CS156: The Calculus of Computation

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Chapter 9: Quantifier-free Equality and Data Structures

The Theory of Equality T_E

$$\Sigma_E : \{=,\ a,\ b,\ c,\ \ldots,\ f,\ g,\ h,\ \ldots,\ p,\ q,\ r,\ \ldots\}$$

uninterpreted symbols:

- constants $a, b, c, \ldots$
- functions $f, g, h, \ldots$
- predicates $p, q, r, \ldots$

Example:

$x = y \ \wedge\ f(x) \neq f(y)$ T_E-unsatisfiable

$f(x) = f(y) \ \wedge\ x \neq y$ T_E-satisfiable

$f(f(f(a))) = a \ \wedge\ f(f(f(f(f(a))))) = a \ \wedge\ f(a) \neq a$

T_E-unsatisfiable

$x = g(y, z) \rightarrow f(x) = f(g(y, z))$ T_E-valid

Axioms of T_E

1. $\forall x.\ x = x$ (reflexivity)
2. $\forall x, y.\ x = y \ \rightarrow\ y = x$ (symmetry)
3. $\forall x, y, z.\ x = y \ \wedge\ y = z \ \rightarrow\ x = z$ (transitivity)

define $=$ to be an equivalence relation.

Axiom schema

4. for each positive integer n and n-ary function symbol f,

$$\forall \bar{x}, \bar{y}.\ \left(\bigwedge_{i=1}^{n} x_i = y_i \right) \rightarrow f(\bar{x}) = f(\bar{y})$$

(function)

For example, for unary f, the axiom is

$$\forall x', y'.\ x' = y' \ \rightarrow\ f(x') = f(y')$$

Therefore,

$$x = g(y, z) \ \rightarrow\ f(x) = f(g(y, z))$$

is T_E-valid. $(x' \rightarrow x, y' \rightarrow g(y, z))$.

Axiom schema

5. for each positive integer n and n-ary predicate symbol p,

$$\forall \bar{x}, \bar{y}.\ \left(\bigwedge_{i=1}^{n} x_i = y_i \right) \rightarrow (p(\bar{x}) \leftrightarrow p(\bar{y}))$$

(predicate)

Thus, for unary p, the axiom is

$$\forall x', y'. x' = y' \ \rightarrow \ (p(x') \ \leftrightarrow \ p(y'))$$

Therefore,

$$a = b \rightarrow (p(a) \leftrightarrow p(b))$$

is T_E-valid. $(x' \rightarrow a, y' \rightarrow b)$.

We discuss T_E-formulae without predicates

For example, for Σ_E-formula

$$F : \; p(x) \;\wedge\; q(x,y) \;\wedge\; q(y,z) \;\rightarrow\; \neg q(x,z)$$

introduce fresh constant $\bullet$ and fresh functions f_p and f_q, and transform F to

$$G : \; f_p(x) = \bullet \;\wedge\; f_q(x,y) = \bullet \;\wedge\; f_q(y,z) = \bullet \;\rightarrow\; f_q(x,z) \neq \bullet \; .$$

Equivalence and Congruence Relations: Basics

Binary relation R over set S

- is an equivalence relation if
  - reflexive: $\forall s \in S.\ s\ R\ s$;
  - symmetric: $\forall s_1, s_2 \in S.\ s_1\ R\ s_2\ \rightarrow\ s_2\ R\ s_1$;
  - transitive: $\forall s_1, s_2, s_3 \in S.\ s_1\ R\ s_2\ \wedge\ s_2\ R\ s_3\ \rightarrow\ s_1\ R\ s_3$.

Example:
Define the binary relation $\equiv_2$ over the set $\mathbb{Z}$ of integers

$$m \equiv_2 n \quad \text{iff} \quad (m \bmod 2) = (n \bmod 2)$$

That is, $m, n \in \mathbb{Z}$ are related iff they are both even or both odd.
$\equiv_2$ is an equivalence relation

- is a congruence relation if in addition

$$\forall \overline{s}, \overline{t}.\ \bigwedge_{i=1}^{n} s_i\ R\ t_i\ \rightarrow\ f(\overline{s})\ R\ f(\overline{t})\ .$$

Classes

For $\left\{\begin{array}{c}\text{equivalence}\\ \text{congruence}\end{array}\right\}$ relation R over set S,

the $\left\{\begin{array}{c}\underline{\text{equivalence}}\\ \underline{\text{congruence}}\end{array}\right\}$ class of $s \in S$ under R is

$$[s]_R \stackrel{\text{def}}{=} \{s' \in S \ : \ sRs'\} .$$

Example:

The equivalence class of 3 under $\equiv_2$ over $\mathbb{Z}$ is

$$[3]_{\equiv_2} = \{n \in \mathbb{Z} \ : \ n \text{ is odd}\} .$$

Partitions

A partition P of S is a set of subsets of S that is

- total $\left(\bigcup_{S' \in P} S'\right) = S$
- disjoint $\forall S_1, S_2 \in P.\ S_1 \neq S_2 \rightarrow S_1 \cap S_2 = \emptyset$

Quotient

The quotient S/R of S by $\left\{\begin{array}{c}\text{equivalence}\\ \text{congruence}\end{array}\right\}$relation R is the partition of S into $\left\{\begin{array}{c}\text{equivalence}\\ \text{congruence}\end{array}\right\}$classes

$$S/R \;=\; \{[s]_R \;:\; s \in S\}\ .$$

It satisfies total and disjoint conditions.

Example: The quotient $\mathbb{Z}/\equiv_2$ is a partition of $\mathbb{Z}$. The set of equivalence classes

$$\{\{n \in \mathbb{Z} \;:\; n \text{ is odd}\},\ \{n \in \mathbb{Z} \;:\; n \text{ is even}\}\}$$

Note duality between relations and classes

Refinements

Two binary relations R_1 and R_2 over set S.
R_1 is a refinement of R_2, $R_1 \prec R_2$, if

$$\forall s_1, s_2 \in S.\ s_1 R_1 s_2 \ \rightarrow\ s_1 R_2 s_2 \ .$$

R_1 refines R_2.

Examples:

- For $S = \{a, b\}$,
  $R_1 : \{aR_1 b\} \qquad R_2 : \{aR_2 b,\ bR_2 b\}$
  Then $R_1 \prec R_2$
- For set $\mathbb{Z}$
  $R_1 : \{xR_1 y \ : \ x \bmod 2 = y \bmod 2\}$
  $R_2 : \{xR_2 y \ : \ x \bmod 4 = y \bmod 4\}$
  Then $R_2 \prec R_1$.

Closures

Given binary relation R over S.
The equivalence closure R^E of R is the equivalence relation s.t.

- R refines R^E, i.e. $R \prec R^E$;
- for all other equivalence relations R' s.t. $R \prec R'$, either $R' = R^E$ or $R^E \prec R'$

That is, R^E is the "smallest" equivalence relation that "covers" R.

Example: If $S = \{a, b, c, d\}$ and $R = \{aRb, bRc, dRd\}$, then

- $aR^E b, bR^E c, dR^E d$ since $R \subseteq R^E$;
- $aR^E a, bR^E b, cR^E c$ by reflexivity;
- $bR^E a, cR^E b$ by symmetry;
- $aR^E c$ by transitivity;
- $cR^E a$ by symmetry.

Similarly, the congruence closure R^C of R is the "smallest" congruence relation that "covers" R.

T_E-satisfiability and Congruence Classes I

Definition: For Σ_E-formula

$$F : \ s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m \ \wedge \ s_{m+1} \neq t_{m+1} \ \wedge \ \cdots \ \wedge \ s_n \neq t_n$$

the subterm set S_F of F is the set that contains precisely the subterms of F.

Example: The subterm set of

$$F : \ f(a, b) = a \ \wedge \ f(f(a, b), b) \neq a$$

is

$$S_F = \{a, \ b, \ f(a, b), \ f(f(a, b), b)\} \ .$$

Note: we consider only quantifier-free conjunctive Σ_E-formulae. Convert non-conjunctive formula F to DNF $\bigvee_i F_i$, where each disjunct F_i is a conjunction of $=$, $\neq$. Check each disjunct F_i. F is T_E-satisfiable iff at least one disjunct F_i is T_E-satisfiable.

T_E-satisfiability and Congruence Classes II

Given Σ_E-formula F

$$F: \; s_1 = t_1 \;\wedge\; \cdots \;\wedge\; s_m = t_m \;\wedge\; s_{m+1} \neq t_{m+1} \;\wedge\; \cdots \;\wedge\; s_n \neq t_n$$

with subterm set S_F, F is T_E-satisfiable iff there exists a congruence relation $\sim$ over S_F such that

- for each $i \in \{1, \ldots, m\}$, $s_i \sim t_i$;
- for each $i \in \{m+1, \ldots, n\}$, $s_i \not\sim t_i$.

Such congruence relation $\sim$ defines T_E-interpretation $I : (D_I, \alpha_I)$ of F. D_I consists of $|S_F/\sim|$ elements, one for each congruence class of S_F under $\sim$.

Instead of writing $I \models F$ for this T_E-interpretation, we abbreviate

$$\sim \models F$$

The goal of the algorithm is to construct the congruence relation over S_F, or to prove that no congruence relation exists.

Congruence Closure Algorithm

$$F: \underbrace{s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m}_{\text{generate congruence closure}} \ \wedge \ \underbrace{s_{m+1} \neq t_{m+1} \ \wedge \ \cdots \ \wedge \ s_n \neq t_n}_{\text{search for contradiction}}$$

Decide if F is T_E-satisfiable.

The algorithm performs the following steps:

1. Construct the congruence closure $\sim$ of

   $$\{s_1 = t_1, \ldots, s_m = t_m\}$$

   over the subterm set S_F. Then

   $$\sim \models \ s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m \ .$$

2. If for any $i \in \{m+1, \ldots, n\}$, $s_i \sim t_i$, return unsatisfiable.
3. Otherwise, $\sim \models F$, so return satisfiable.

How do we actually construct the congruence closure in Step 1?

Congruence Closure Algorithm (Details)

Initially, begin with the finest congruence relation $\sim_0$ given by the partition

$$\{\{s\} \ : \ s \in S_F\} \ .$$

That is, let each term over S_F be its own congruence class.

Then, for each $i \in \{1, \ldots, m\}$, impose $s_i = t_i$ by merging the congruence classes

$$[s_i]_{\sim_{i-1}} \quad \text{and} \quad [t_i]_{\sim_{i-1}}$$

to form a new congruence relation $\sim_i$.
To accomplish this merging,

- form the union of $[s_i]_{\sim_{i-1}}$ and $[t_i]_{\sim_{i-1}}$
- propagate any new congruences that arise within this union.

The new relation $\sim_i$ is a congruence relation in which $s_i \sim t_i$.

Congruence Closure Algorithm: Example 1 I

Given Σ_E-formula

$$F : \; f(a, b) = a \;\wedge\; f(f(a, b), b) \neq a$$

Construct initial partition by letting each member of the subterm set S_F be its own class:

1. $\{\{a\},\ \{b\},\ \{f(a, b)\},\ \{f(f(a, b), b)\}\}$

According to the first literal $f(a, b) = a$, merge

$$\{f(a, b)\} \quad \text{and} \quad \{a\}$$

to form partition

2. $\{\{a, f(a, b)\},\ \{b\},\ \{f(f(a, b), b)\}\}$

According to the (function) congruence axiom,

$$f(a, b) \sim a,\ \ b \sim b \quad \text{implies} \quad f(f(a, b), b) \sim f(a, b)\ ,$$

resulting in the new partition

3. $\{\{a, f(a, b), f(f(a, b), b)\},\ \{b\}\}$

Congruence Closure Algorithm: Example 1 II

This partition represents the congruence closure of S_F.

Is it the case that

$$\{\{a, f(a, b), f(f(a, b), b)\},\ \{b\}\} \ \models\ F\ ?$$

No, as $f(f(a, b), b) \sim a$ but F asserts that $f(f(a, b), b) \neq a$.
Hence, F is T_E-unsatisfiable.

Congruence Closure Algorithm: Example 2 I

Example: Given Σ_E-formula

$$F : \; f(f(f(a))) = a \;\wedge\; f(f(f(f(f(a))))) = a \;\wedge\; f(a) \neq a$$

From the subterm set S_F, the initial partition is

1. $\{\{a\}, \{f(a)\}, \{f^2(a)\}, \{f^3(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$

where, for example, $f^3(a)$ abbreviates $f(f(f(a)))$.

According to the literal $f^3(a) = a$, merge

$$\{f^3(a)\} \quad \text{and} \quad \{a\} \; .$$

From the union,

2. $\{\{a, f^3(a)\}, \{f(a)\}, \{f^2(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$

deduce the following congruence propagations:

$$f^3(a) \sim a \;\Rightarrow\; f(f^3(a)) \sim f(a) \quad \text{i.e.} \quad f^4(a) \sim f(a)$$

and

$$f^4(a) \sim f(a) \;\Rightarrow\; f(f^4(a)) \sim f(f(a)) \;\; \text{i.e.} \;\; f^5(a) \sim f^2(a)$$

Thus, the final partition for this iteration is the following:

3. $\{\{a, f^3(a)\}, \{f(a), f^4(a)\}, \{f^2(a), f^5(a)\}\}$.

Congruence Closure Algorithm: Example 2 II

3. $\{\{a, f^3(a)\}, \ \{f(a), f^4(a)\}, \ \{f^2(a), f^5(a)\}\}$.

From the second literal, $f^5(a) = a$, merge

$$\{f^2(a), f^5(a)\} \quad \text{and} \quad \{a, f^3(a)\}$$

to form the partition

4. $\{\{a, f^2(a), f^3(a), f^5(a)\}, \ \{f(a), f^4(a)\}\}$.

Propagating the congruence

$$f^3(a) \sim f^2(a) \ \Rightarrow \ f(f^3(a)) \sim f(f^2(a)) \text{ i.e. } f^4(a) \sim f^3(a)$$

yields the partition

5. $\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\}$,

which represents the congruence closure in which all of S_F are equal. Now,

$$\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\} \ \models \ F \ ?$$

No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E-unsatisfiable.

Congruence Closure Algorithm: Example 3

Given Σ_E-formula

$$F: \; f(x) = f(y) \; \wedge \; x \neq y \; .$$

The subterm set S_F induces the following initial partition:

1. $\{\{x\}, \; \{y\}, \; \{f(x)\}, \; \{f(y)\}\}$.

Then $f(x) = f(y)$ indicates to merge

$\{f(x)\}$ and $\{f(y)\}$.

The union $\{f(x), f(y)\}$ does not yield any new congruences, so the final partition is

2. $\{\{x\}, \; \{y\}, \; \{f(x), f(y)\}\}$.

Does

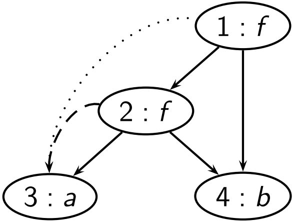
$$\{\{x\}, \; \{y\}, \; \{f(x), f(y)\}\} \; \models \; F \; ?$$

Yes, as $x \not\sim y$, agreeing with $x \neq y$. Hence, F is T_E-satisfiable.

Implementation of Algorithm

Directed Acyclic Graph (DAG)

For Σ_E-formula F, graph-based data structure for representing the subterms of S_F (and congruence relation between them).



$f(f(a, b), b)$

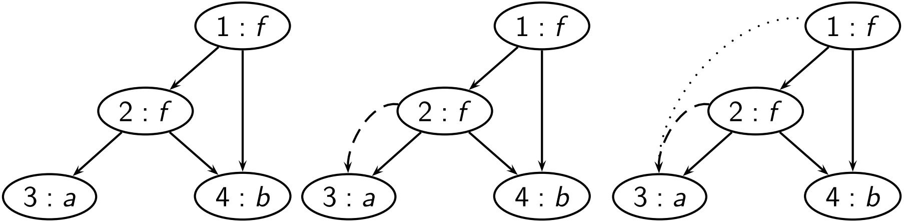
$f(a, b)$

$a \qquad b$

Efficient way for computing the congruence closure.

Summary of idea

$$f(a,b) = a \;\wedge\; f(f(a,b),b) \neq a$$



Initial DAG

$f(a,b) = a \;\Rightarrow$
$\textsc{merge}\; f(a,b)\; a$

$f(a,b) \sim a,\;\; b \sim b \;\Rightarrow$
$\quad f(f(a,b),b) \sim f(a,b)$
$\textsc{merge}\; f(f(a,b),b)$
$\quad f(a,b)$

_ _ _ explicit equation ...... by congruence

$$\left.\begin{array}{r} \textsc{find}\; f(f(a,b),b) = a = \textsc{find}\; a \\ f(f(a,b),b) \neq a \end{array}\right\} \Rightarrow \textbf{Unsatisfiable}$$

DAG representation

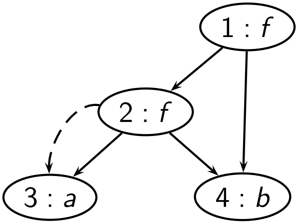
type **node** = {

`id`	:	**id**
		node's unique identification number
`fn`	:	**string**
		constant or function name
`args`	:	**id list**
		list of function arguments
`mutable find`	:	**id**
		the representative of the congruence class
`mutable ccpar`	:	**id set**
		if the node is the representative for its congruence class, then its `ccpar` (congruence closure parents) are all parents of nodes in its congruence class

}

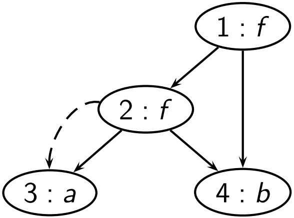
DAG Representation of node 2

$$
\begin{array}{llll}
\texttt{type}\ \textbf{node} = \{ & & & \\
\quad \texttt{id} & : & \textbf{id} & \ldots 2 \\
\quad \texttt{fn} & : & \textbf{string} & \ldots f \\
\quad \texttt{args} & : & \textbf{id list} & \ldots [3,4] \\
\quad \texttt{mutable find} & : & \textbf{id} & \ldots 3 \\
\quad \texttt{mutable ccpar} & : & \textbf{id set} & \ldots \emptyset \\
\} & & &
\end{array}
$$



DAG Representation of node 3

```
type node = {
    id            : id        ... 3
    fn            : string    ... a
    args          : id list   ... []
    mutable find  : id        ... 3
    mutable ccpar : id set    ... {1,2}
  }
```

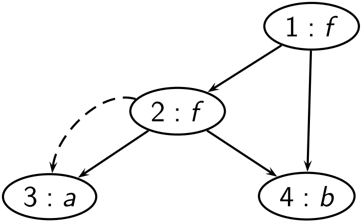


The Implementation I

<u>FIND function</u>

returns the representative of node's congruence class

```
let rec FIND i =
  let n = NODE i in
  if n.find = i then i else FIND n.find
```



<u>Example</u>: FIND $2 = 3$
FIND $3 = 3$

3 is the representative of $\{2, 3\}$.

The Implementation II

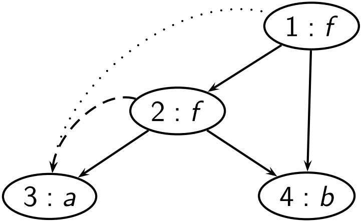
UNION function

$$
\begin{array}{l}
\texttt{let}\ \textsc{union}\ i_1\ i_2 = \\
\quad \texttt{let}\ n_1 = \textsc{node}\,(\textsc{find}\ i_1)\ \texttt{in} \\
\quad \texttt{let}\ n_2 = \textsc{node}\,(\textsc{find}\ i_2)\ \texttt{in} \\
\quad n_1.\texttt{find} \leftarrow n_2.\texttt{find}; \\
\quad n_2.\texttt{ccpar} \leftarrow n_1.\texttt{ccpar} \cup n_2.\texttt{ccpar}; \\
\quad n_1.\texttt{ccpar} \leftarrow \emptyset
\end{array}
$$

n_2 is the representative of the union class

The Implementation III

Example



UNION 1 2 $\quad n_1 = 1 \quad n_2 = 3$

$1.\texttt{find} \leftarrow 3$

$3.\texttt{ccpar} \leftarrow \{1, 2\}$

$1.\texttt{ccpar} \leftarrow \emptyset$

The Implementation IV

CCPAR function

Returns parents of all nodes in i's congruence class

$$\texttt{let}\ \textsc{ccpar}\ i = $$
$$(\textsc{node}\ (\textsc{find}\ i)).\texttt{ccpar}$$

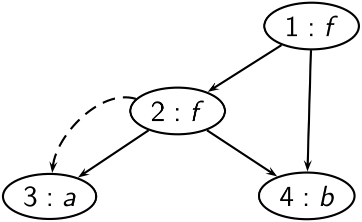
CONGRUENT predicate

Test whether i_1 and i_2 are congruent

$$
\begin{aligned}
&\texttt{let}\ \textsc{congruent}\ i_1\ i_2 = \\
&\quad \texttt{let}\ n_1 = \textsc{node}\ i_1\ \texttt{in} \\
&\quad \texttt{let}\ n_2 = \textsc{node}\ i_2\ \texttt{in} \\
&\quad n_1.\texttt{fn} = n_2.\texttt{fn} \\
&\qquad \land\ |n_1.\texttt{args}| = |n_2.\texttt{args}| \\
&\qquad \land\ \forall i \in \{1, \ldots, |n_1.\texttt{args}|\}.\ \textsc{find}\ n_1.\texttt{args}[i] = \textsc{find}\ n_2.\texttt{args}[i]
\end{aligned}
$$

The Implementation V

Example:



Are 1 and 2 congruent?

`fn` fields	— both f
# of arguments	— same
left arguments $f(a, b)$ and a	— both congruent to 3
right arguments b and b	— both 4 (congruent)

Therefore 1 and 2 are congruent.

The Implementation VI

MERGE function

```
let rec MERGE i1 i2 =
  if FIND i1 ≠ FIND i2 then begin
    let P_i1 = CCPAR i1 in
    let P_i2 = CCPAR i2 in
    UNION i1 i2;
    foreach t1 ∈ P_i1, t2 ∈ P_i2 do
      if FIND t1 ≠ FIND t2 ∧ CONGRUENT t1 t2
      then MERGE t1 t2
    done
  end
```

P_{i_1} and P_{i_2} store the values of CCPAR i_1 and CCPAR i_2 (before the union).

Decision Procedure: T_E-satisfiability

Given Σ_E-formula

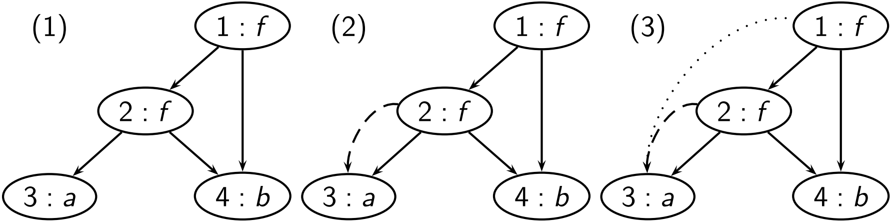
$$F: \; s_1 = t_1 \;\wedge\; \cdots \;\wedge\; s_m = t_m \;\wedge\; s_{m+1} \neq t_{m+1} \;\wedge\; \cdots \;\wedge\; s_n \neq t_n \;,$$

with subterm set S_F, perform the following steps:

1. Construct the initial DAG for the subterm set S_F.
2. For $i \in \{1, \ldots, m\}$, $\textsc{merge}\ s_i\ t_i$.
3. If $\textsc{find}\ s_i = \textsc{find}\ t_i$ for some $i \in \{m+1, \ldots, n\}$, return unsatisfiable.
4. Otherwise (if $\textsc{find}\ s_i \neq \textsc{find}\ t_i$ for all $i \in \{m+1, \ldots, n\}$) return satisfiable.

Example 1: T_E-Satisfiability

$$f(a,b) = a \ \wedge \ f(f(a,b),b) \neq a$$



Initial DAG

MERGE 2 3

$P_2 = \{1\}$

$P_3 = \{2\}$

UNION 2 3

CONGRUENT 1 2

MERGE 1 2

$P_1 = \{\}$

$P_2 = \{1, 2\}$

UNION 1 2

FIND $f(f(a,b),b) = a =$ FIND $a \Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F : \; f(a,b) = a \;\wedge\; f(f(a,b),b) \neq a \; .$$

The subterm set is

$$S_F = \{a, \; b, \; f(a,b), \; f(f(a,b),b)\} \; ,$$

resulting in the initial partition

$$(1) \quad \{\{a\}, \; \{b\}, \; \{f(a,b)\}, \; \{f(f(a,b),b)\}\}$$

in which each term is its own congruence class. Fig (1).

Final partition (Fig (3))

$$(2) \quad \{\{a, f(a,b), f(f(a,b),b)\}, \; \{b\}\}$$

Note: dash edge _ _ _ _ merge dictated by equalities in F
dotted edge .... deduced merge

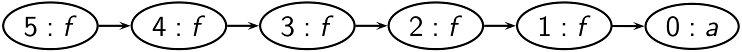
Does

$$\{\{a, f(a,b), f(f(a,b),b)\}, \; \{b\}\} \;\models\; F \; ?$$

No, as $f(f(a,b),b) \sim a$, but F asserts that $f(f(a,b),b) \neq a$.
Hence, F is T_E-unsatisfiable.

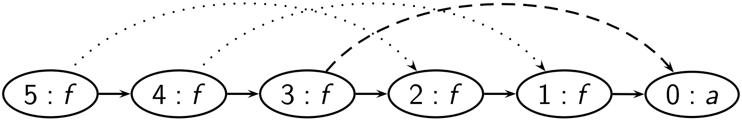
Example 2: T_E-Satisfiability

$$f(f(f(a))) = a \ \wedge \ f(f(f(f(f(a))))) = a \ \wedge \ f(a) \neq a$$



(1)

Initial DAG



(2)

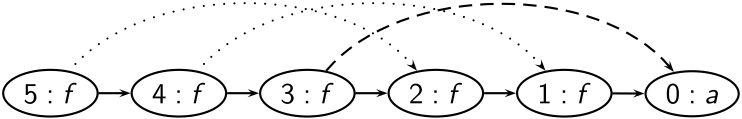
$f(f(f(a))) = a \quad \Rightarrow \quad$ MERGE 3 0 : $\quad P_3 = \{4\} \quad P_0 = \{1\}$ $\quad$ UNION 3 0

$\Rightarrow \quad$ MERGE 4 1 : $\quad P_4 = \{5\} \quad P_1 = \{2\}$ $\quad$ UNION 4 1

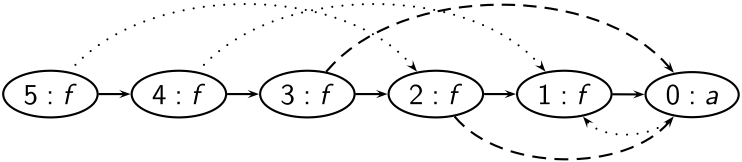
$\Rightarrow \quad$ MERGE 5 2 : $\quad P_5 = \{\} \quad P_2 = \{3\}$ $\quad$ UNION 5 2

Example 2: T_E-Satisfiability

$$f(f(f(a))) = a \ \wedge \ f(f(f(f(f(a))))) = a \ \wedge \ f(a) \neq a$$



(2)



(3)

$f(f(f(f(f(a))))) = a \quad \Rightarrow$ MERGE 5 0 : $\quad P_5 = \{3\} \quad P_0 = \{1, 4\}$

UNION 5 0

$\Rightarrow$ MERGE 3 1 : *STOP.Why?*

UNION 3 1

FIND $f(a) = f(a) =$ FIND $a \ \Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F : \; f(f(f(a))) = a \;\wedge\; f(f(f(f(f(a))))) = a \;\wedge\; f(a) \neq a \;,$$

which induces the initial partition

1. $\{\{a\},\ \{f(a)\},\ \{f^2(a)\},\ \{f^3(a)\},\ \{f^4(a)\},\ \{f^5(a)\}\}$.
   The equality $f^3(a) = a$ induces the partition
2. $\{\{a,\ f^3(a)\},\ \{f(a),\ f^4(a)\},\ \{f^2(a),\ f^5(a)\}\}$.
   The equality $f^5(a) = a$ induces the partition
3. $\{\{a,\ f(a),\ f^2(a),\ f^3(a),\ f^4(a),\ f^5(a)\}\}$.
   Now, does

   $$\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\} \;\models\; F \;?$$

   No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E-unsatisfiable.

Theorem (Sound and Complete)

Quantifier-free conjunctive Σ_E-formula F is T_E-satisfiable iff the congruence closure algorithm returns satisfiable.

Recursive Data Structures

Quantifier-free Theory of Lists T_{cons}

Σ_{cons} : $\{\text{cons}, \text{car}, \text{cdr}, \text{atom}, =\}$

- constructor cons : $\text{cons}(x, y)$ list constructed by appending y to x
- left projector car : $\text{car}(\text{cons}(x, y)) = x$
- right projector cdr : $\text{cdr}(\text{cons}(x, y)) = y$
- atom : unary predicate

Axioms of T_{cons}

- reflexivity, symmetry, transitivity
- function (congruence) axioms:

$$\begin{aligned}
&\forall x_1, x_2, y_1, y_2.\ x_1 = x_2 \ \wedge\ y_1 = y_2 \ \rightarrow\ \text{cons}(x_1, y_1) = \text{cons}(x_2, y_2)\\
&\forall x, y.\ x = y \ \rightarrow\ \text{car}(x) = \text{car}(y)\\
&\forall x, y.\ x = y \ \rightarrow\ \text{cdr}(x) = \text{cdr}(y)
\end{aligned}$$

- predicate (congruence) axiom:

$$\forall x, y.\ x = y \ \rightarrow\ (\text{atom}(x) \ \leftrightarrow\ \text{atom}(y))$$

-

$$\begin{aligned}
&(A1)\ \forall x, y.\ \text{car}(\text{cons}(x, y)) = x && \text{(left projection)}\\
&(A2)\ \forall x, y.\ \text{cdr}(\text{cons}(x, y)) = y && \text{(right projection)}\\
&(A3)\ \forall x.\ \neg\text{atom}(x) \rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) = x && \text{(construction)}\\
&(A4)\ \forall x, y.\ \neg\text{atom}(\text{cons}(x, y)) && \text{(atom)}
\end{aligned}$$

Simplifications

- Consider only quantifier-free conjunctive Σ_{cons}-formulae. Convert non-conjunctive formula to DNF and check each disjunct.
- $\neg\text{atom}(u_i)$ literals are removed:

$$\boxed{\text{replace} \quad \neg\text{atom}(u_i) \quad \text{with} \quad u_i = \text{cons}(u_i^1, u_i^2)}$$

  by the (construction) axiom.
- Result of a conjunctive Σ_{cons}-formula with literals

$$s = t \qquad s \neq t \qquad \text{atom}(u)$$

- Because of similarity to Σ_E, we sometimes combine $\Sigma_{\text{cons}} \cup \Sigma_E$.

Algorithm: T_{cons}-Satisfiability (the idea)

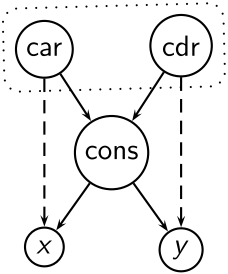
$$F: \quad \underbrace{s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m}_{\text{generate congruence closure}}$$

$$\wedge \quad \underbrace{s_{m+1} \neq t_{m+1} \ \wedge \ \cdots \ \wedge \ s_n \neq t_n}_{\text{search for contradiction}}$$

$$\wedge \quad \underbrace{\text{atom}(u_1) \ \wedge \ \cdots \ \wedge \ \text{atom}(u_\ell)}_{\text{search for contradiction}}$$

where s_i, t_i, and u_i are T_{cons}-terms

Algorithm: T_{cons}-Satisfiability



1. Construct the initial DAG for S_F
2. for each node n with $n.\texttt{fn} = \text{cons}$
   - add $\text{car}(n)$ and MERGE $\text{car}(n)$ $n.\texttt{args}[1]$
   - add $\text{cdr}(n)$ and MERGE $\text{cdr}(n)$ $n.\texttt{args}[2]$

   by axioms (A1), (A2)
3. for $1 \leq i \leq m$, MERGE s_i t_i
4. for $m + 1 \leq i \leq n$, if FIND s_i = FIND t_i, return **unsatisfiable**
5. for $1 \leq i \leq \ell$, if $\exists v.$ FIND v = FIND u_i $\wedge$ $v.\texttt{fn} = \text{cons}$, return **unsatisfiable**
6. Otherwise, return **satisfiable**

Example

Given $(\Sigma_{\text{cons}} \cup \Sigma_E)$-formula

$$F: \quad \begin{array}{c} \text{car}(x) = \text{car}(y) \;\wedge\; \text{cdr}(x) = \text{cdr}(y) \\ \wedge\; \neg\text{atom}(x) \;\wedge\; \neg\text{atom}(y) \;\wedge\; f(x) \neq f(y) \end{array}$$

where the function symbol f is in Σ_E

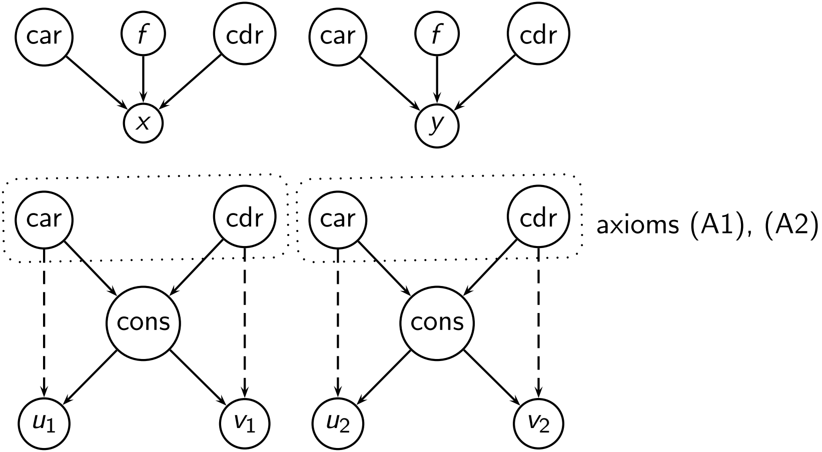
$$F': \quad \begin{array}{ll} \text{car}(x) = \text{car}(y) \quad \wedge & (1) \\ \text{cdr}(x) = \text{cdr}(y) \quad \wedge & (2) \\ x = \text{cons}(u_1, v_1) \quad \wedge & (3) \\ y = \text{cons}(u_2, v_2) \quad \wedge & (4) \\ f(x) \neq f(y) & (5) \end{array}$$

Recall the projection axioms:

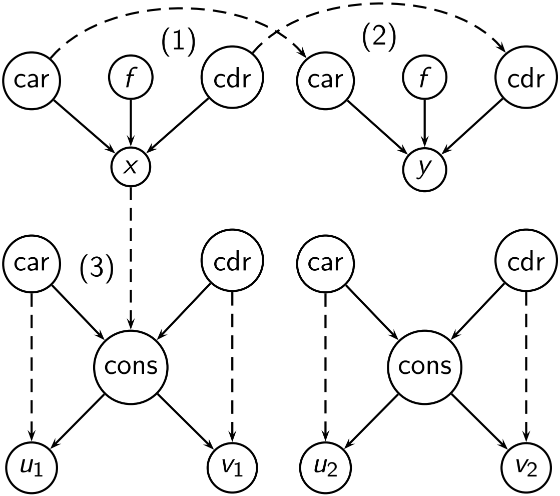
$(A1)$ $\forall x, y.\ \text{car}(\text{cons}(x, y)) = x$

$(A2)$ $\forall x, y.\ \text{cdr}(\text{cons}(x, y)) = y$

Example (cont): Initial DAG



Example (cont): MERGE



- - explicit equation

. . . by congruence

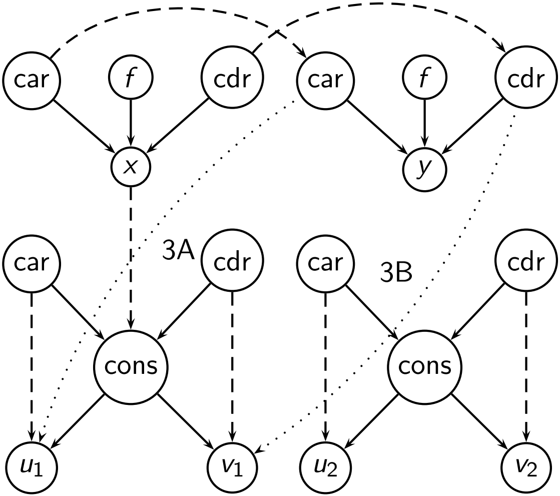
1 : MERGE car(x) car(y)

2 : MERGE cdr(x) cdr(y)

3 : MERGE x cons(u_1, v_1)

$\Downarrow$

Example (cont): Propagation



Congruent:

car(x) car(cons(u_1, v_1))

FIND car(x) = car(y)

FIND car(cons(...)) = u_1

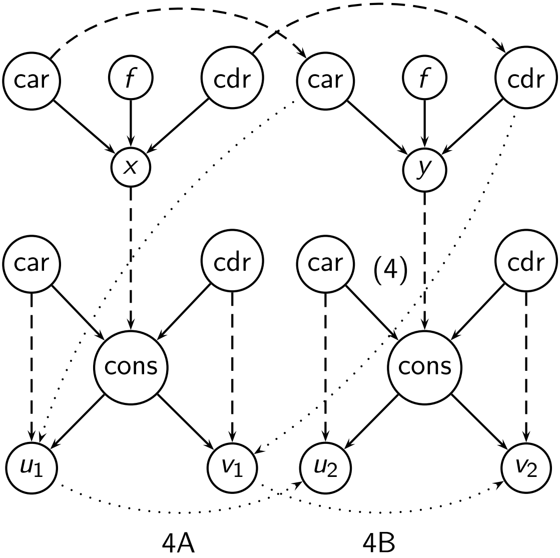
Congruent:

cdr(x) cdr(cons(u_1, v_1))

FIND cdr(x) = cdr(y)

FIND cdr(cons(...)) = v_1

Example (cont): MERGE



4 : MERGE y cons(u_2, v_2)

$\Downarrow$

Congruent:

car(y) car(cons(u_2, v_2))

FIND car$(y) = u_1$

FIND car(cons$(\ldots)$) $= u_2$

Congruent:

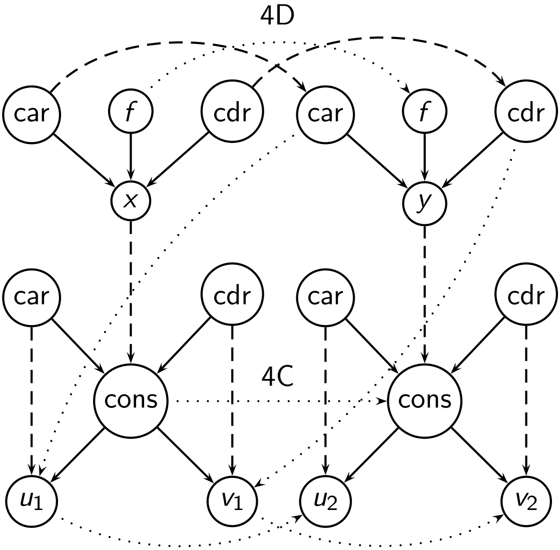
cdr(y) cdr(cons(u_2, v_2))

FIND cdr$(y) = v_1$

FIND cdr(cons$(\ldots)$) $= v_2$

$\Downarrow$

Example (cont): CONGRUENCE



Congruent:

cons(u_1, v_1) cons(u_2, v_2)

Congruent: $f(x)$ $f(y)$

5 : FIND $f(x) = f(y)$

FIND $f(y) = f(y)$

$\Downarrow$

F is **unsatisfiable**