Exercises
Exercises should be completed on your own.

1. (2 pt.) Consider the graph $G$ below.

```
A-----2-----B
 |           |
4          7
|           |
D-----3-----E
```

(a) (1 pt.) What MST does Prim’s algorithm return? In what order does Prim’s algorithm add edges to the MST when started from vertex C?

(b) (1 pt.) What MST does Kruskal’s algorithm return? In what order does Kruskal’s algorithm add edges to the MST?

[We are expecting: For both, just a list of edges. No justification is required.]

2. (4 pt.) At a Thanksgiving dinner, there are $n$ food items $f_0, \ldots, f_{n-1}$. Each food item has a tastiness $t_i > 0$ (measured in units of deliciousness per ounce) and a quantity $q_i > 0$ (measured in ounces). There are $q_i$ ounces of food $f_i$ available to you, and for any real number $x \in [0, q_i]$, the total deliciousness that you derive from eating $x$ ounces of food $f_i$ is $x \cdot t_i$. (Notice that $x$ here doesn’t have to be an integer).

Unfortunately, you only have capacity for $Q$ ounces of food in your belly, and you would like to maximize deliciousness subject to this constraint.

(a) (2 pt.) Design a greedy algorithm which takes as input the tuples $(f_i, t_i, q_i)$, and outputs tuples $(f_i, x_i)$ so that $0 \leq x_i \leq q_i$, $\sum x_i \leq Q$, and $\sum x_i t_i$ is as large as possible. Your algorithm should take time $O(n \log(n))$.

[We are expecting: Pseudocode and a short English explanation.]

(b) (2 pt.) Fill in the inductive step below to prove that your algorithm is correct.

- **Inductive hypothesis:** After making the $t$’th greedy choice, there is an optimal solution that extends the solution that the algorithm has constructed so far.
- **Base case:** Any optimal solution extends the empty solution, so the inductive hypothesis holds for $t = 0$.
- **Inductive step:** (you fill in)
- **Conclusion:** At the end of the algorithm, the algorithm returns an set $S^*$ of tuples $(f_i, x_i)$ so that $\sum x_i = Q$. Thus, there is no solution extending $S^*$ other than $S^*$ itself. Thus, the inductive hypothesis implies that $S^*$ is optimal.

[We are expecting: A proof of the inductive step: assuming the inductive hypothesis holds for $t - 1$, prove that it holds for $t$.]
Problems

You may talk with your fellow CS161-ers about the problems. However:

- Try the problems on your own before collaborating.
- Write up your answers yourself, in your own words. You should never share your typed-up solutions with your collaborators.
- If you collaborated, list the names of the students you collaborated with at the beginning of each problem.

1. (6 pt.) Consider the problem of making change. Suppose that coins come in denominations \( P = \{p_0, \ldots, p_m\} \) cents (for example, in the US, this would be \( P = \{1, 5, 10, 25\} \), corresponding to pennies, nickels, dimes, and quarters). Given \( n \) cents (where \( n \) is a non-negative integer), you would like to find the way to represent \( n \) using the fewest coins possible. For example, in the US system, 55 cents is minimally represented using three coins, two quarters and a nickel.

   (a) (3 pt.) Suppose that the denominations are \( P = \{1, 10, 25\} \) (aka, the US ran out of nickels). Your friend uses the following greedy strategy for making change:

   Algorithm 1: makeChange(n)

   ```python
   Input: n and P
   coins = []:
   while n > 0 do
       \( p^* \leftarrow \max\{p \in P : p \le n\}; \)
       \( n = n - p^*; \)
       coins.append(p^*)
   return coins
   ```

   Your friend acknowledges that this won’t work for general \( P \) (for example if \( P = \{2\} \) then we simply can’t make any odd \( n \)), but claims that for this particular \( P \) it does work. That is, your friend claims that this algorithm will always return a way to make \( n \) out of the denominations in \( P \) with the fewest coins possible.

   Is your friend correct for \( P = \{1, 10, 25\} \)?

   [We are expecting: Your answer, and either a proof or a counterexample. If you do a proof by induction, make sure to explicitly state your inductive hypothesis, base case, inductive step, and conclusion.]

   (b) (3 pt.) Your friend says that additionally their algorithm should work for any \( P \) of the form \( P = \{1, 2, 4, 8, \ldots, 2^s\} \).

   Is your friend correct for \( P = \{1, 2, 4, \ldots, 2^s\} \)?

   [We are expecting: Your answer, and either a proof or a counterexample. If you do a proof by induction, make sure to explicitly state your inductive hypothesis, base case, inductive step, and conclusion.]
2. (7 pt.) On Thanksgiving day, you arrive on an island with $n$ turkeys. You’ve already had thanksgiving dinner (and maybe you prefer tofu instead anyway), so you don’t want to eat the turkeys; but you do want to wish them all a Happy Thanksgiving. However, the turkeys each have very different sleep schedules. Turkey $i$ is awake only in a single closed interval $[a_i, b_i]$. Your plan is to stand in the center of the island and say loudly “Happy Thanksgiving!” at certain times $t_1, \ldots, t_m$. Any turkey who is awake at one of the times $t_j$ will hear the message. It’s okay if a turkey hears the message more than once, but you want to be sure that every turkey hears the message at least once.

(a) (3 pt.) Design a greedy algorithm which takes as input the list of intervals $[a_i, b_i]$ and outputs a list of times $t_1, \ldots, t_m$ so that $m$ is as small as possible and so that every turkey hears the message at least once. Your algorithm should run in time $O(n \log(n))$.

[We are expecting: Pseudocode and an English description of the main idea of your algorithm, as well as a short justification of the running time.]

(b) (4 pt.) Prove that your algorithm is correct.

[We are expecting: A proof by induction]

3. (6 pt.) Let $G$ be a connected weighted undirected graph. In class, we defined a minimum spanning tree of $G$ as a spanning tree $T$ of $G$ which minimizes the quantity

$$X = \sum_{e \in T} w_e,$$

where the sum is over all the edges in $T$, and $w_e$ is the weight of edge $e$. Define a “minimum-maximum spanning tree” to be a spanning tree that minimizes the quantity

$$Y = \max_{e \in T} w_e.$$

That is, a minimum-maximum spanning tree has the smallest maximum edge weight out of all possible spanning trees.

(a) (4 pt.) Prove that a minimum spanning tree in a connected weighted undirected graph $G$ is always a minimum-maximum spanning tree for $G$.

[We are expecting: A short but formal proof.]

(b) (2 pt.) Show that the converse to part (a) is not true. That is, a minimum-maximum spanning tree is not necessarily a minimum spanning tree.

[We are expecting: A counter-example, with an explanation of why it is a counter-example.]

(c) (1 bonus pt.) Give a deterministic $O(m)$ algorithm to find a minimum-maximum spanning tree in a connected weighted undirected graph $G$ with $n$ vertices and $m$ edges.

[We are expecting: Pseudocode, along with an English description of the idea of the algorithm, and an informal justification of correctness and running time.]