Practice Midterm Solutions

With solutions by Mehran Sahami and Chris Piech

1. a. The answer to this question is simply a multinomial coefficient, which can be written/computed in numerous ways:

\[
\binom{12}{5, 4, 3} = \frac{12!}{5!4!3!} = \binom{12}{5} \binom{7}{3} = \binom{12}{5} \binom{7}{4}
\]

b. \(\binom{10}{3, 4, 3} + \binom{10}{5, 2, 3} + \binom{10}{5, 4, 1}\)

We select (remove) two candies of the same type to give to Larry and Sergey (there is only 1 way to do this for each type of candy). The remaining 10 candies are then distributed to the remaining 10 students. The three terms above correspond respectively to Twix Bars, Almond Joys, and Swedish Fish being given to Larry and Sergey.

Note that each of the multinomial coefficients could have been written in different ways (analogously to what was shown in part (a)).

c. \(\binom{10}{3, 4, 3} + \binom{10}{5, 2, 3} + \binom{10}{5, 4, 1} + \binom{10}{4, 3, 3} + \binom{10}{4, 4, 2} + \binom{10}{5, 3, 2}\)

We select two candies to remain in the bag and the remaining 10 candies are then distributed to the 10 students. The six terms above correspond respectively to the cases where the two candies left in the bag are: (a) 2 Twix Bars, (b) 2 Almond Joys, (c) 2 Swedish Fish, (d) 1 Twix Bar and 1 Almond Joy, (e) 1 Twix Bar and 1 Swedish Fish, and (f) 1 Almond Joy and 1 Swedish Fish.

Note that each of the multinomial coefficients could have been written in different ways (analogously to what was shown in part (a)).

2. There are multiple ways to obtain this answer; here are two:

The first (common) method is to let \(X = \) number of slices of pizza eaten immediately after last slice of cheese pizza is eaten. Note that \(X \sim \text{NegBin}(12, 0.5)\) since there are 12 slices of cheese pizza and slices of the two pizzas are equally likely to get eaten.

Now, we want to consider all cases where \(12 \leq X \leq 21\), since at least 12 slices of pizza must be eaten in order for there to be a chance that the last cheese slice was eaten, and if no more than 21 (out of 24) slices are eaten when the last cheese slice is eaten, then at least 3 slices of pepperoni must remain. Thus, the probability we want is given by the expression:

\[
\sum_{i=12}^{21} P(X = i) = \sum_{i=12}^{21} \left( i - 1 \atop 11 \right) \left( \frac{1}{2} \right)^i \left( \frac{1}{2} \right)^{12 - i} = \sum_{i=12}^{21} \left( i - 1 \atop 11 \right) \left( \frac{1}{2} \right)^i
\]

A second method to compute the answer is to use a set of Binomial variables defined as: \(Y_i = \) number of cheese slices eaten at time when \(i\) total slices have been eaten. We have \(Y_i \sim \text{Bin}(i, 0.5)\), since we have \(i\) trials (slices of pizza eaten), where there is a 50% chance that
each slice eaten is cheese. Here, we want to compute: 
\[ \frac{1}{2} \sum_{i=11}^{20} P(Y_i = 11) = \frac{1}{2} \sum_{i=11}^{20} \binom{i}{11} \left( \frac{1}{2} \right)^{11} \left( \frac{1}{2} \right)^{i-11} = \frac{1}{2} \sum_{i=11}^{20} \binom{i}{11} \left( \frac{1}{2} \right)^i \]

And just to show the equivalence of this result, if we let \( j = i + 1 \), we can rewrite the expression immediately above in the same way we computed it in the first method:

\[ \sum_{i=11}^{20} \binom{i}{11} \left( \frac{1}{2} \right)^i = \sum_{j=12}^{21} \binom{j-1}{11} \left( \frac{1}{2} \right)^j \]

3. a. Let \( X \) = the number of times the randomly chosen song is played. 
Here the probability \( p \) of selecting the particular song = 1/500 and the number of independent trials (song selections) \( n = 200 \). So, we have \( X \sim \text{Bin}(200, 1/500) \). We want to compute:

\[ P(X > 4) = 1 - P(X \leq 4) = 1 - \sum_{i=0}^{4} P(X = i) = 1 - \sum_{i=0}^{4} \binom{200}{i} \left( \frac{1}{500} \right)^i \left( \frac{499}{500} \right)^{200-i} \]

b. Let \( p \) = probability that a randomly chosen song is played more than 4 times. As determined in part (a): \( p = 1 - \sum_{i=0}^{4} \binom{200}{i} \left( \frac{1}{500} \right)^i \left( \frac{499}{500} \right)^{200-i} \)

Now, let \( Y \) = the number of songs that have been heard more than 4 times. Here, this problem set-up fits the Poisson paradigm (it is really the same as computing if 3 buckets in a hash table each have more the 4 strings hashed to them). Thus, we have: \( Y \sim \text{Poi}(\lambda) \) where \( \lambda = 500p \), and \( p \) is defined as above.

\[ P(Y = 3) = e^{-\lambda} \frac{\lambda^3}{3!} \quad \text{where} \quad \lambda = 500 \left( 1 - \sum_{i=0}^{4} \binom{200}{i} \left( \frac{1}{500} \right)^i \left( \frac{499}{500} \right)^{200-i} \right) . \]

Note that a normal approximation is not as appropriate as a Poisson approximation here since \( p \) is a very small value.

4. a. Let \( X_i \) = the value rolled on die \( i \), where \( 1 \leq i \leq 4 \). \( P(X \geq k) = P(X_1 \geq k, X_2 \geq k, X_3 \geq k, X_4 \geq k) = \left( \frac{6-k+1}{6} \right)^4 \), since all four rolls must be greater than or equal to \( k \).

b. Using the definition of expectation:

\[ E[X] = \sum_{x=1}^{6} x \cdot P(X = x) = \sum_{x=1}^{6} x \cdot \left[ P(X \geq x) - P(X \geq x + 1) \right] \]

\[ = \sum_{x=1}^{6} x \cdot \left[ \left( \frac{6-x+1}{6} \right)^4 - \left( \frac{6-x}{6} \right)^4 \right] \]
Alternatively, one can use a property covered in Lecture 11, which is that if \( X \) is non-negative, then:

\[
E[X] = \sum_{x=1}^{6} P(X \geq x) = \sum_{x=1}^{6} \left( \frac{6 - x + 1}{6} \right)^4 = \left( \frac{6}{6} \right)^4 + \left( \frac{5}{6} \right)^4 + \left( \frac{4}{6} \right)^4 + \left( \frac{3}{6} \right)^4 + \left( \frac{2}{6} \right)^4 + \left( \frac{1}{6} \right)^4
\]

The two expressions to compute \( E[X] \) above are, indeed, equivalent.


Let \( X_i \) be the value rolled on die \( i \), where \( 1 \leq i \leq 4 \). As computed in class, we know that \( E[X_i] = 3.5 \) for all \( 1 \leq i \leq 4 \).

\( E[T] = E[X_1 + X_2 + X_3 + X_4] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = 4(3.5) = 14 \)

So, \( E[S] = 14 - E[X] \), where \( E[X] \) is as computed in part (b).

5. a. Define the event \( L_i \) as liking movie \( T_i \), and \( G \) as liking the “Tearjerker” genre; therefore \( P(L_i|G) = p_i \) and \( P(L_i|G^c) = q_i \).

We want to compute \( P(L_1 L_2 L_3|G) \). By the definition of conditional independence,

\[
P(L_1 L_2 L_3|G) = P(L_1|G)P(L_2|G)P(L_3|G) = p_1 \cdot p_2 \cdot p_3.
\]

b. Using the definitions of events from before, we want to compute \( P(L_1 \cup L_2 \cup L_3|G) \).

\[
P(L_1 \cup L_2 \cup L_3|G) = 1 - P((L_1 \cup L_2 \cup L_3)^C|G) \quad \text{(Complement)}
\]

\[
= 1 - P(L_1^C \cap L_2^C \cap L_3^C|G) \quad \text{(DeMorgan's)}
\]

\[
= 1 - (1 - p_1)(1 - p_2)(1 - p_3)
\]

c. Using Bayes’ Theorem:

\[
P(G|L_1 L_2 L_3) = \frac{P(L_1 L_2 L_3|G)P(G)}{P(L_1 L_2 L_3|G)P(G) + P(L_1 L_2 L_3|G^c)P(G^c)}
\]

\[
= \frac{p_1 p_2 p_3 (0.6)}{(p_1 p_2 p_3)(0.6) + (q_1 q_2 q_3)(0.4)}
\]

6. Let \( X = \) lifetime of screen in our laptop.

Let event \( A = \) manufacturer A produced the screen.

Let event \( B = \) manufacturer B produced the screen.

a. We want to compute \( P(A \mid X > 18) \). Using Bayes Theorem, we have:

\[
P(A \mid X > 18) = \frac{P(X > 18 \mid A)P(A)}{P(X > 18)} = \frac{(1 - P(X \leq 18 \mid A)) \cdot 0.5}{P(X > 18)}
\]
Noting that $(X \mid A) \sim \text{N}(20, 4)$, we have:

\[ P(A \mid X > 18) = \frac{(0.5) \left(1 - P \left(\frac{X - 20}{2} \leq \frac{18 - 20}{2}\right)\right)}{P(X > 18)} \]

\[ = \frac{(0.5)\Phi(1)}{P(X > 18)} = \frac{(0.5)(0.8413)}{P(X > 18)} \]

Now, we need to compute $P(X > 18)$:

\[ P(X > 18) = P(X > 18 \mid A)P(A) + P(X > 18 \mid B)P(B) \]

\[ = P(X > 18 \mid A)(0.5) + P(X > 18 \mid B)(0.5) \]

\[ = 0.5 \cdot \left(1 - P \left(\frac{X - 20}{2} \leq \frac{18 - 20}{2}\right)\right) + 0.5 \left[1 - \left(1 - e^{-\frac{18}{20}}\right)\right] \]

\[ = 0.5 \cdot (1 - P(Z \leq -1)) + 0.5e^{-\frac{9}{20}} \]

\[ = 0.5 \cdot (1 - (1 - P(Z \leq 1))) + 0.5e^{-\frac{9}{20}} \]

\[ = 0.5\Phi(1) + 0.5e^{-\frac{9}{20}} \]

\[ = 0.5 \cdot 0.8413 + 0.5e^{-\frac{9}{20}} \]

Substituting $P(X > 18)$ into the expression for $P(A \mid X > 18)$, yields the answer:

\[ P(A \mid X > 18) = \frac{(0.5)(0.8413)}{0.8413 + e^{-\frac{9}{20}}} \]

b. Here, we want to compute $P(B \mid X > 18)$. Using Bayes Theorem, we have:

\[ P(B \mid X > 18) = \frac{P(X > 18 \mid B)P(B)}{P(X > 18)} = \frac{(1 - P(X \leq 18 \mid B)) \cdot 0.5}{P(X > 18)} \]

Noting that $(X \mid B) \sim \text{Exp}(1/20)$, we have:

\[ P(B \mid X > 18) = \frac{0.5 \left(1 - \left(1 - e^{-\frac{18}{20}}\right)\right)}{P(X > 18)} = \frac{0.5e^{-\frac{9}{20}}}{P(X > 18)} \]

Substituting the previously computed value for $P(X > 18)$ into the expression for $P(B \mid X > 18)$, yields the final answer:

\[ P(B \mid X > 18) = \frac{0.5e^{-\frac{9}{20}}}{0.8413 + e^{-\frac{9}{20}}} \]