



Normal Distribution

CS109, Stanford University

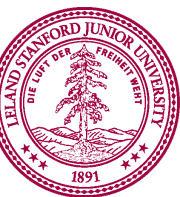
Announcement – Sign Up For PEP

Personal Exam Prep (PEP)

- 15 minute 1:1 meeting with a TA twice a quarter.
- Happens the week before each exam.
- Midterm PEP is next week (Mon 2/2-Weds 2/4)!!
- Required part of the course.

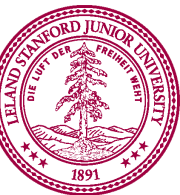
[Sign up Link is Live](#) !!

Signups are first come first serve.



Announcement – Sign Up For PEP

PEP Frequently Asked Questions:

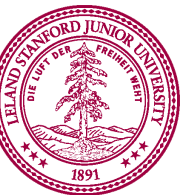


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PEP Frequently Asked Questions:

(1) **Am I graded on my PEP?**

Yes !! Like section. We will assume you are caught up through Friday's lecture (1/30). Don't need to solve everything perfectly, but need to be able to engage meaningfully.



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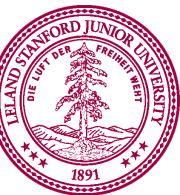
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(2) What can I do to prepare for PEP?

Nothing special. As long as you are caught up with course content and are ready to talk about it, you are all set! *TAs each spend 7 hours of their time next week doing PEPs – so we want to use that as effectively as we can.*



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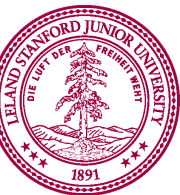
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(3) Is PEP a required part of the course?

Yes.



Enough Servers?

You are running a massive website. On the busiest minute you receive:

Average of 10^6 requests

Variance of 10^4 requests.

You are going to buy n servers

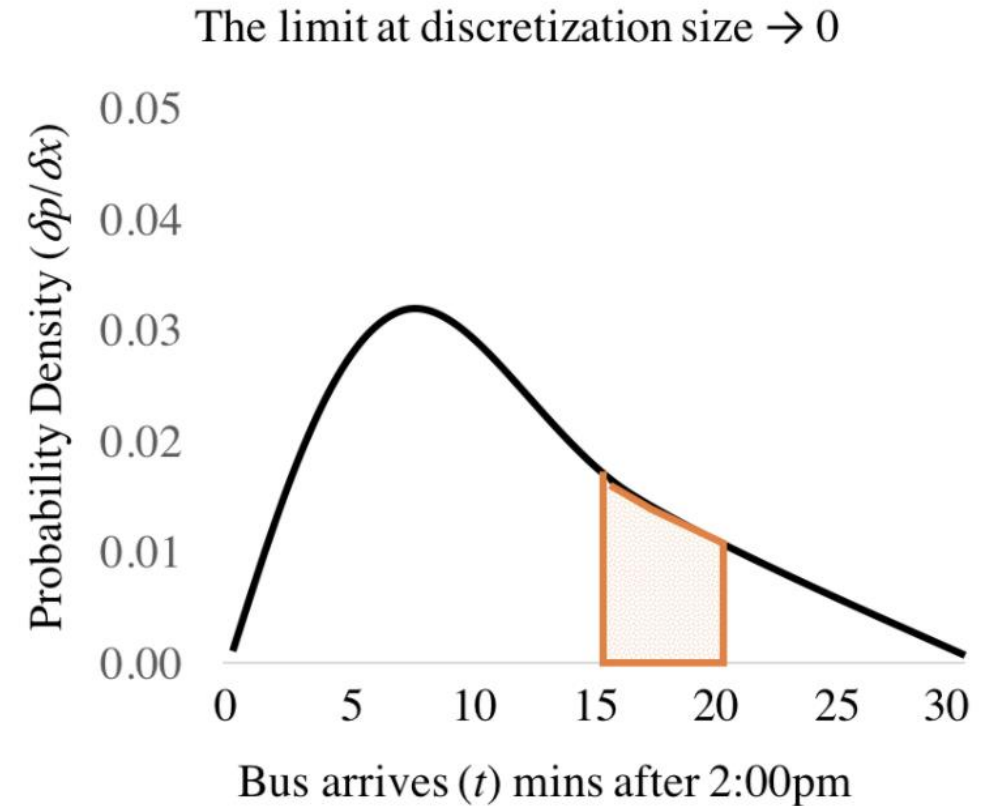
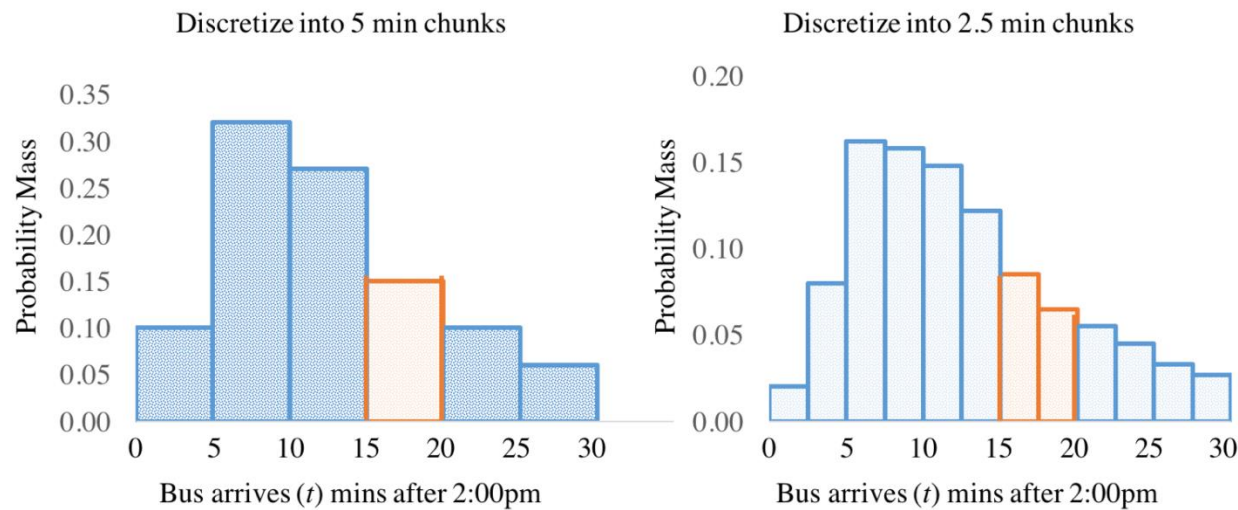
Each server can handle 10,000 requests per min, otherwise you drop requests

What is the smallest value of n such that $P(\text{drop}) < 0.0001$

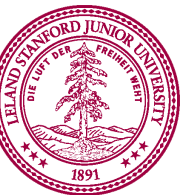


Review

Review: Probability Density Function



Recall the definition of a derivative:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$


What do you get if you
integrate over a
probability *density* function?

A probability!

Review: Probability Density Function



The **probability density function** (PDF) of a continuous random variable represents the **derivative** of probability at a given point.

Units of probability *divided by units of X*.
Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^b f(X = x) dx$$

Uniform and Exponential Distributions

Uniform Random Variable

Notation: $X \sim \text{Uni}(\alpha, \beta)$

Description: A continuous random variable that takes on values, with equal likelihood, between α and β

Parameters: $\alpha \in \mathbb{R}$, the minimum value of the variable.
 $\beta \in \mathbb{R}$, $\beta > \alpha$, the maximum value of the variable.

Support: $x \in [\alpha, \beta]$

PDF equation: $f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{else} \end{cases}$

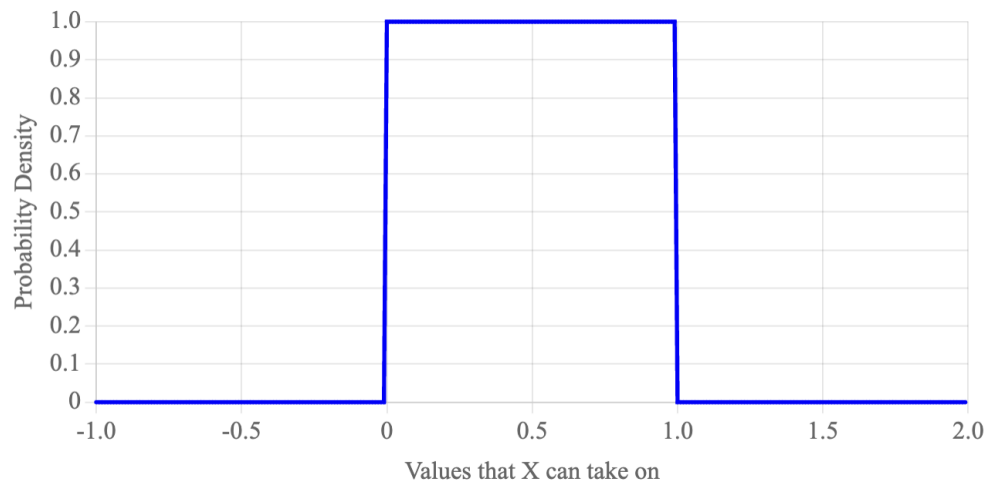
CDF equation: $F(x) = \begin{cases} \frac{x - \alpha}{\beta - \alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{for } x < \alpha \\ 1 & \text{for } x > \beta \end{cases}$

Expectation: $E[X] = \frac{1}{2}(\alpha + \beta)$

Variance: $\text{Var}(X) = \frac{1}{12}(\beta - \alpha)^2$

PDF graph:

Parameter α : Parameter β :



Exponential Random Variable

Notation: $X \sim \text{Exp}(\lambda)$

Description: Time until next events if (a) the events occur with a constant mean rate and (b) they occur independently of time since last event.

Parameters: $\lambda \in \{0, 1, \dots\}$, the constant average rate.

Support: $x \in \mathbb{R}^+$

PDF equation: $f(x) = \lambda e^{-\lambda x}$

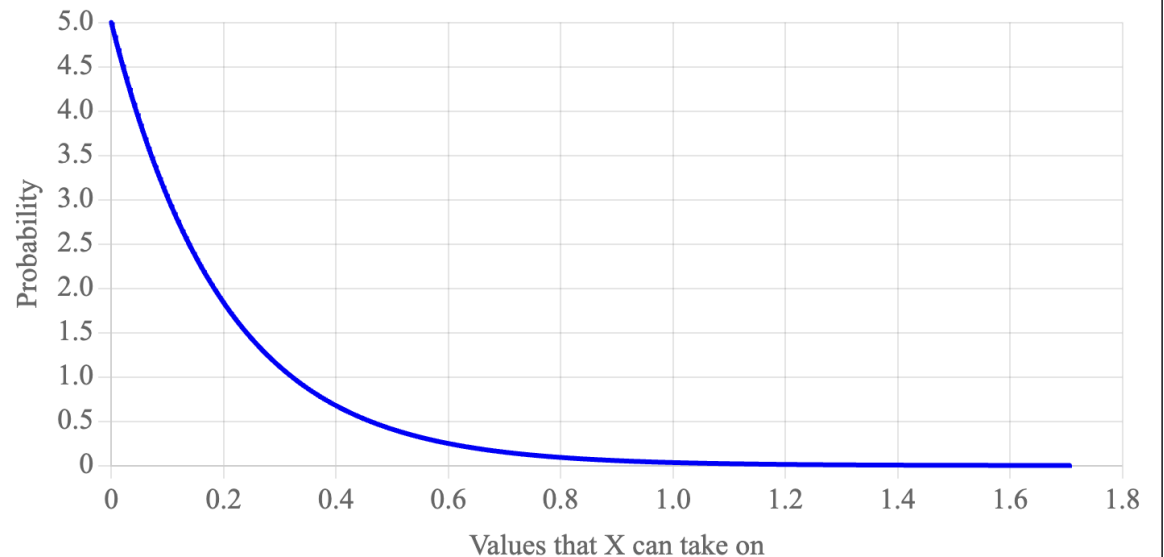
CDF equation: $F(x) = 1 - e^{-\lambda x}$

Expectation: $E[X] = 1/\lambda$

Variance: $\text{Var}(X) = 1/\lambda^2$

PDF graph:

Parameter λ :



Cumulative Distribution Function

A cumulative distribution function (CDF) is a “closed form” equation for the probability that a random variable is less than a given value

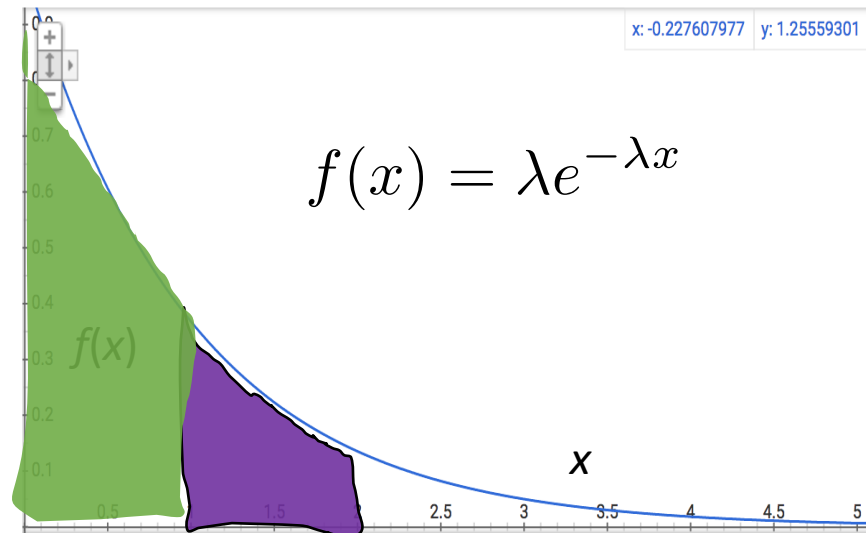
$$F(x) = P(X < x)$$



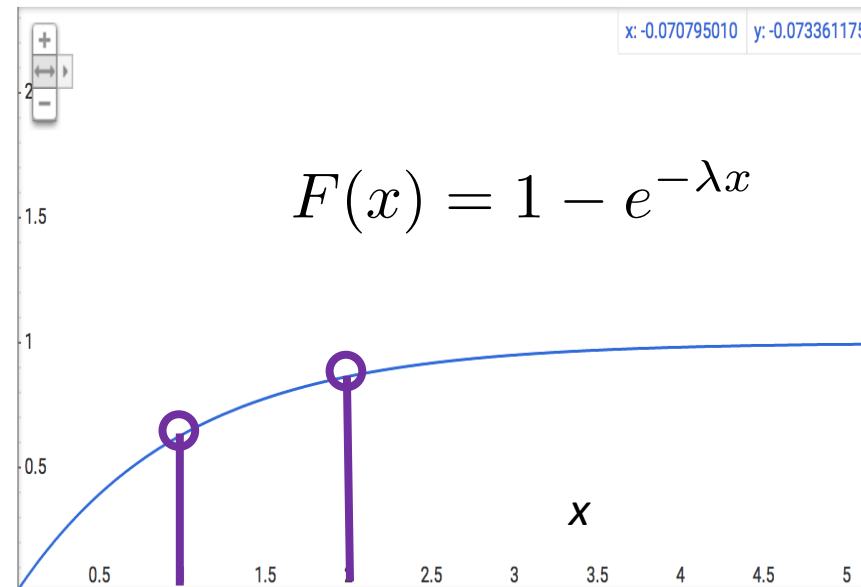
If you learn how to use a cumulative distribution function, you can avoid integrals!

Using CDF Example. X is $\text{Exp}(\lambda = 1)$

Probability
density
function



Cumulative
distribution
function



$$F_X(x) = P(X < x)$$
$$= \int_{y=-\infty}^x f(y) dy$$

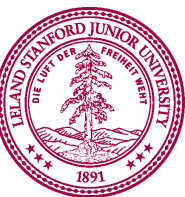
$$P(1 < X < 2)$$

$$= \int_{x=1}^2 f(x) dx$$

or

$$= F(2) - F(1)$$

$$= (1 - e^{-2})$$
$$- (1 - e^{-1})$$
$$\approx 0.23$$



Did you know? Exponential is Memoryless!

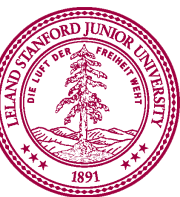
$$F(x) = 1 - e^{-\lambda x}$$

$$X \sim \text{Exp}(\lambda)$$

X = time until the next event

$$P(X > s + t | X > s) = P(X > t)$$

What if s time has passed?



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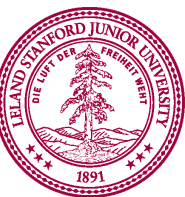
What if s time has passed?

$$P(X > 8 | X > 5) = P(X > 3)$$

If I have already waited 5 minutes for an event, what is the probability I will have to wait more than 3 minutes?

The same as the probability of waiting 3 minutes. It doesn't care about how long you have already waited!

No “memory”!!



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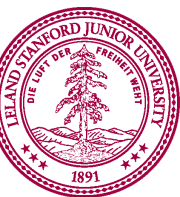
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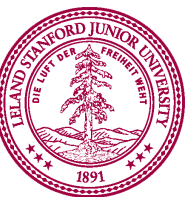
What if s time has passed?

Which is something we can prove:

$$\begin{aligned} P(X > s + t | X > s) &= \frac{P(X > s + t \text{ and } X > s)}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} \end{aligned}$$

Def of conditional prob.

Because $X > s + t$ implies $X > s$



Did you know? Exponential is Memoryless!

$$F(x) = 1 - e^{-\lambda x}$$

$$X \sim \text{Exp}(\lambda)$$

X = time until the next event

$$P(X > s + t | X > s) = P(X > t)$$

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Which is something we can prove:

$$P(X > s + t | X > s) = \frac{P(X > s + t \text{ and } X > s)}{P(X > s)}$$

Def of conditional prob.

$$= \frac{P(X > s + t)}{P(X > s)}$$

Because $X > s + t$ implies $X > s$

$$= \frac{1 - F_X(s + t)}{1 - F_X(s)}$$

Def of CDF

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

By CDF of Exp

$$= e^{-\lambda t}$$

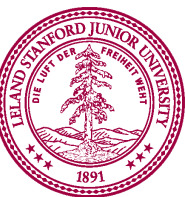
Simplify

$$= 1 - F_X(t)$$

By CDF of Exp

$$= P(X > t)$$

Def of CDF



I am going to use these two properties later in class today

Properties of Expectation

Property: Expectation of a Linear Transform

$$E[aX + b] = aE[X] + b$$

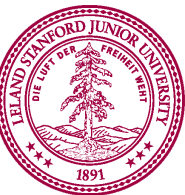
Where a and b are constants and not random variables.

Properties of Variance

Property: Variance of a Linear Transform

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

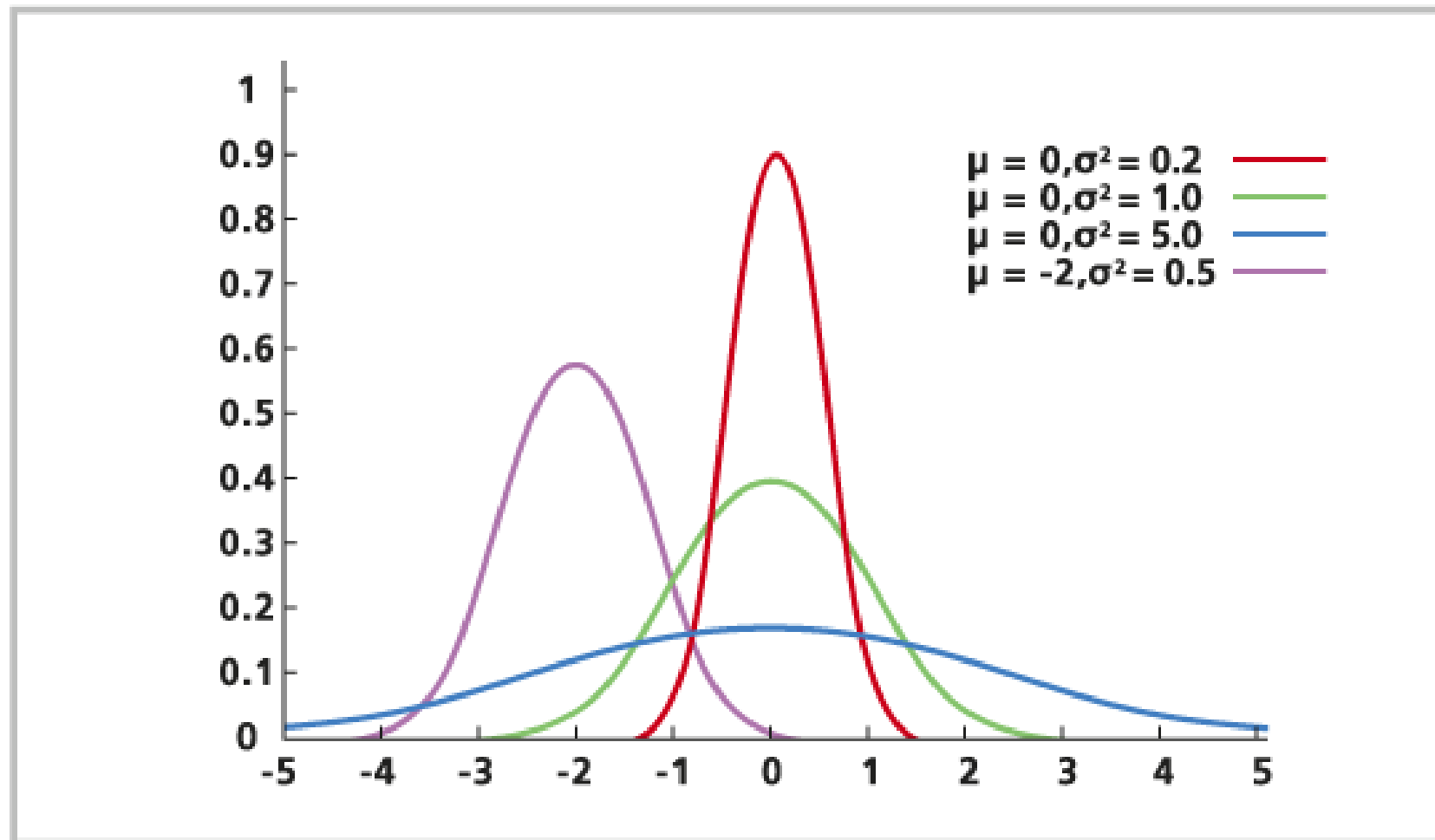
Where a and b are constants and not random variables.



/Review

Big Day

NormCore: A Few Normal Examples



Normal Random Variable

def An **Normal** random variable X is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

PDF

Expectation

Variance

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

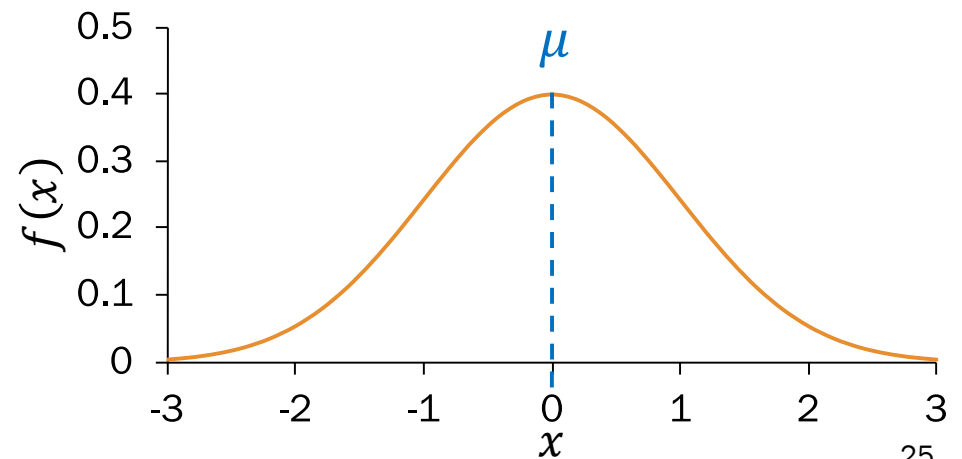
$$E[X] = \mu$$

$$\text{Var}(X) = \sigma^2$$

Other names: **Gaussian** random variable

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician.



Johann Carl Friedrich Gauss ([/ɡaʊs/](#); **German:** *Gauß* [\[ɡaʊs\]](#) ([listen](#)); **Latin:** *Carolus Fridericus Gauss*; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields, including [algebra](#), [analysis](#), [astronomy](#), [differential geometry](#), [electrostatics](#), [geodesy](#), [geophysics](#), [magnetic fields](#), [matrix theory](#), [mechanics](#), [number theory](#), [optics](#) and [statistics](#).

Sometimes referred to as the *Princeps mathematicorum*^[1] ([Latin](#) for "the foremost of mathematicians") and ["the greatest mathematician since antiquity"](#). Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.^[2]

Did not invent Normal distribution but rather popularized it

Why the Normal?

- Common for natural phenomena: height, weight, etc.

These are log-normal

- Most noise in the world is Normal

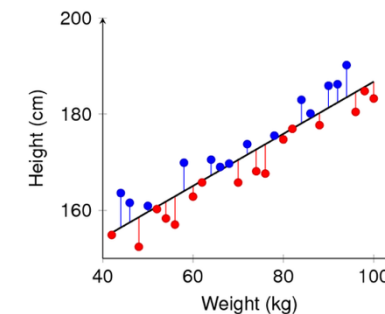
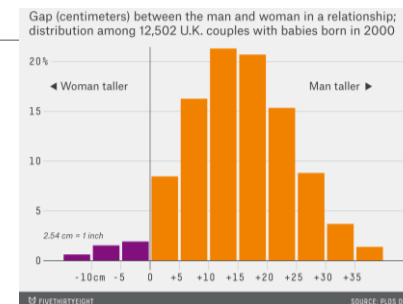
Most noise is assumed normal

- Often results from the sum of many random variables

Only if they are equally weighted and independent

- Sample means are distributed normally

That is actually true...



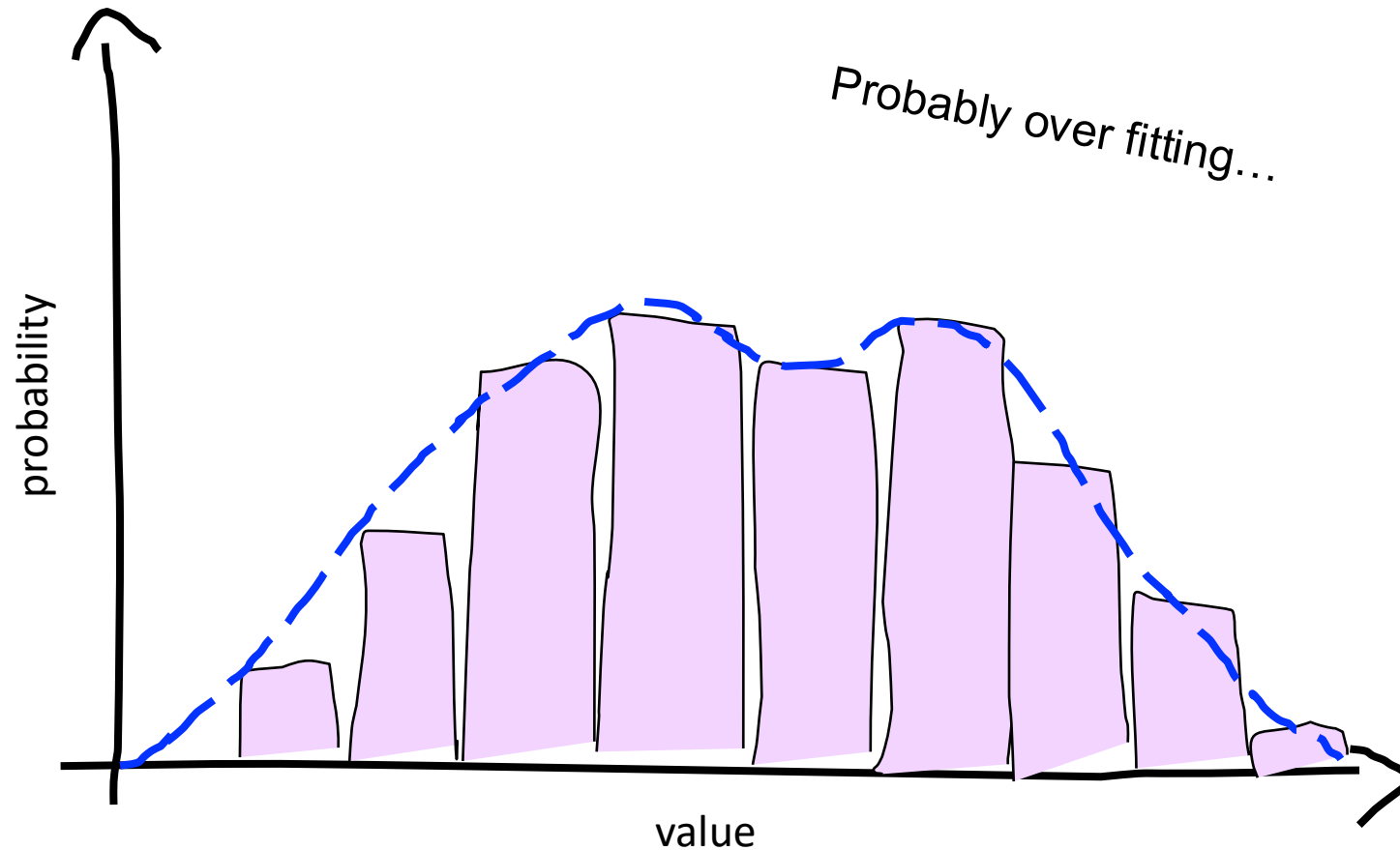
That's what they want you to believe...



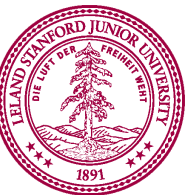


“The simplest explanation is usually the best one”

Complexity is Tempting

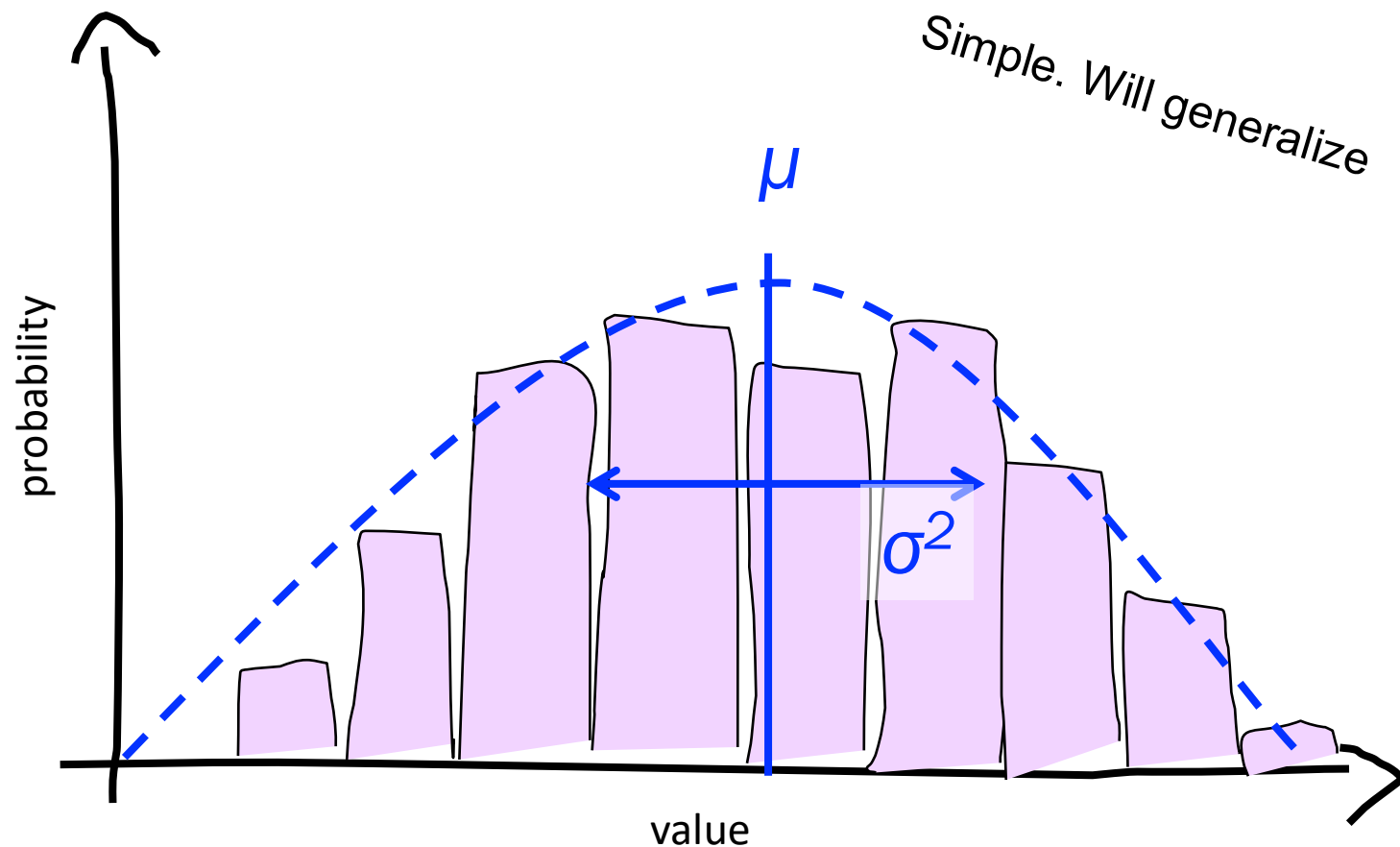


* That describes the training data, but will it **generalize**?

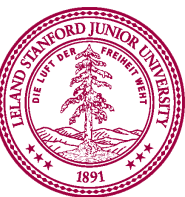


Fewest Assumptions

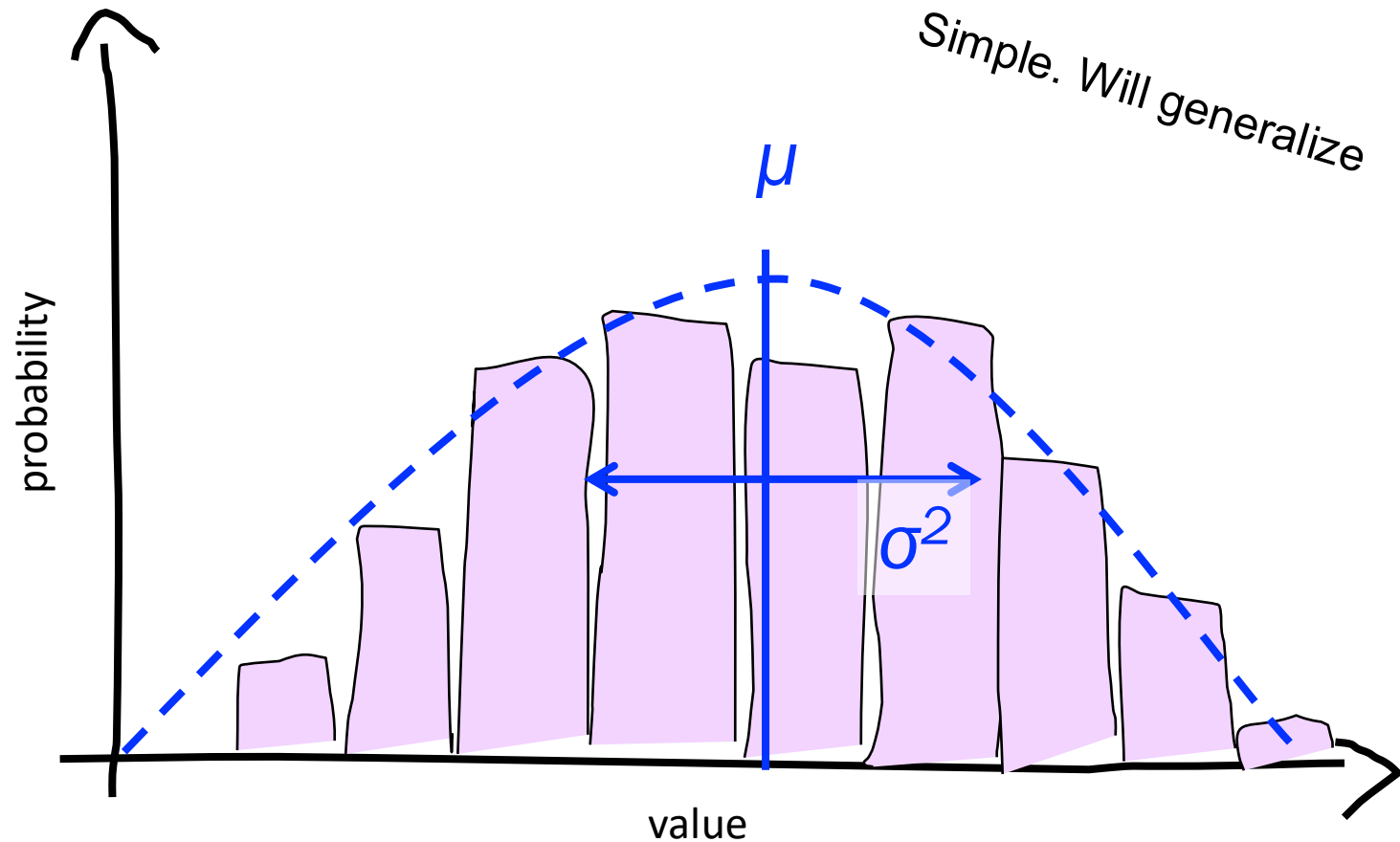
$$H(X) = - \sum_{i=1}^n P(x_i) \log P(x_i)$$



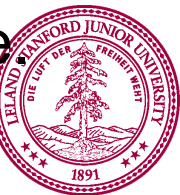
* A Gaussian **maximizes entropy** for a given mean and variance



Fewest Assumptions



* A Gaussian makes the **fewest assumptions** after matching mean and variance

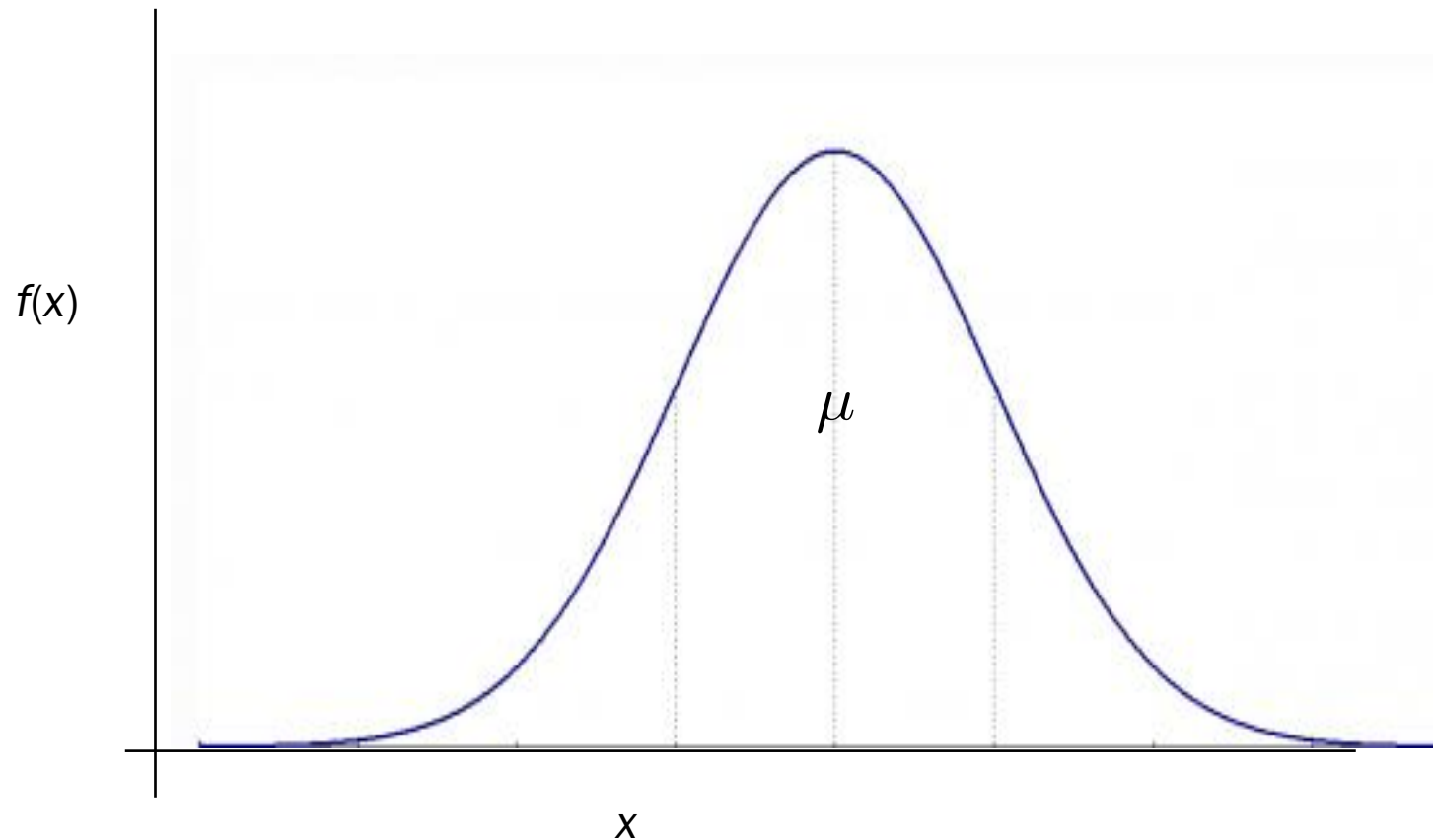


Normal is Beautiful!

Normal Probability Density Function

$$\mathcal{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



Anatomy of a Beautiful Equation

$$\mathcal{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

“exponential”

the distance to the mean

probability density at x

a constant

sigma shows up twice



Does it look less scary like this?

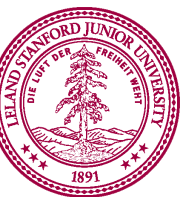
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

This means "e to the power of" and is
common function in code math libraries

$$f(x) \propto \frac{1}{\sigma} \cdot \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

This means "proportional to". There is a constant but
there are many cases where we don't care what it is!

What if you had to take the log of this function?



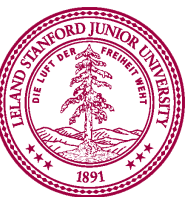
Lets go!

Let's Try It Out: Submarine Manufacturing

Your team is tasked with producing the side panels for Deep Sea Submarines. Physics requires all panels to be built within 10 micros of 500. Panel thickness is normally distributed. You check how precise your manufacturing is, and find these stats:

- Average panel thickness: $\mu = 500$ microns
- Variance of thickness: $\sigma^2 = 36$ microns²

What fraction of the panels you manufacture will meet standards?



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$$X \sim \mathcal{N}(\mu = 500, \sigma^2 = 36)$$



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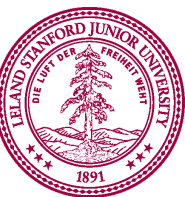
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$$P(490 \leq X \leq 510) = \int_{490}^{510} f(X = x) dx = \int_{490}^{510} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx$$



Let's Try It Out: Submarine Manufacturing

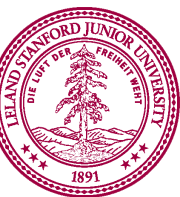
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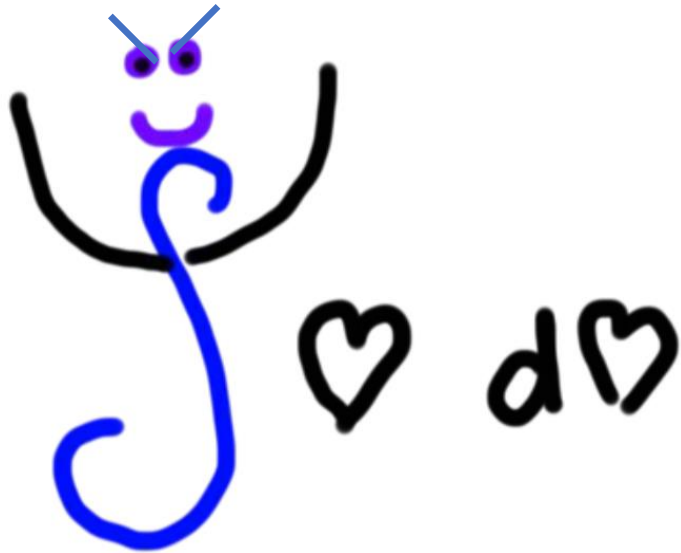
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Submarine Manufacturing

$$P(490 \leq X \leq 510) = \int_{490}^{510} f(X = x) dx = \int_{490}^{510} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx$$



Loving, not scary
...except this time

No closed form for the integral

No closed form for $F(x)$

Spoiler: Numerically Solved CDF

$$\mathcal{N}(\mu, \sigma^2)$$

A function that has been solved for numerically

$$F(x) = \Phi \left(\frac{x - \mu}{\sigma} \right)$$

The cumulative density
function of any normal

* We are going to spend the next few slides getting here



Linear Transform of a Normal is... Normal!

Let $X \sim \mathcal{N}(\mu, \sigma^2)$

$Y = aX + b$ is also Normal

$$\begin{aligned} E[Y] &= E[aX + b] \\ &= aE[X] + b \\ &= a\mu + b \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(aX + b) \\ &= a^2 \text{Var}(X) \\ &= a^2 \sigma^2 \end{aligned}$$

$$Y \sim \mathcal{N}(a\mu + b, a^2 \sigma^2)$$

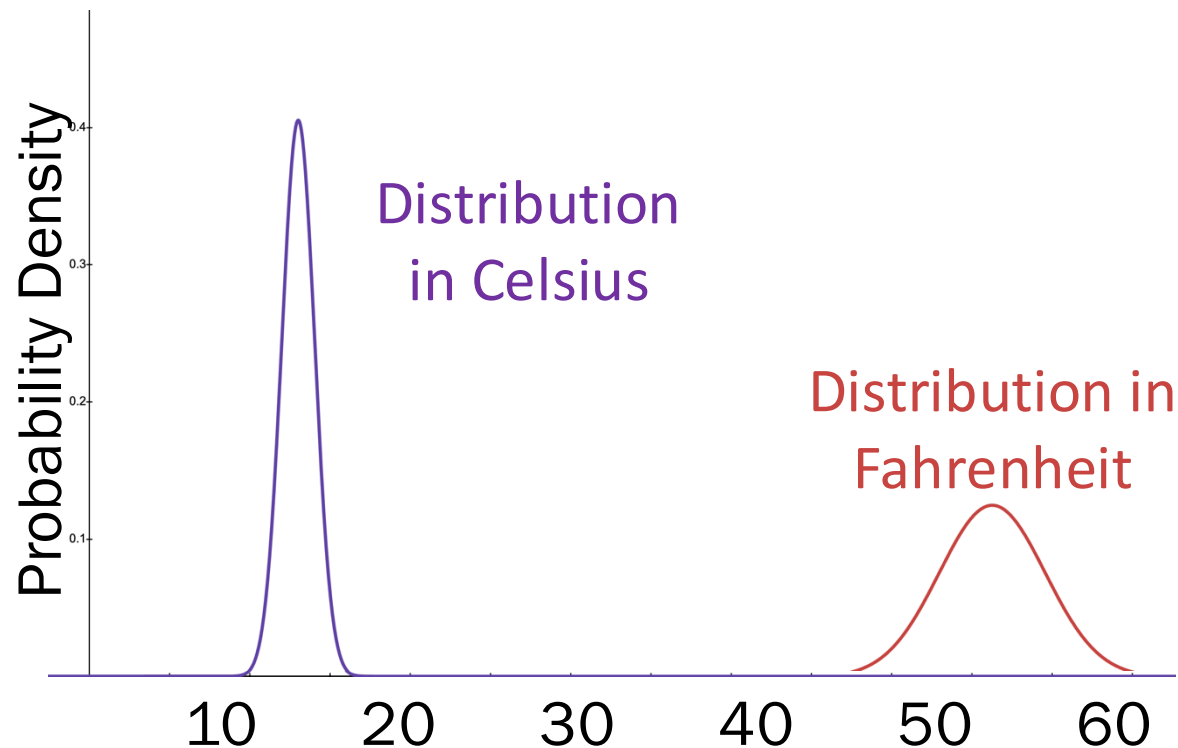


Aside: Celsius to Fahrenheit

$$Y = aX + b \quad Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Average temp in Palo Alto (on Jan 29th)
in Celsius:

$$X \sim N(\mu = 13, \sigma^2 = 1)$$



What is the distribution in Fahrenheit?

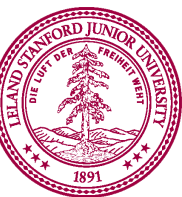
Let $Y = 1.8X + 32$

be the temperature in Fahrenheit.

Because this is a linear transform...

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

$$Y \sim N(\mu = 55.4, \sigma^2 = 3.24)$$



Linear Transform of a Normal is... Normal!

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$$\begin{aligned} \text{Var}(Y) &= \text{Var}(aX + b) \\ &= a^2 \text{Var}(X) \\ &= a^2 \sigma^2 \end{aligned}$$

$$Y \sim \mathcal{N}(a\mu + b, a^2 \sigma^2)$$



The cutest linear transform

Let $X \sim \mathcal{N}(\mu, \sigma^2)$

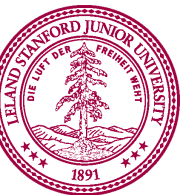
$Y = aX + b$ is also Normal

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

There is a special case of linear transform for any X :

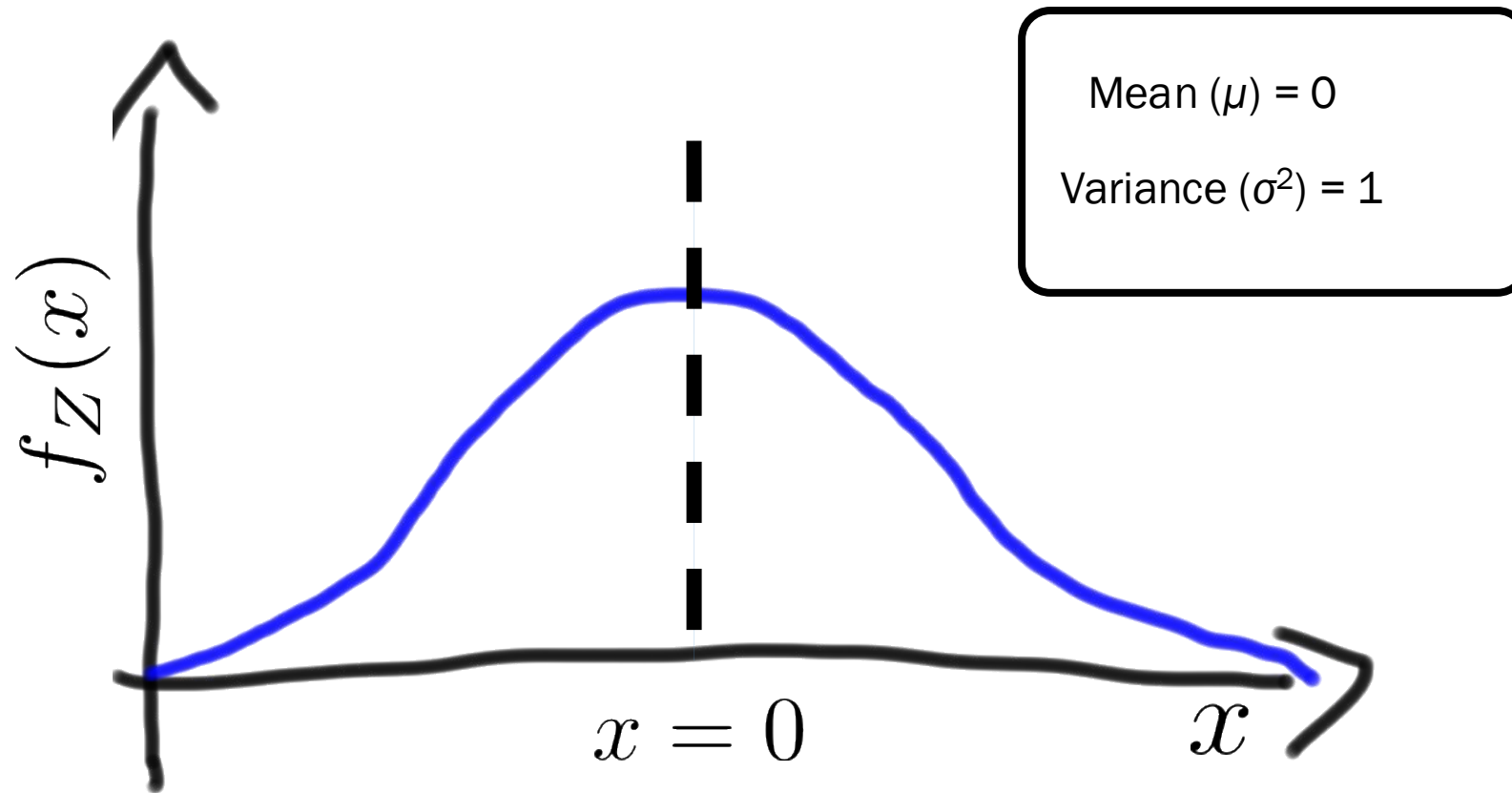
$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \quad a = \frac{1}{\sigma} \quad b = -\frac{\mu}{\sigma}$$

$$\begin{aligned} Z &\sim \mathcal{N}(a\mu + b, a^2\sigma^2) \\ &\sim \mathcal{N}\left(\frac{\mu}{\sigma} - \frac{\mu}{\sigma}, \frac{\sigma^2}{\sigma^2}\right) \\ &\sim \mathcal{N}(0, 1) \end{aligned}$$

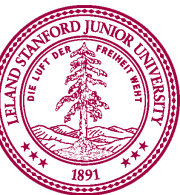


The Standard Normal

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$



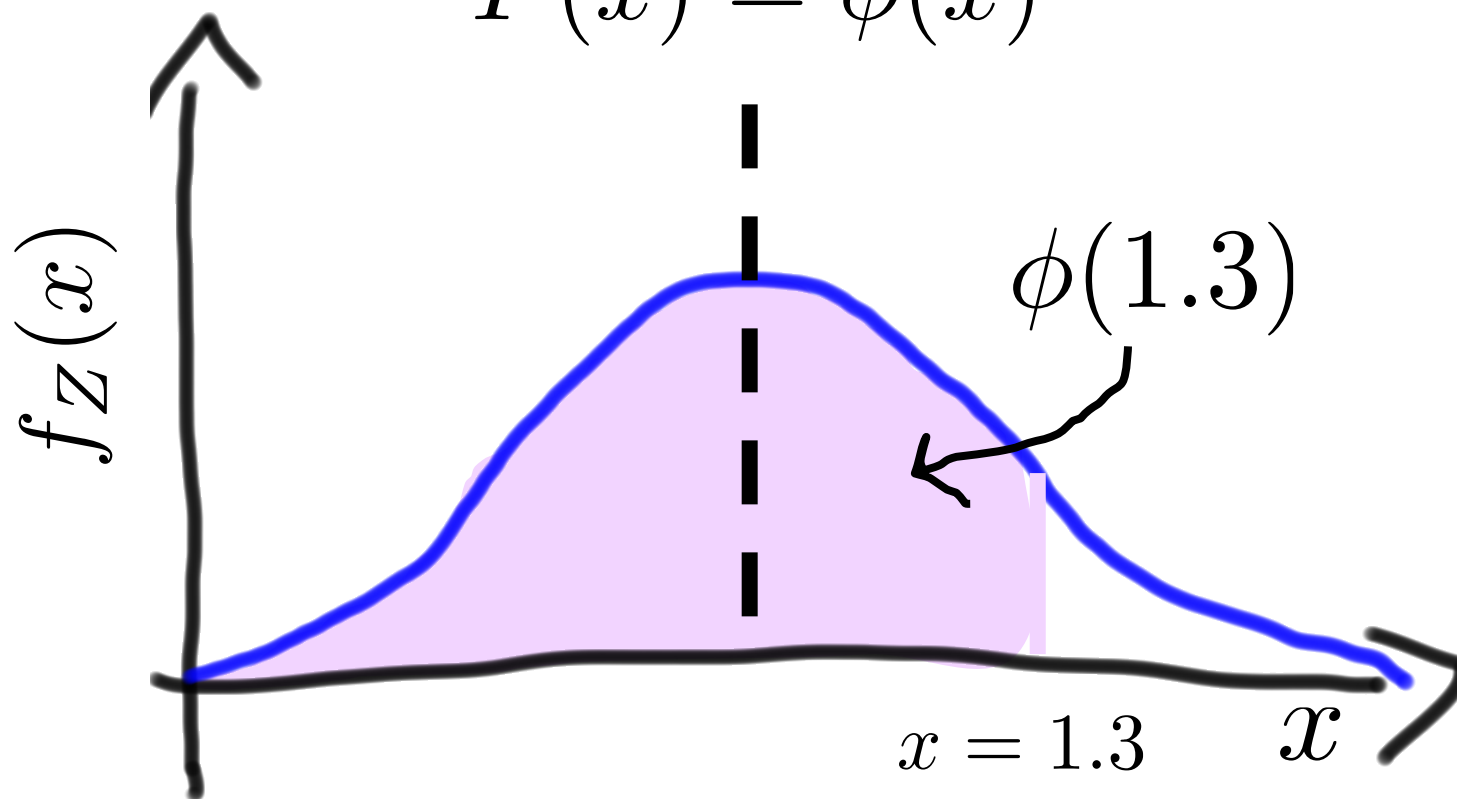
*This is the probability density function for the standard normal



Phi

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$

$$F(x) = \phi(x)$$



*This is the probability density function for the standard normal



Using Table of Φ

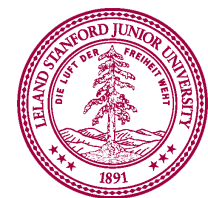
Standard Normal Cumulative Probability Table

$$\Phi(1.31) = 0.9049$$



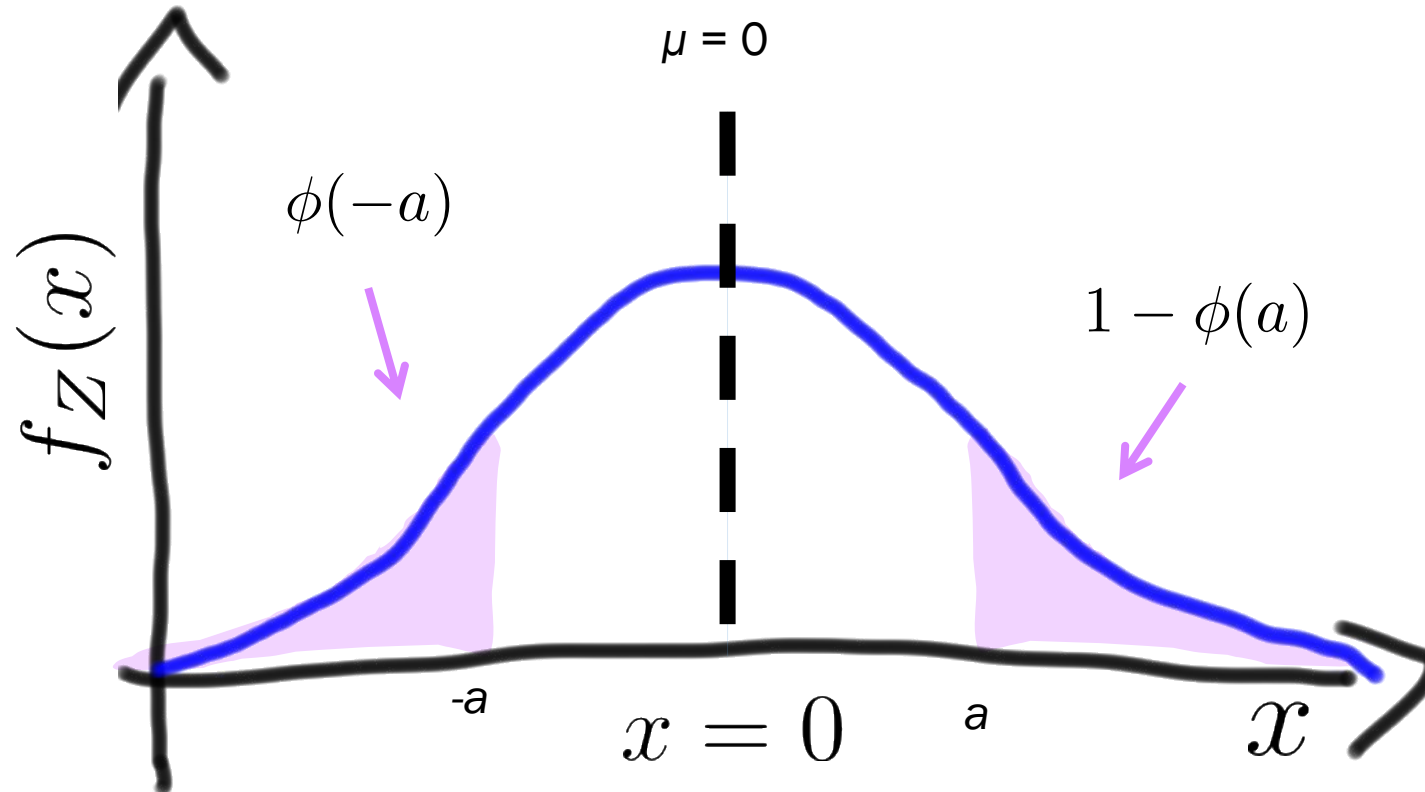
Cumulative probabilities for **POSITIVE** z-values are shown in the following table:

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |

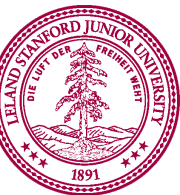


Symmetry of Phi

$$\phi(-a) = 1 - \phi(a)$$

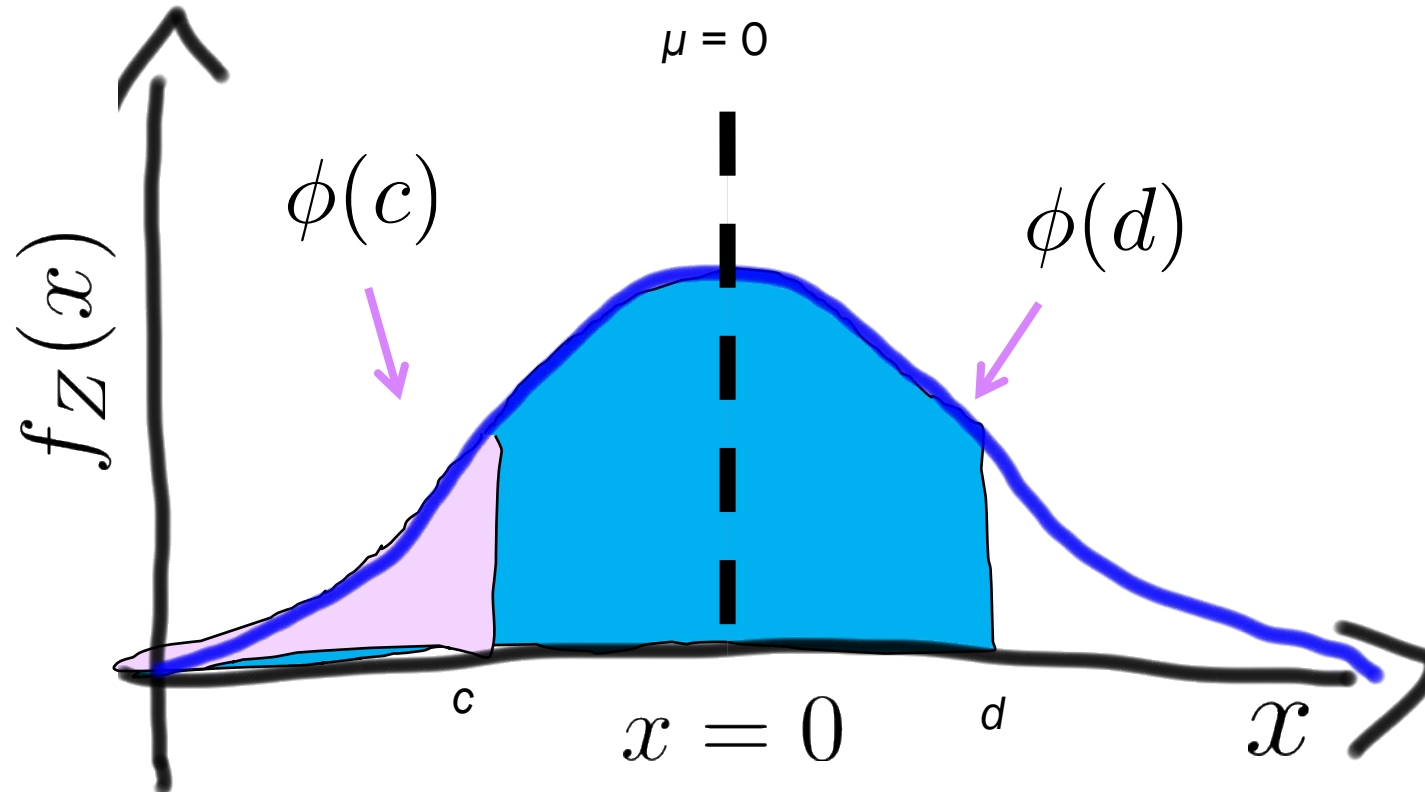


*This is the probability density function for the standard normal



Interval of Phi

$$P(c < Z < d) = \phi(d) - \phi(c)$$

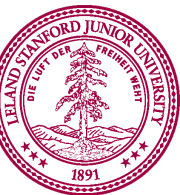


Compute $F(x)$ via Transform

$$\text{Let } X \sim \mathcal{N}(\mu, \sigma^2) \qquad Z = \frac{X - \mu}{\sigma}$$

Use Z to compute $F(x)$

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(X - \mu \leq x - \mu) \\ &= P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\ &= P\left(Z \leq \frac{x - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{x - \mu}{\sigma}\right) \end{aligned}$$





For normal distribution,
 $F(x)$ is computed using
the phi transform.

And here we are

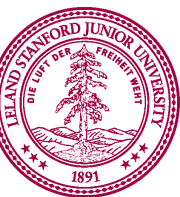
$$\mathcal{N}(\mu, \sigma^2)$$

CDF of Standard Normal: A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

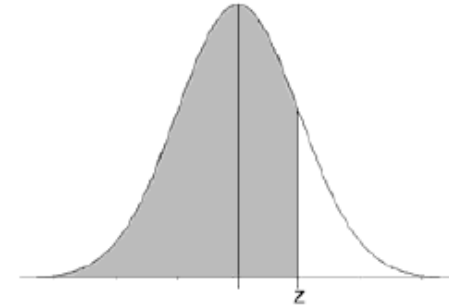
Table of $\Phi(\mathbf{Z})$ values in textbook, p. 201 and handout



Using the Phi Table

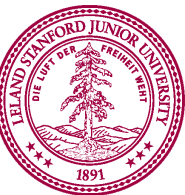
Standard Normal Cumulative Probability Table

$$\Phi(0.54) = 0.7054$$



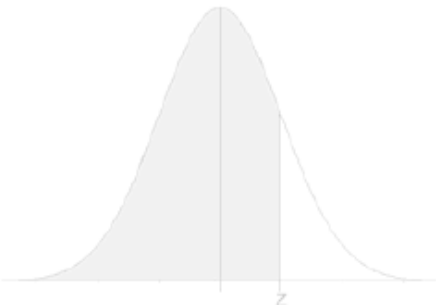
Cumulative probabilities for **POSITIVE** z-values are shown in the following table:

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| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
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| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
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| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |



Do We Have To Use The Table??

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
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| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7122 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7421 | 0.7453 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7968 | 0.7996 | 0.8025 | 0.8053 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
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| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |



Table is kinda old school



We Are Computer Scientists!

Every modern programming language has phi stored in a library:

```
from scipy import stats  
stats.norm.cdf(x, mean, std)
```

$= P(X < x)$ where $X \sim \mathcal{N}(\mu, \sigma^2)$



We Are Computer Scientists!

Every modern programming language has phi stored in a library:

```
from scipy import stats  
stats.norm.cdf(x, mean, std)
```

not variance!!!

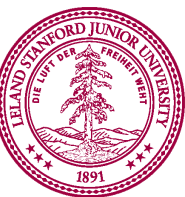
$$= P(X < x) \text{ where } X \sim \mathcal{N}(\mu, \sigma^2)$$



We Made One For You

The screenshot shows a web browser window with the address bar displaying `chrispiech.github.io/probabilityForComputerScientists/en/intro/calculators/`. The page features a dark sidebar on the left with the title "Course Reader for CS109" and a search bar. Below the search bar are links for "Notation Reference", "Random Variable Reference", and "Calculators". The "Calculators" section is expanded, showing a list of topics under "Part 1: Core Probability": Counting, Combinatorics, Definition of Probability, Equally Likely Outcomes, Probability of or, Conditional Probability, Independence, Probability of and, Law of Total Probability, Bayes' Theorem, Log Probabilities, Many Coin Flips, Worked Examples, and Enigma Machine. The main content area contains three calculator tools:

- Phi Calculator, $\Phi(x)$** : A text input for `x` with the value `0.7` and a blue button labeled `phi(x)`.
- Inverse Phi Calculator, $\Phi^{-1}(y)$** : A text input for `y` with the value `0.7` and a blue button labeled `inverse_phi(y)`.
- Norm CDF Calculator**: Three text inputs for `x` (0.0), `mu` (0), and `std` (1), and a blue button labeled `norm.cdf(x, mu, std)`.



Practice: Submarine Manufacturing

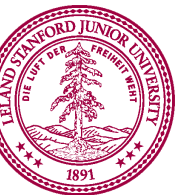
Your team is tasked with producing the side panels for Deep Sea Submarines. Physics requires all panels to be built within 10 microns of 500. You check how precise your manufacturing is, and find these stats:

- Average panel thickness: $\mu = 500$ microns
- Variance of thickness: $\sigma^2 = 36$ microns²

What fraction of the panels you manufacture will meet standards?

$$X \sim \mathcal{N}(\mu = 500, \sigma^2 = 36)$$

$$P(490 \leq X \leq 510) = \int_{490}^{510} f(X = x) dx$$



Practice: Submarine Manufacturing

$$\text{If } X \sim \mathcal{N}(\mu, \sigma^2), F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

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- Average panel thickness: $\mu = 500$ microns
- Variance of thickness: $\sigma^2 = 36$ microns²



Now using the CDF!

What fraction of the panels you manufacture will meet standards?

$$X \sim \mathcal{N}(\mu = 500, \sigma^2 = 36)$$

$$P(490 \leq X \leq 510) = ?$$

Practice: Submarine Manufacturing

$$\text{If } X \sim \mathcal{N}(\mu, \sigma^2), F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Your team is tasked with producing the side panels for Deep Sea Submarines. Physics requires all panels to be built within 10 micros of 500. You check how precise your manufacturing is, and find these stats:

- Average panel thickness: $\mu = 500$ microns
- Variance of thickness: $\sigma^2 = 36$ microns²



Now using the CDF!

What fraction of the panels you manufacture will meet standards?

$$X \sim \mathcal{N}(\mu = 500, \sigma^2 = 36)$$

$$P(490 \leq X \leq 510) = P(X < 510) - P(X < 490) = \Phi\left(\frac{510 - 500}{6}\right) - \Phi\left(\frac{490 - 500}{6}\right)$$

subtract mean, divide by std. dev.



Practice: Submarine Manufacturing

$$\text{If } X \sim \mathcal{N}(\mu, \sigma^2), F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Your team is tasked with producing the side panels for Deep Sea Submarines. Physics requires all panels to be built within 10 microns of 500. You check how precise your manufacturing is, and find these stats:

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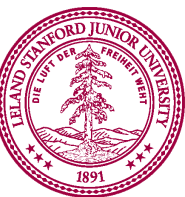


Now using the CDF!

What fraction of the panels you manufacture will meet standards?

$$X \sim \mathcal{N}(\mu = 500, \sigma^2 = 36)$$

$$\begin{aligned} P(490 \leq X \leq 510) &= P(X < 510) - P(X < 490) = \Phi\left(\frac{510 - 500}{6}\right) - \Phi\left(\frac{490 - 500}{6}\right) \\ &= \Phi\left(\frac{5}{3}\right) - \left(1 - \Phi\left(\frac{5}{3}\right)\right) = 2 \Phi\left(\frac{5}{3}\right) - 1 \approx 0.904 \end{aligned}$$



Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$





huntrixboi



Are you ready for something different?

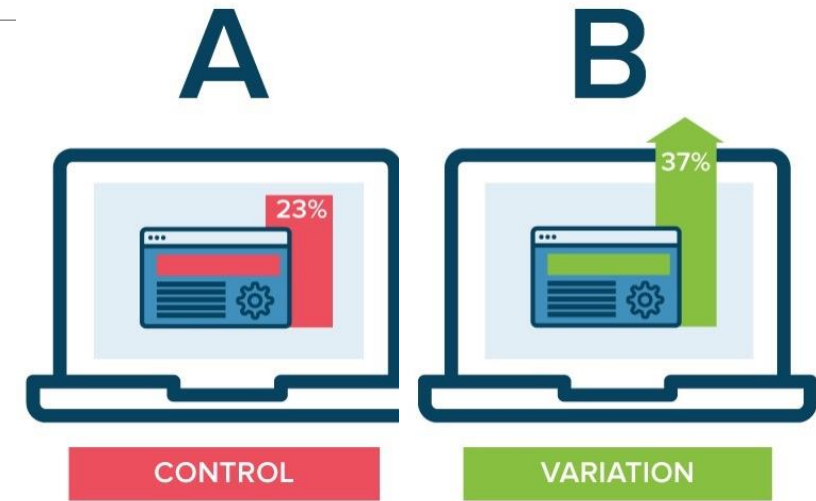
Pop quiz!
(jk)

Midterm: Website Testing

A new website design is tested out on 1M users.

- Let X be the number of users whose time on the site increases with the new design.
- The CEO will endorse the new design if $X \geq 501k$.

What is $P(\text{CEO endorses change} \mid \text{it has no effect})$?

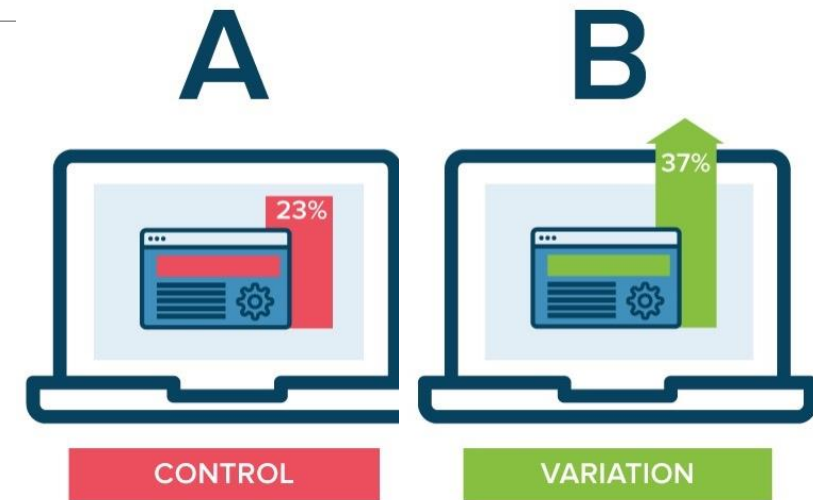


Midterm: Website Testing

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What is $P(\text{CEO endorses change} \mid \text{it has no effect})$?



$$X \sim \text{Bin}(n = 10^6, p = 0.5)$$

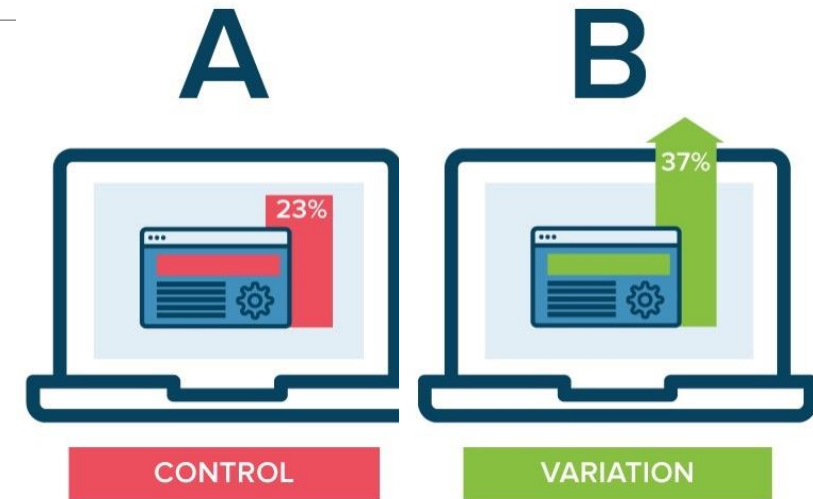
$$P(X > 501000) = \sum_{i=501000}^{10^6} \binom{10^6}{i} (0.5)^i (0.5)^{10^6-i}$$

Midterm: Website Testing

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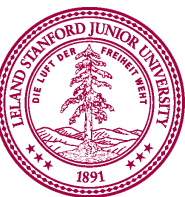
What is $P(\text{CEO endorses change} \mid \text{it has no effect})$?



$$X \sim \text{Bin}(n = 10^6, p = 0.5)$$

```
>>> math.comb(1000000, 501000)
```

$$P(X > 501000) = \sum_{i=501000}^{10^6} \binom{10^6}{i} (0.5)^i (0.5)^{10^6-i}$$

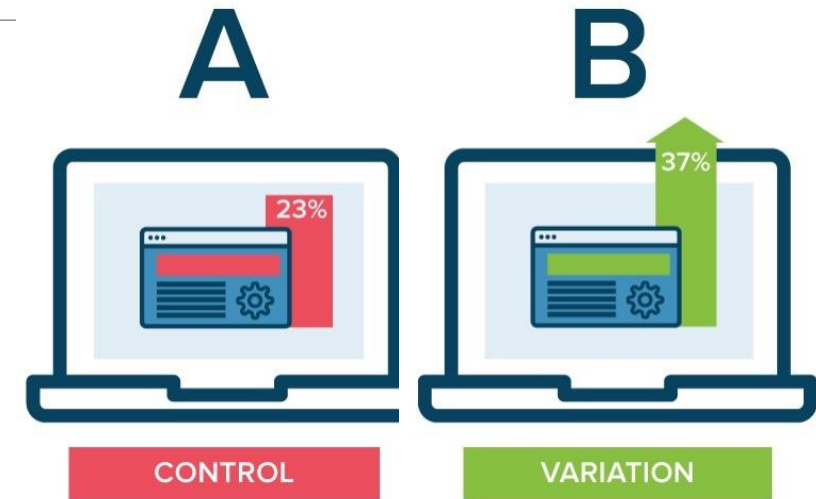


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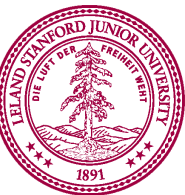
$$X \sim \text{Bin}(n = 10^6, p = 0.5)$$

$$P(X > 501000) = \sum_{i=501000}^{10^6} \binom{10^6}{i} (0.5)^i (0.5)^{10^6-i}$$

```
>>> math.comb(1000000, 501000)
```

ValueError: Exceeds the limit (4300 digits) for integer string conversion; use `sys.set_int_max_str_digits()` to increase the limit

```
>>>
```

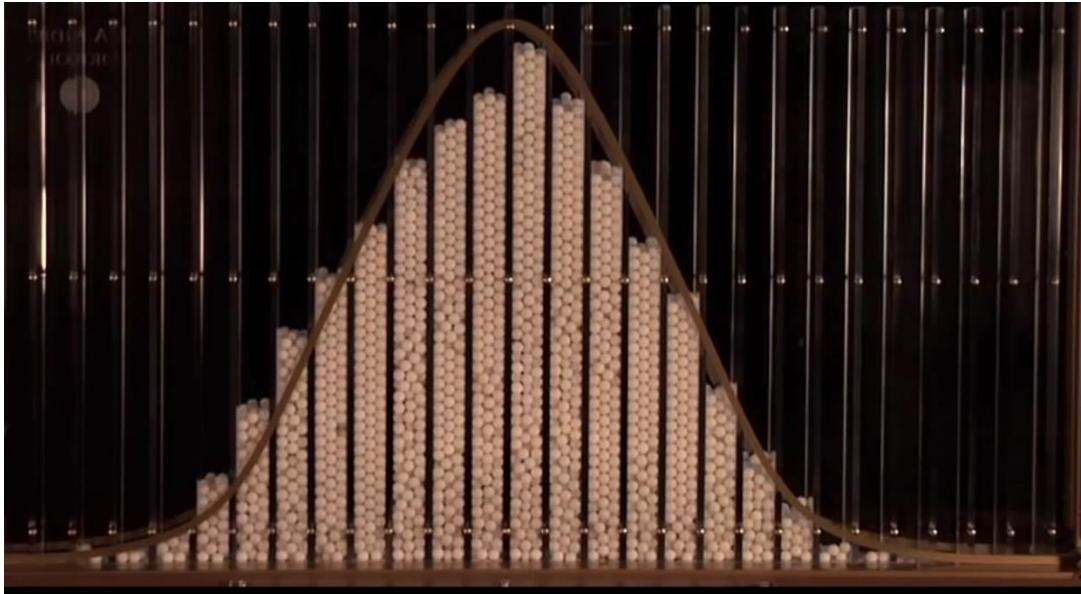


$$P(X > 505000) = \sum_{i=505000}^{10^6} \binom{10^6}{i} (0.5)^i (0.5)^{10^6-i}$$

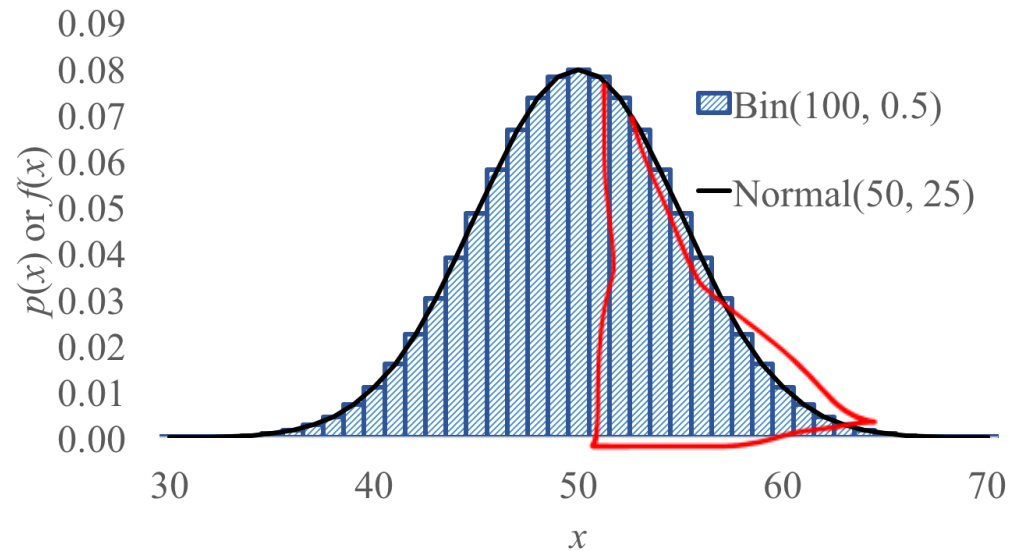


THIS IS
FINE

Don't worry, Normal approximates Binomial

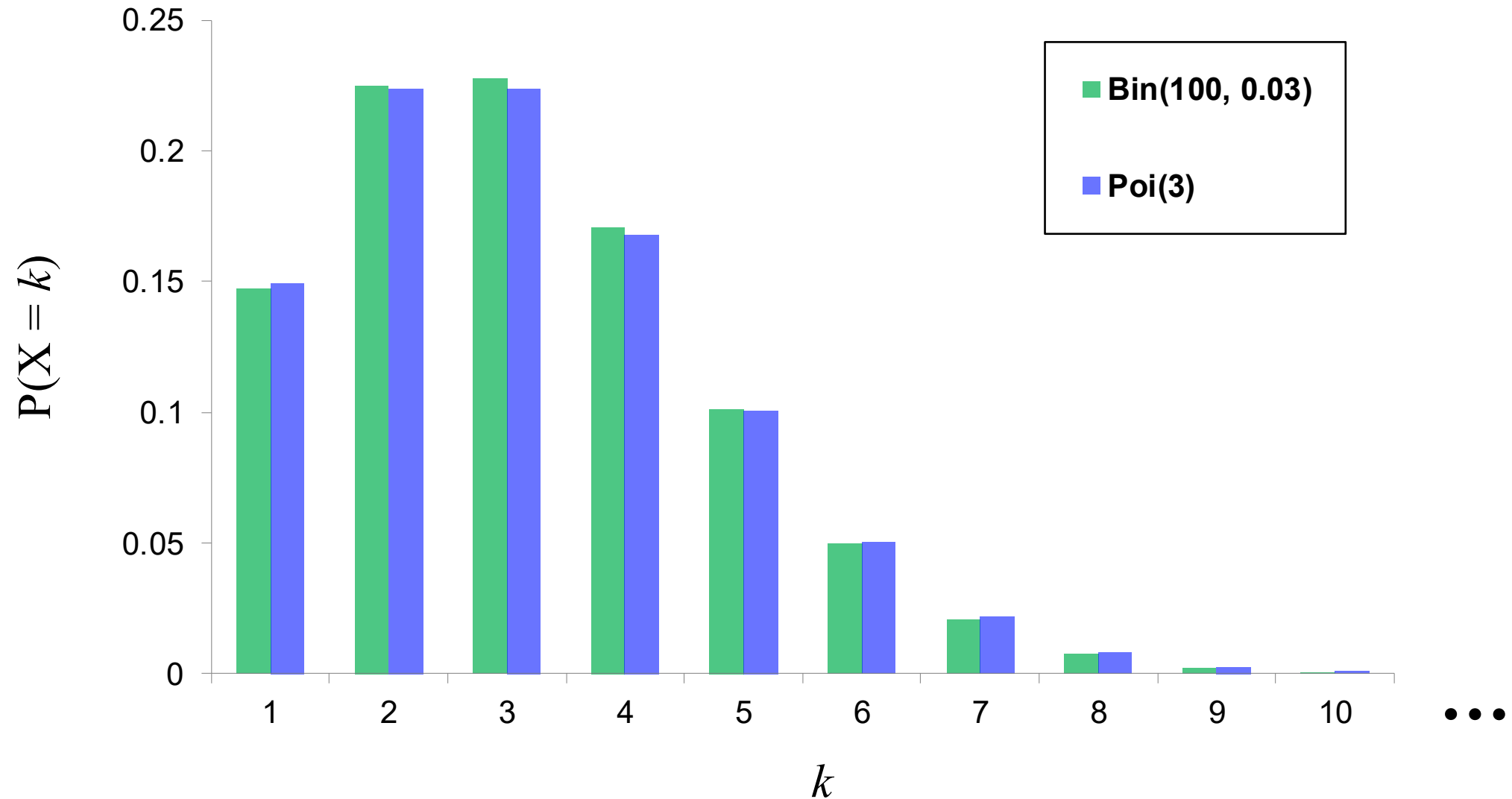


Galton Board



(We'll explain *why*
in 2 weeks' time)

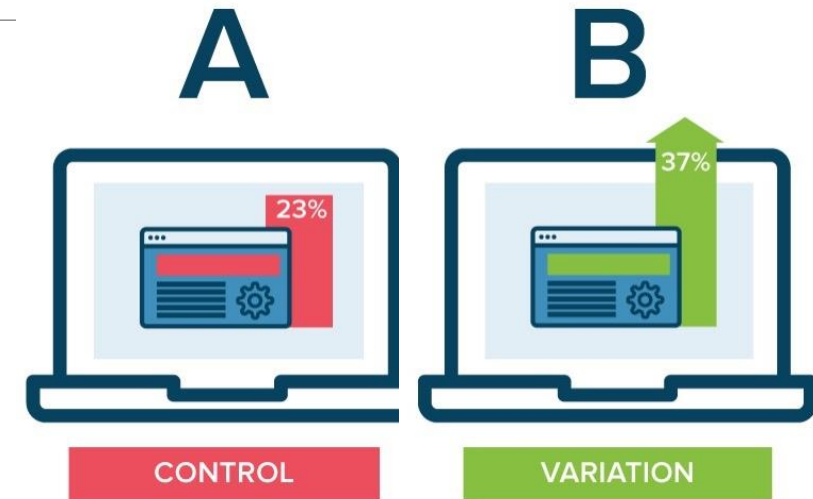
Poisson Approximates Binomial, With Extreme n and p



Midterm: Website Testing

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What is $P(\text{CEO endorses change} \mid \text{it has no effect})$?

$$X \sim \text{Bin}(n = 10^6, p = 0.5)$$

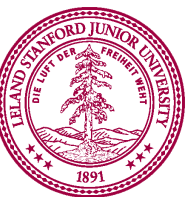
$$Y \sim N(\mu = 500000, \sigma^2 = 250000)$$

$n \cdot p$ $n \cdot p \cdot (1 - p)$

$$\begin{aligned} P(X > 501000) &\approx P(Y > 501000) \\ &\approx 1 - P(Y < 501000) \\ &\approx 1 - F_Y(501000) \approx 0.02275 \end{aligned}$$

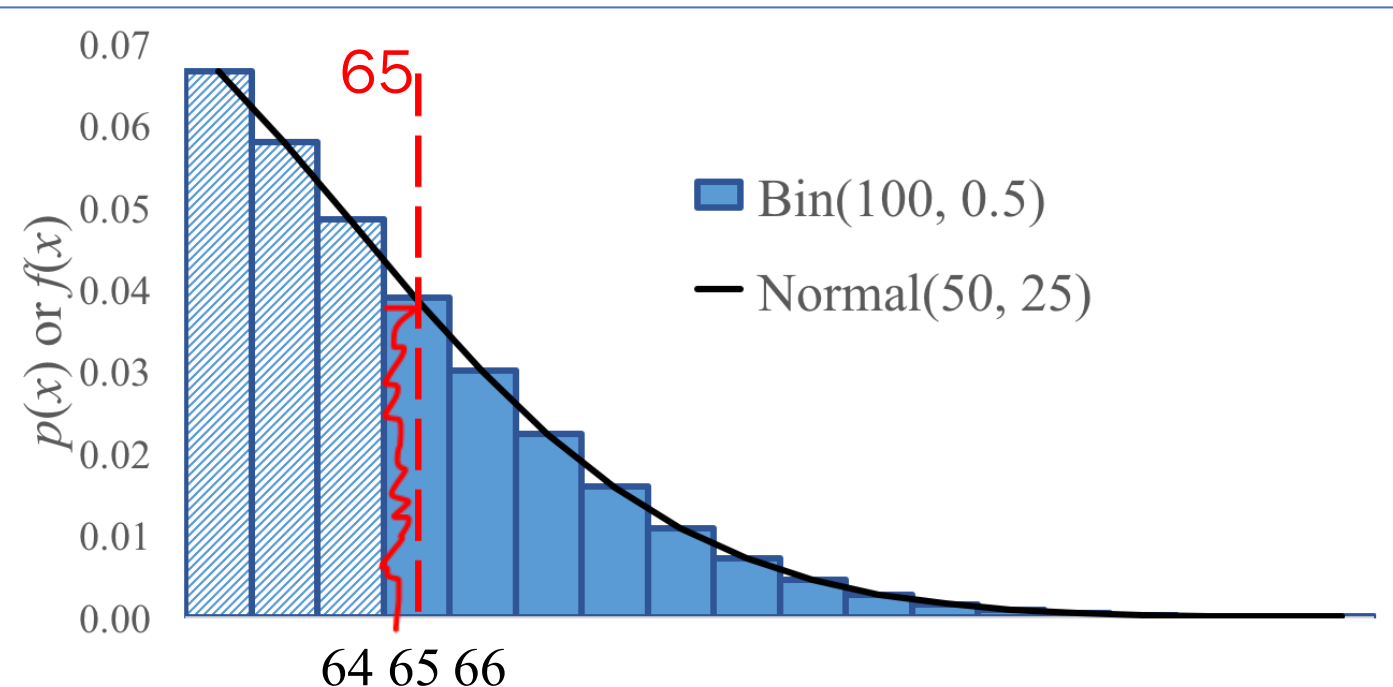
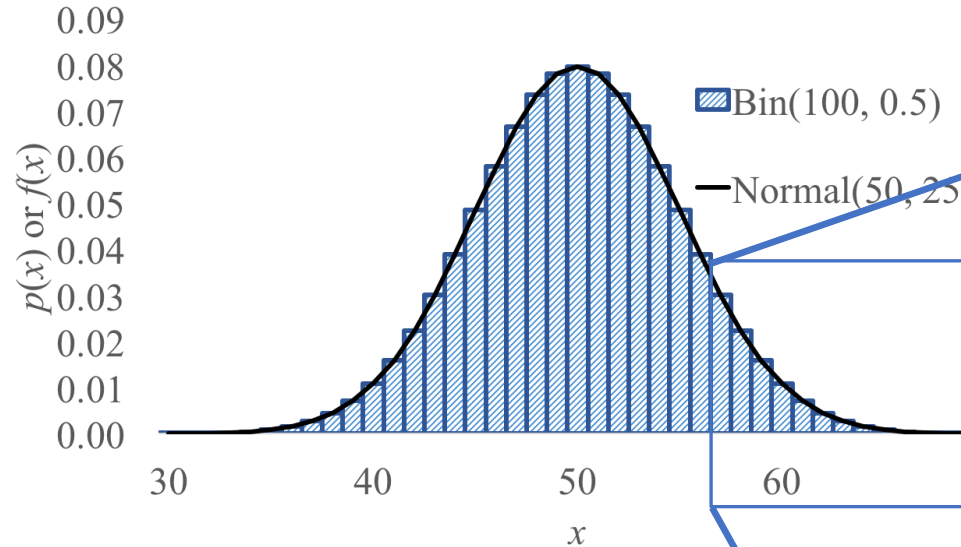


Correct answer is 0.02270



Normal Approximation (with continuity correction)

In our website testing, $Y \sim \mathcal{N}(50, 25)$ approximates $X \sim \text{Bin}(100, 0.5)$.



$$P(X \geq 65) \text{ Binomial}$$

$$\approx P(Y \geq 64.5) \text{ Normal}$$

$$\approx 0.0018$$



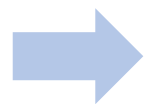
You must perform a **continuity correction** when approximating a Binomial RV with a Normal RV.



Continuity correction

If $Y \sim \mathcal{N}(np, np(1 - p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

Discrete (e.g., Binomial)
probability question



Continuous (Normal)
probability question

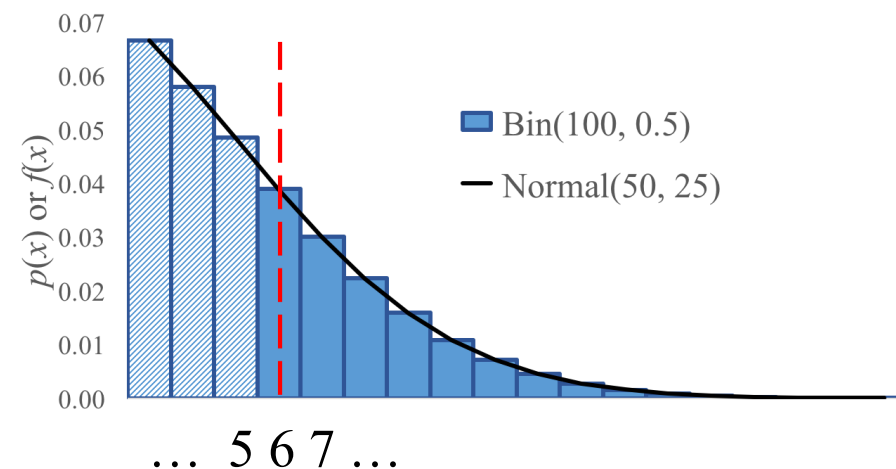
$$P(X = 6)$$

$$P(X \geq 6)$$

$$P(X > 6)$$

$$P(X < 6)$$

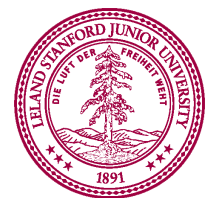
$$P(X \leq 6)$$



Continuity correction

If $Y \sim \mathcal{N}(np, np(1 - p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

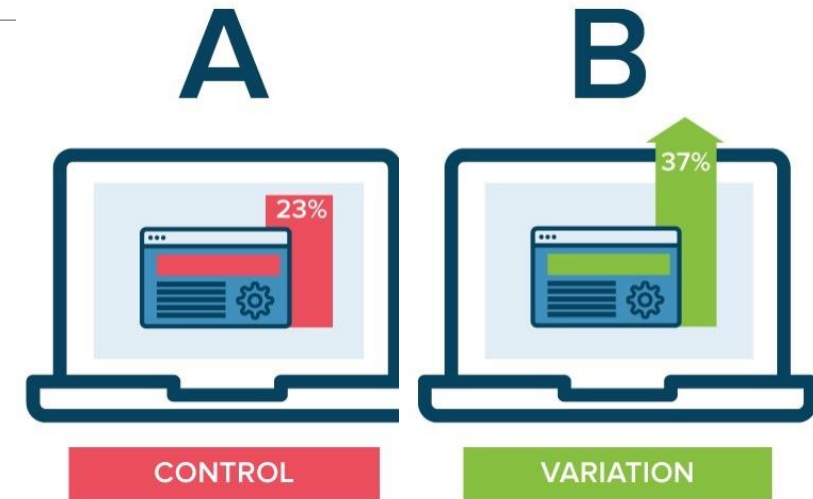
| Discrete (e.g., Binomial) probability question | Continuous (Normal) probability question |
|---|---|
| $P(X = 6)$ | $P(5.5 \leq Y \leq 6.5)$ |
| $P(X \geq 6)$ | $P(Y \geq 5.5)$ |
| $P(X > 6)$ | $P(Y \geq 6.5)$ |
| $P(X < 6)$ | $P(Y \leq 5.5)$ |
| $P(X \leq 6)$ | $P(Y \leq 6.5)$ |



Midterm: Website Testing

A new website design is tested out on 1M users.

- Let X be the number of users whose time on the site increases with the new design.
- The CEO will endorse the new design if $X \geq 501k$.



What is $P(\text{CEO endorses change} \mid \text{it has no effect})$?

$$X \sim \text{Bin}(n = 10^6, p = 0.5)$$

$$Y \sim N(\mu = 500000, \sigma^2 = 250000)$$

$n \cdot p$ $n \cdot p \cdot (1 - p)$

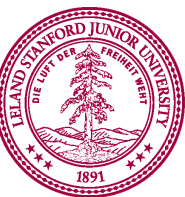
$$P(X > 501000) \approx P(Y > 501000.5)$$

$$\approx 1 - P(Y < 501000.5)$$

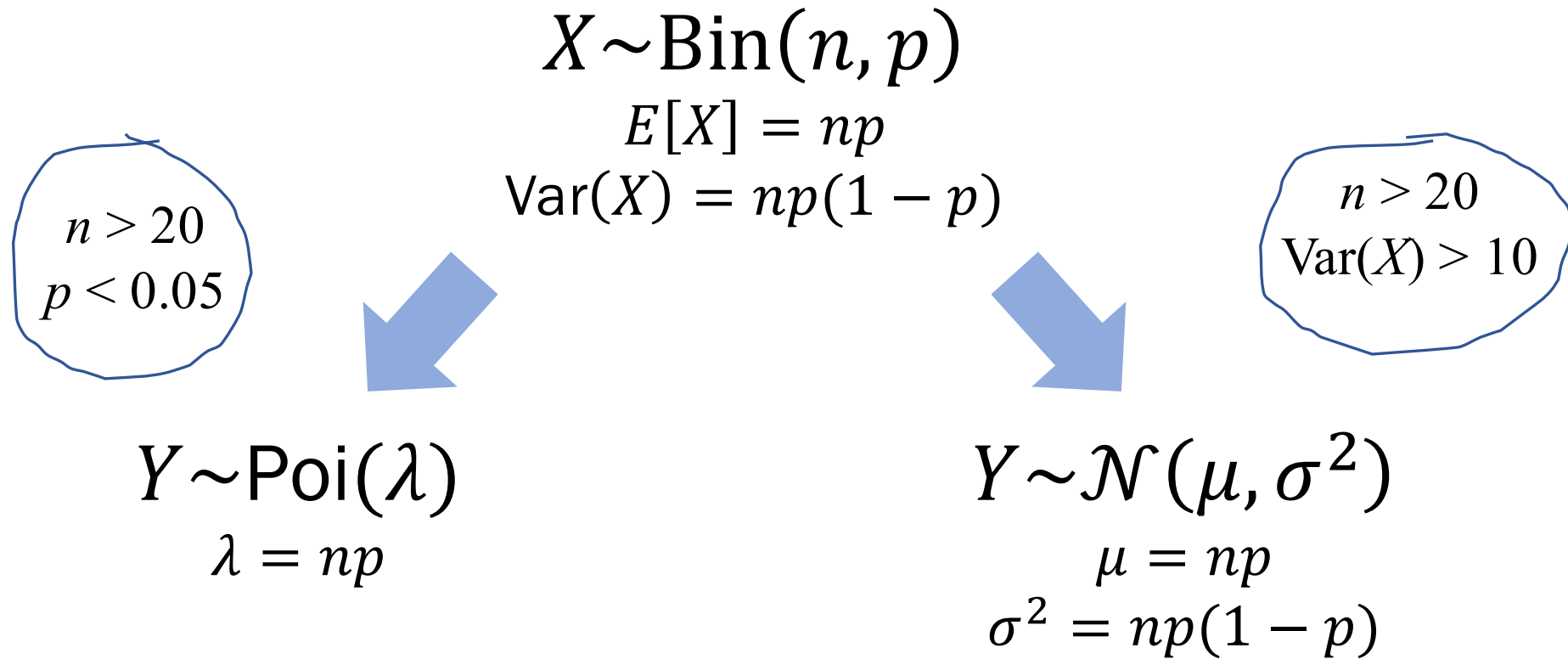
$$\approx 1 - F_Y(501000.5) \approx 0.02270$$



YOU ARE AMAZING!



Two Ways To Approximate The Binomial



Poisson approximation for big n , small p .
Normal approximation for big n , medium p .

Just Invented the Normal
Approximation

Stanford Admissions (a while back)

Stanford accepts 2480 students.

- Each admitted student matriculates w.p. 0.68 (independent trials)
- Let $X = \#$ of students who will attend

What is $P(X > 1745)$? *Give a numerical approximation.*

- Strategy:
- A. Just Binomial
 - B. Poisson
 - C. Normal
 - D. None/other



(by yourself)

Stanford Admissions

Stanford accepts 2480 students.

- Each admitted student matriculates w.p. 0.68 (independent trials)
- Let $X = \#$ of students who will attend

What is $P(X > 1745)$? *Give a numerical approximation.*

Strategy:

- A. Just Binomial not an approximation (also computationally expensive)
- B. Poisson $p = 0.68$, not small enough
- ☒ C. Normal ☒ Variance $np(1 - p) = 540 > 10$
- D. None/other

Define an approximation

Let $Y \sim \mathcal{N}(E[X], \text{Var}(X))$

$$E[X] = np = 1686$$

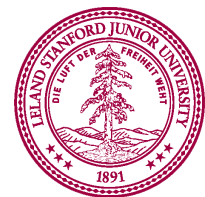
$$\text{Var}(X) = np(1 - p) \approx 540 \rightarrow \sigma = 23.3$$

$$P(X > 1745) \approx P(Y \geq 1745.5) \quad \text{! Continuity correction}$$

Solve

SciPy can do this

$$\begin{aligned} P(Y \geq 1745.5) &= 1 - F(1745.5) \\ &= 1 - \Phi\left(\frac{1745.5 - 1686}{23.3}\right) \\ &= 1 - \Phi(2.54) \approx 0.0055 \end{aligned}$$

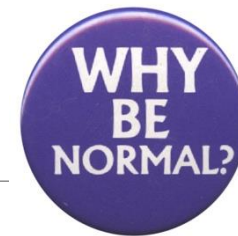


Pedagogical Pause

Great questions!
Great thinkers start with great
questions. Ask away!!!

Super Question:

Why Be Normal? 68% rule

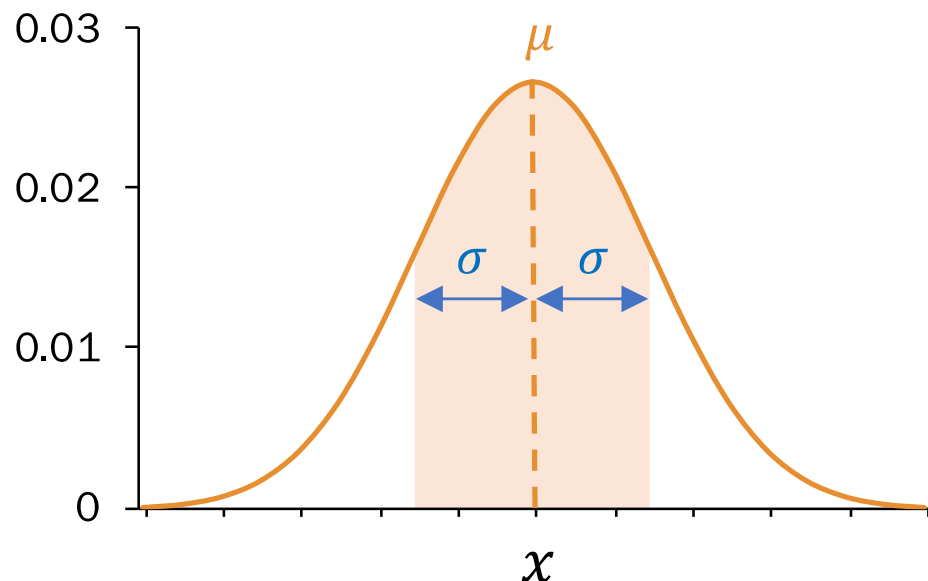


You may have heard the statement:

“68% of the class will fall within 1 standard deviation of the exam average.”

In general, this is only true of **normal distributions**:

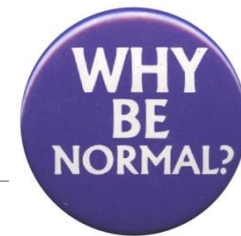
Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF F .



$$\begin{aligned} P(|X - \mu| < \sigma) &= P(\mu - \sigma < X < \mu + \sigma) \\ &= F(\mu + \sigma) - F(\mu - \sigma) \\ &= \Phi\left(\frac{(\mu + \sigma) - \mu}{\sigma}\right) - \Phi\left(\frac{(\mu - \sigma) - \mu}{\sigma}\right) \\ &= \Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1)) \\ &= 2\Phi(1) - 1 \approx 2(0.8413) - 1 = \mathbf{0.6826} \end{aligned}$$



Why Be Normal? 68% rule

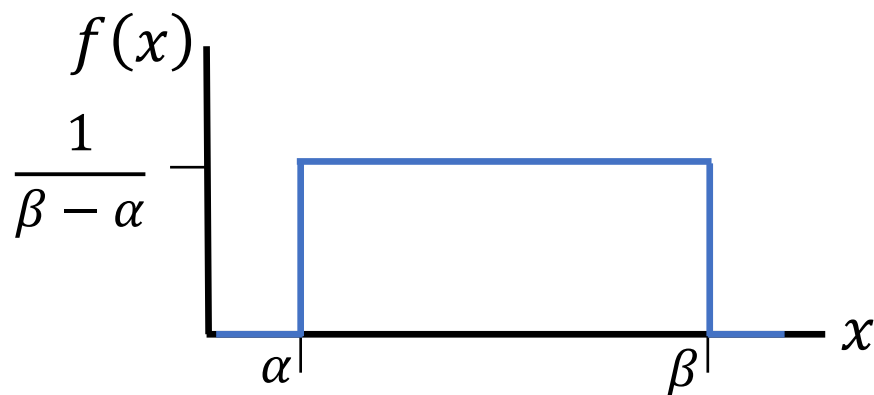


You may have heard the statement:

“68% of the class will fall within 1 standard deviation of the exam average.”

In general, this is only true of **normal distributions**:

Counterexample: Let $X \sim \text{Uni}(\alpha, \beta)$.



$$\mu = E[X] = \frac{\alpha + \beta}{2}$$

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12} \Rightarrow \sigma = \text{SD}(X) = \frac{\beta - \alpha}{\sqrt{12}}$$

$$P(|X - \mu| < \sigma) = P(\mu - \sigma < X < \mu + \sigma)$$

$$= \frac{1}{\beta - \alpha} \cdot [(\mu + \sigma) - (\mu - \sigma)]$$

$$= \frac{1}{\beta - \alpha} [2\sigma] = \frac{1}{\beta - \alpha} \cdot \left[2 \cdot \frac{\beta - \alpha}{\sqrt{12}} \right]$$

$$= 2/\sqrt{12} \approx 0.58$$



Challenge

光棍节

SINGLE'S DAY

SALE

双11

福

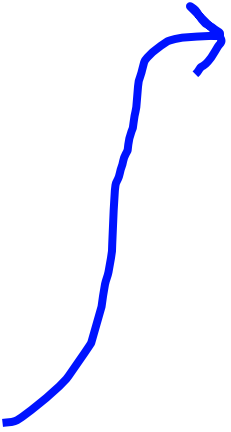
Enough Servers?

You receive $R \sim N(\mu = 10^6, \sigma = 10^4)$ requests in the busiest min

You are going to buy n servers

Each server can handle 10,000 requests per min, otherwise you drop requests

What is the smallest value of n such that $P(\text{drop}) < 0.0001$

$$\begin{aligned} P(\text{enough}) &> 0.9999 \\ P(R < 10^4 \cdot n) &> 0.9999 \\ \phi\left(\frac{10^4 \cdot n - 10^6}{10^4}\right) &> 0.9999 \\ \frac{10^4 \cdot n - 10^6}{10^4} &> \phi^{-1}(0.9999) \end{aligned}$$

$$\begin{aligned} \frac{10^4 \cdot n - 10^6}{10^4} &> \phi^{-1}(0.9999) \\ \frac{10^4 \cdot n - 10^6}{10^4} &> 3.72 \\ n &> \frac{10^6 + 3.72 \cdot 10^4}{10^4} \end{aligned}$$

