you used a prompt like this:

f"Dice simulator output. Sum of two random 6 sided dice: {outcome}"

Could you test an LLMs understanding of probability?

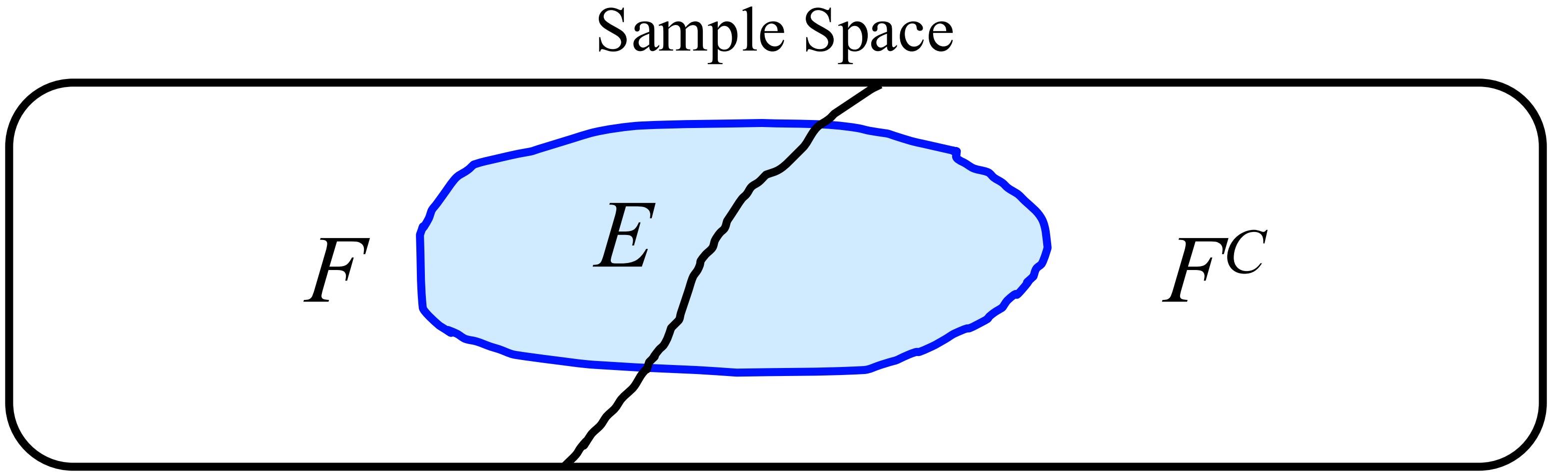
Outcome	string_pr
2	3.0×10^{-30}
⋮	⋮
7	9.9×10^{-29}
⋮	⋮
12	2.4×10^{-30}

Come on Friday!

Stories + Make History

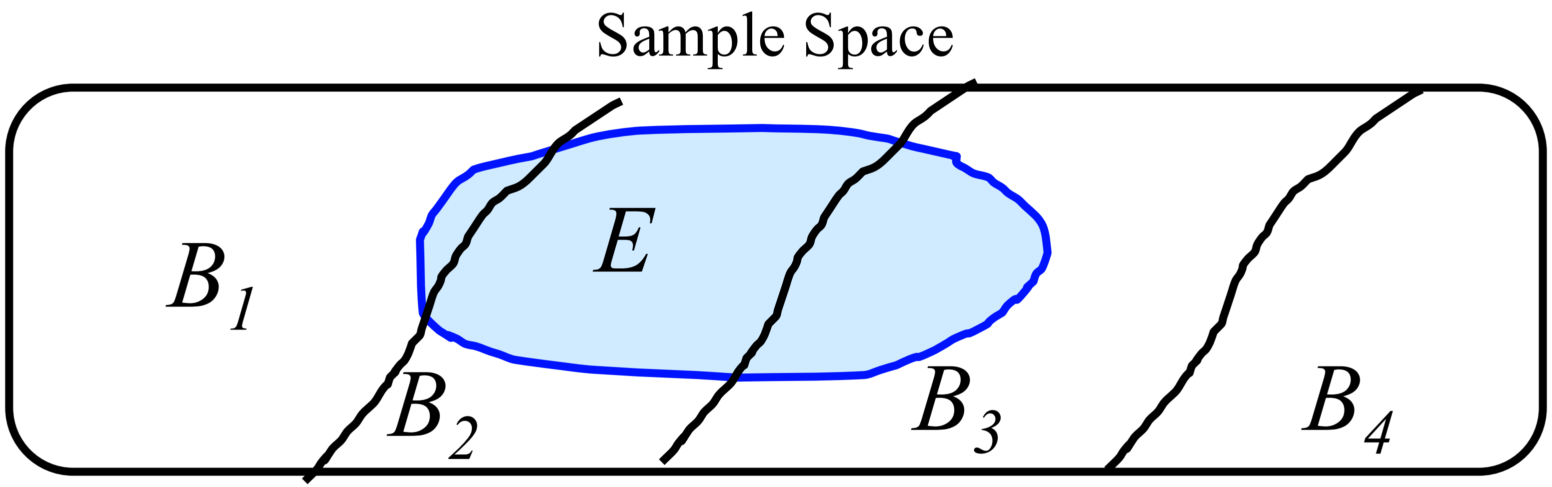
Sneak Peek…

Law of Total Probability



$$P(E) = P(EF) + P(EF^C)$$
$$= P(E|F)P(F) + P(E|F^C)P(F^C)$$

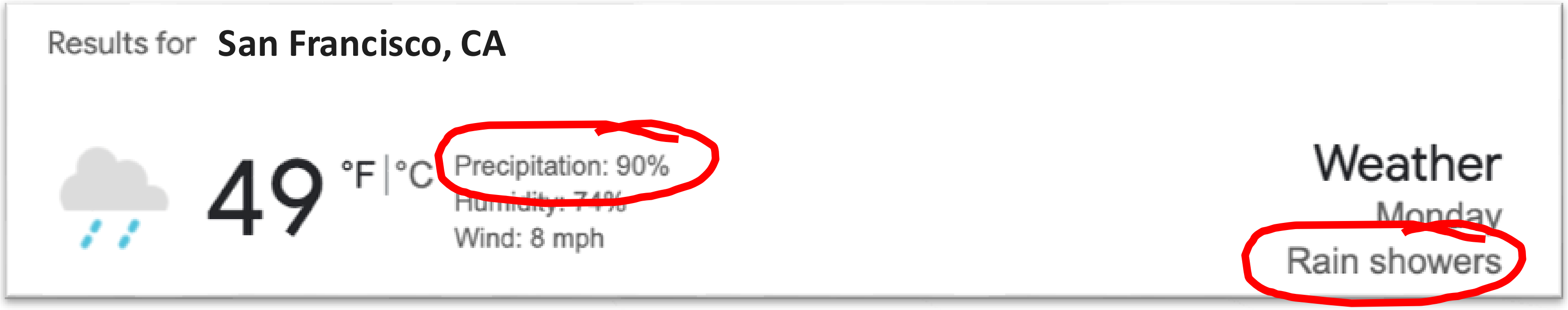
Law of Total Probability



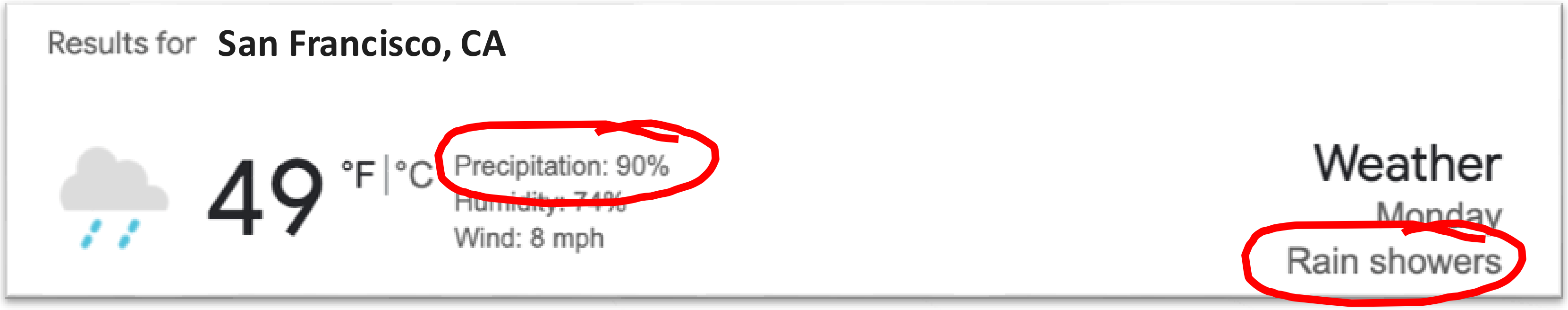
Thm For mutually exclusive events $B_1, B_2, \ldots, B_n$
s.t. $B_1 \cup B_2 \cup \cdots \cup B_n = S$,

$$P(E) = \sum_i P(B_i \cap E)$$

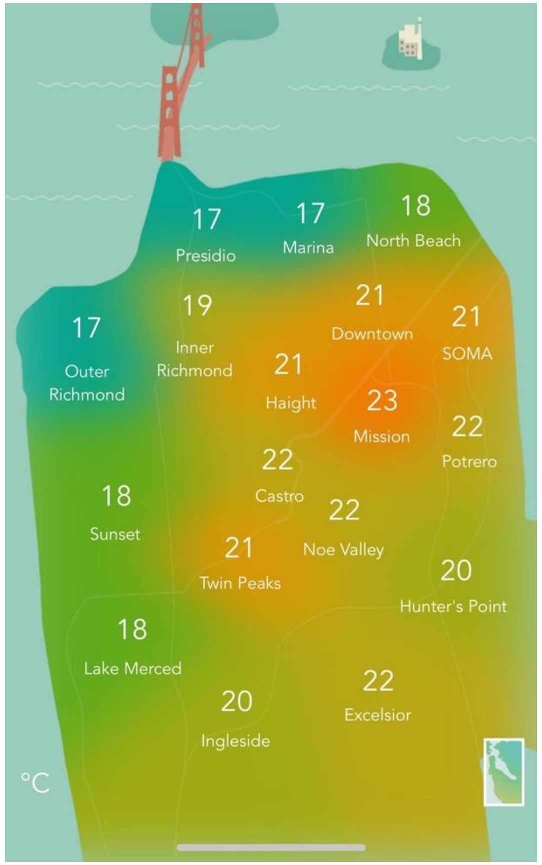
$$= \sum_i P(E|B_i)P(B_i)$$



Background event. Where is the person in San Francisco?



Background event.
Where is the person in San Francisco?



From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

It rains tomorrow

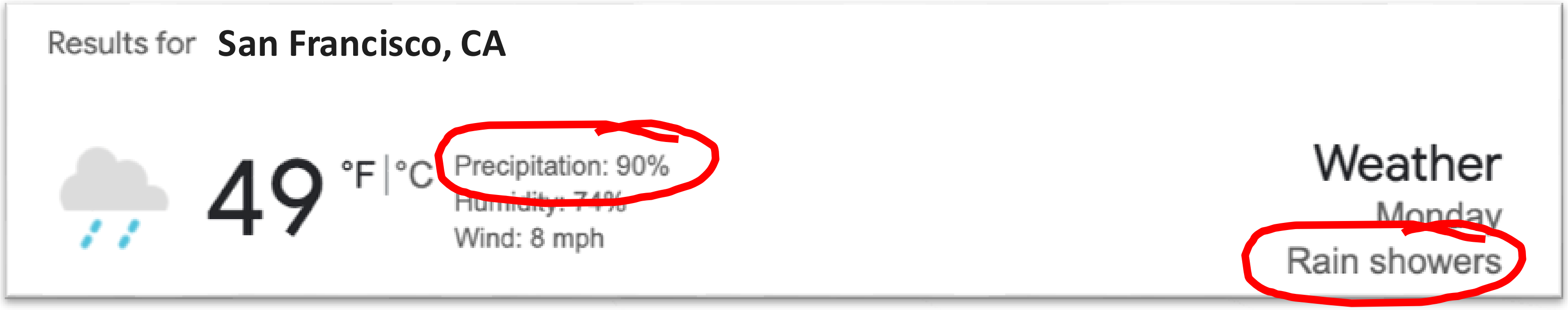
$$P(R|D_i)$$

Person is in district *i*

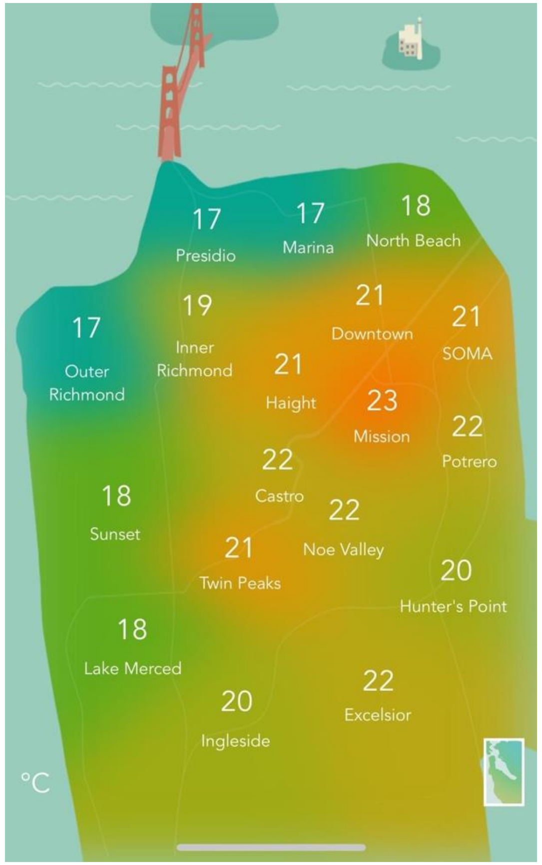
$$P(D_i)$$

Want:

$$P(R)$$



Background event.
Where is the person in San Francisco?



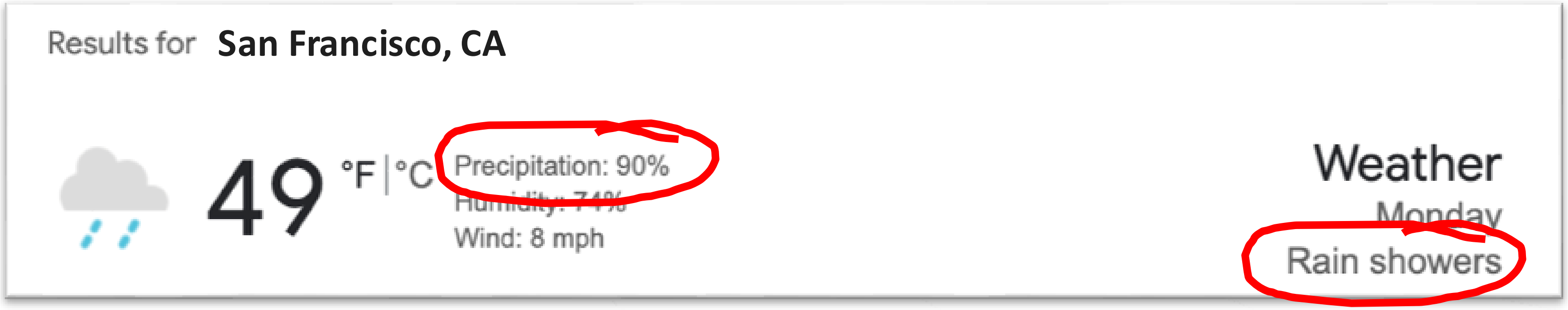
From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

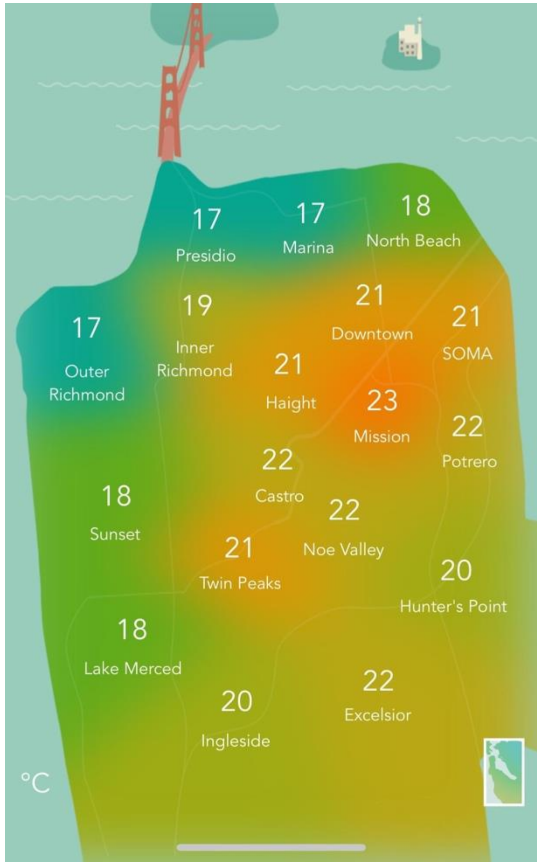
	Mission District	Presidio		SOMA
$P(R\|D_i)$	0.23	0.84	…	0.52
$P(D_i)$	0.15	0.02		0.24

Want:

$$P(R)$$



Background event.
Where is the person in San Francisco?



From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

	Mission District	Presidio		SOMA
$P(R\|D_i)$	0.23	0.84	…	0.52
$P(D_i)$	0.15	0.02		0.24

Want:

$$P(R) = \sum_{\text{district } i} P(R \text{ and } D_i) = \sum_{\text{district } i} P(R|D_i) \cdot P(D_i)$$

SARS Virus Testing

A test is 98% effective at detecting SARS

- However, test has a "false positive" rate of 1%
- 0.5% of US population has SARS
- Let E = you test positive for SARS with this test
- Let F = you actually have SARS
- What is P(F | E)?

Solution:

$$P(F \mid E) = \frac{P(E \mid F)\, P(F)}{P(E \mid F)\, P(F) + P(E \mid F^c)\, P(F^c)}$$

$$P(F \mid E) = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \approx 0.330$$

Pedagogic Pause

Multiple Choice Theory

Let's consider the relationship between **knowing** the concepts used in a multiple choice midterm question, and getting the question correct, taking into account guessing and making silly mistakes.

Let 3/4 be the prior probability that a learner knows the concepts to a midterm question.

Let 1/4 be the probability that a learner gets the answer **correct** if they **don't** know the concepts.

Let 9/10 be the probability that a learner gets the question **correct** given they **do** know the concepts.

What is the probability they know the concept, given they answered correct?

Monty Hall Problem

Monty Hall Problem

Monty Hall Problem aka Let's Make a Deal

Behind one door is a prize (equally likely to be any door).

Behind the other two doors is nothing

1. We choose a door
2. Host opens 1 of other 2 doors, revealing nothing
3. We are given an option to change to the other door.

Should we switch?

Doors A,B,C

Note: If we don't switch,

$$P(\text{Win}) = 1/3$$

In the world where we switch

Without loss of generality, say we pick A (out of Doors A,B,C).

1/3   1/3   1/3

A: Prize in Door A
- Host opens B or C
- We switch
- We <u>always lose</u>

P(Win | A) = 0

B: Prize in Door B
- Host must open C
- We switch to B
- We <u>always win</u>

P(Win | B) = 1

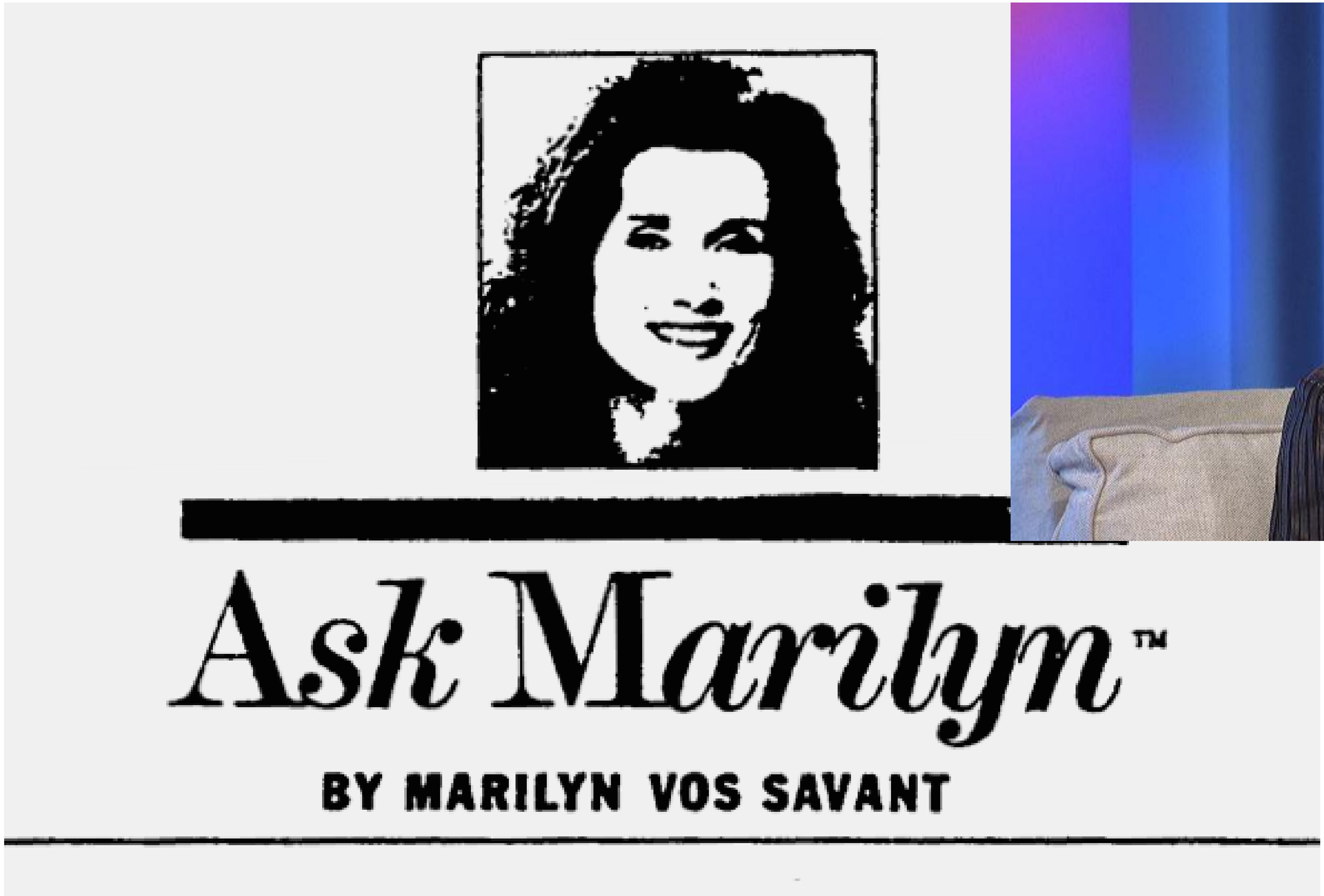
C: Prize in Door C
- Host must open B
- We switch to C
- We <u>always win</u>

P(Win | C) = 1

$$P(\text{Win}) = P(\text{Win}|A)P(A) + P(\text{Win}|B)P(B) + P(\text{Win}|C)P(C)$$
$$= \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 1 = \frac{2}{3}$$

You should switch!

Marilyn Vos Savant



Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

$$\frac{1}{1000} = \text{P(envelope is prize)}$$

$$\frac{999}{1000} = \text{P(other 999 envelopes have prize)}$$

2. I open 998 of remaining 999 (showing they are empty).

$$\frac{999}{1000} = \text{P(998 empty envelopes had prize)} + \text{P(last other envelope has prize)}$$

$$= \text{P(last other envelope has prize)}$$

3. Should you switch?

No: $\text{P(win without switching)} = \dfrac{1}{\text{original \# envelopes}}$

Yes: $\text{P(win with new knowledge)} = \dfrac{\text{original \# envelopes - 1}}{\text{original \# envelopes}}$