1. **Random Number of Random Variables**: *law of total expectation*

Let $N$ be a non-negative integer-valued random variable; that is, takes values in $\{0, 1, 2, \ldots\}$. Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of iid random variables (independent of $N$), each with mean $\mu$, and $X = \sum_{i=1}^{N} X_i$ be the sum of the first $N$ of them. Before doing any work, what do you think $E[X]$ will turn out to be? Show it mathematically.

\[
E[X] = E \left[ \sum_{i=1}^{N} X_i \right] = \sum_{n} E \left[ \sum_{i=1}^{N} X_i \mid N = n \right] p_N(n) = \sum_{n} E \left[ \sum_{i=1}^{n} X_i \mid N = n \right] p_N(n) \\
= \sum_{n} E \left[ \sum_{i=1}^{n} X_i \right] p_N(n) = \sum_{n} n \mu p_N(n) = \mu \sum_{n} n p_N(n) = \mu E[N]
\]

Alternatively,

\[
E[X] = E[E[X|N]] = E[N\mu] = \mu E[N]
\]

2. **Beta Sum**: *beta distribution and sum of RVs*

What is the distribution of the sum of 100 IID Betas? Let $X$ be the sum

\[
X = \sum_{i=1}^{100} X_i \quad \text{where each } X_i \sim \text{Beta}(a = 3, b = 4)
\]

Note the variance of a Beta:

\[
\text{Var}(X_i) = \frac{ab}{(a+b)^2(a+b+1)} \quad \text{where } X_i \sim \text{Beta}(a, b)
\]

By the Central Limit Theorem, the sum of equally weighted IID random variables will be Normally distributed. We calculate the expectation and variance of $X_i$ using the beta
formulas:

\[
E(X_i) = \frac{a}{a + b} \quad \text{Expectation of a Beta}
\]

\[
= \frac{3}{7} \approx 0.43
\]

\[
\text{Var}(X_i) = \frac{ab}{(a + b)^2(a + b + 1)} \quad \text{Variance of a Beta}
\]

\[
= \frac{3 \cdot 4}{(3 + 4)^2(3 + 4 + 1)}
\]

\[
= \frac{12}{49 \cdot 8} \approx 0.03
\]

\[
X \sim N(\mu = n \cdot E[X_i], \sigma^2 = n \cdot \text{Var}(X_i))
\]

\[
\sim N(\mu = 43, \sigma^2 = 3)
\]

3. Medicine Doses:

Megha has a health condition that requires unpredictable amounts of medication. Every day, there is a 20% chance that she feels perfectly fine and requires no medicine. Otherwise, she needs to take a dose of medication. The necessary dose is equally likely to be any value in the continuous range 1 to 5 ounces. How much medicine she needs on any given day is independent of all other days.

Megha’s insurance will fully cover 90 ounces of medicine for each 30-day period. What is the probability that 90 ounces will be enough for the next 30 days? Make your life easier by using Central Limit Theorem.

Let \( M \) be the amount of medicine Megha will need in the next thirty days. Let \( M_i \) be the amount of medicine Megha needs on the \( i \)th day. \( M \) is a sum of \( M_1 \) through \( M_{30} \) and can be modeled with the CLT.

To use the CLT, we need to first know the mean and variance of \( M_i \). To do this, let \( D_i \) be the event that she needs to take a dose on the \( i \)th day. Note that \( M_i | D_i \sim Uni(1, 5) \) and \( M_i | D_i^C = 0 \). Using the law of total expectation, we have:

\[
E[M_i] = E[M_i | D_i]P(D_i) + E[M_i | D_i^C]P(D_i^C) = 3 \cdot 0.8 + 0 \cdot 0.2 = 2.4
\]

To find the variance of \( M_i \), we need to know \( E[M_i^2] \). We can use a similar approach as the previous problem along with the law of the unconscious statistician.
\[
E[M_i^2] = E[M_i^2|D_i]P(D_i) + E[M_i^2|D_i^C]P(D_i^C) \\
= \frac{4}{5} \int_{m=1}^{5} m^2 f_M(m) dm + 0 \cdot .2 \\
= \frac{4}{5} \int_{m=1}^{5} m^2 \frac{1}{4} dt \approx 8.267
\]

We then have \( \text{Var}(M_i) = E[M_i^2] - E[M_i]^2 = 8.267 - 2.4^2 = 2.507 \). According to the CLT:
\[
\sum_{i=1}^{30} M_i \approx N(30 \times 2.4, 30 \times 2.507) \implies M \sim N(72, 75.21) P(M < 90) \approx \Phi \left( \frac{90 - 72}{\sqrt{75.21}} \right) \approx 0.98
\]