Please answer each of the following problems. Refer to the course webpage for the collaboration policy, as well as for helpful advice for how to write up your solutions.

1. **Probability refresher** (4 points)

   (a) (1 point) What is the cardinality of the set of all subsets of \( \{1,2,...,n\} \)? [We are expecting a mathematical expression along with one or two sentences explaining why it is correct.]

   (b) (1 point) Suppose we choose a subset of \( \{1,...,n\} \) uniformly at random: that is, every set has an equal probability of being chosen. Let \( X \) be a random variable denoting the cardinality (that is, the size) of a set randomly chosen in this way. Calculate the expected value of \( X \) and show your work. [We are expecting a mathematically rigorous argument establishing your answer. Your solution should not include summation signs.]

   (c) (2 points) Let \( \text{rand}(a,b) \) return an integer uniformly at random from the range \([a,b]\). Each call to \( \text{rand}(a,b) \) is independent. Consider the following function:

   \[
   f(k,n): \\
   \quad \text{if } k \leq 1: \\
   \quad \quad \text{return } \text{rand}(1,n) \\
   \quad \quad \text{return } 2 \cdot f(k/2, n)
   \]

   What is the expected value and variance of \( f(k,n) \)? Assume that \( k \) is a power of 2. Please show your work. [In addition to your answer, we are expecting a mathematical derivation along with a brief (1-2 sentence) analysis of what the pseudocode above does in order to explain why your derivation is the right thing to do.]

2. **Fun with Big-O notation.** (6 points; 1 point each) Mark the following as True or False. Briefly but convincingly justify all of your answers, using the definitions of \( O(\cdot) \), \( \Theta(\cdot) \) and \( \Omega(\cdot) \). [To see the level of detail we are expecting, the first question has been worked out for you.]

   (z) \( n = \Omega(n^2) \). This statement is False. To see this, we will use a proof by contradiction. Suppose that, as per the definition of \( \Omega(\cdot) \), there is some \( n_0 \) and some \( c > 0 \) so that for all \( n \geq n_0, n \geq c \cdot n^2 \). Choose \( n = \max\{1/c, n_0\} + 1 \). Then \( n \geq n_0 \), but we have \( n > 1/c \), which implies that \( c \cdot n^2 > n \). This is a contradiction.

   (a) \( n = O(n \log(n)) \).

   (b) \( n^{1/\log(n)} = \Theta(1) \).

   (c) If

   \[
   f(n) = \begin{cases} 
   5^n & \text{if } n < 2^{1000} \\
   2^{1000} n^2 & \text{if } n \geq 2^{1000}
   \end{cases}
   \]
and \( g(n) = \frac{n^2}{1000} \), then \( f(n) = O(g(n)) \).

(d) For all possible functions \( f(n), g(n) \geq 0 \), if \( f(n) = O(g(n)) \), then \( 2^{f(n)} = O(2^{g(n)}) \).

(e) \( 5^\log(n) = O(\log(n)^2) \)

(f) \( n = \Theta(100^\log(n)) \)

(g) The following two problems are not required but might be fun to think about. We will give you feedback if you do them but they will not affect your grade.

i. (bonus) \(|n| \sin(\pi n/2)| = O(|n| \cos(\pi n/2)|)\)

ii. (bonus) \(|n| \sin(\pi n/2)| = \Omega(|n| \cos(\pi n/2)|)\)

3. n-naught not needed. (3 points) Suppose that \( T(n) = O(n^d) \), and that \( T(n) \) is never equal to \( \infty \). Prove rigorously that there exists a \( c \) so that \( 0 \leq T(n) \leq c \cdot n^d \) for all \( n \geq 1 \). That is, the definition of \( O(\cdot) \) holds with \( n_0 = 1 \). [We are expecting a rigorous proof using the definition of \( O(\cdot) \)].

4. Fun with recurrences. (6 points; 1 point each)

Solve the following recurrence relations; i.e. express each one as \( T(n) = O(f(n)) \) for the tightest possible function \( f(n) \), and give a short justification. Be aware that some parts might be slightly more involved than others. Unless otherwise stated, assume \( T(1) = 1 \). [To see the level of detail expected, we have worked out the first one for you.]

(a) \( T(n) = 2T(n/2) + 3n \)

(b) \( T(n) = 3T(n/4) + \sqrt{n} \)

(c) \( T(n) = 7T(n/2) + \Theta(n^3) \)

(d) \( T(n) = 4T(n/2) + n^2 \log n \)

(e) \( T(n) = 2T(n/3) + n^c \), where \( c \geq 1 \) is a constant (that is, it doesn’t depend on \( n \)).

(f) \( T(n) = 2T(\sqrt{n}) + 1 \), where \( T(2) = 1 \)

5. Different-sized sub-problems. (6 points) (Note: this problem will be easier to do after class on Monday 4/16.) Solve the following recurrence relation.

\[ T(n) = T(n/2) + T(n/4) + T(n/8) + n, \]

where \( T(1) = 1 \). [We are expecting a formal proof. You may state your final running time with \( O(\cdot) \) notation, but do not use it in your proof.]

6. What’s wrong with this proof? (8 points) Consider the following recurrence relation:

\[ T(n) = T(n - 5) + 10 \cdot n \]

for \( n \geq 5 \), where \( T(0) = T(1) = T(2) = T(3) = T(4) = 1 \). Consider the following three arguments.
1. **Claim:** $T(n) = O(n)$. To see this, we will use strong induction. The inductive hypothesis is that $T(k) = O(k)$ for all $5 \leq k < n$. For the base case, we see $T(5) = T(0) + 10 \cdot 5 = 51 = O(1)$. For the inductive step, assume that the inductive hypothesis holds for all $k < n$. Then

$$T(n) = T(n - 5) + 10n,$$

and by induction $T(n - 5) = O(n - 5)$, so

$$T(n) = O(n - 5) + 10n = O(n).$$

This establishes the inductive hypothesis for $n$. Finally, we conclude that $T(n) = O(n)$ for all $n$.

2. **Claim:** $T(n) = O(n)$. To see this, we will use the Master Method. We have $T(n) = a \cdot T(n/b) + O(n^d)$, for $a = d = 1$ and

$$b = \frac{1}{1 - 5/n}.$$ 

Then we have that $a < b^d$ (since $1 < 1/(1 - 5/n)$ for all $n > 0$), and the master theorem says that this takes time $O(n^d) = O(n)$.

3. **Claim:** $T(n) = O(n^2)$. Imagine the recursion tree for this problem. (Notice that it’s not really a “tree,” since the degree is 1). At the top level we have a single problem of size $n$. At the second level we have a single problem of size $n - 5$. At the $t$’th level we have a single problem of size $n - 5t$, and this continues for at most $t = \lfloor n/5 \rfloor + 1$ levels. At the $t$’th level for $t \leq \lfloor n/5 \rfloor$, the amount of work done is $10(n - 5t)$. At the last level the amount of work is at most 1. Thus the total amount of work done is at most

$$1 + \sum_{t=0}^{\lfloor n/5 \rfloor} 10(n - 5t) = O(n^2).$$

(a) (3 points) Which, if any, of these arguments are correct? [We are expecting a single sentence stating which arguments are correct.]

(b) (5 points) For each argument that you said was incorrect, explain why it is incorrect. If you said that all three were incorrect, then give a correct argument. [We are expecting a few sentences of detailed reasoning for each incorrect algorithm; and if you give your own proof we are expecting something with the level of detail of the proofs above—except it should be correct!]