Lecture 15
Minimum Spanning Trees
Announcements

• HW6 due **Friday SUNDAY**
  • It’s a long problem set
  • Next week is Thanksgiving break, so there’s no rush to get started on HW7.
  • You can use late days until Tuesday at 3pm.
  • **HOWEVER:** The course staff also get Thanksgiving break, so take advantage of Piazza/office hours *before Friday.*

• HW7 still released Friday (11/17)
  • Due Friday 12/1 (**NOT** Friday 11/24)
Last time

• **Greedy algorithms**
  • Make a series of choices.
    • Choose this activity, then that one, ..
    • Never backtrack.
  • Show that, at each step, your choice does not rule out success.
    • At every step, there exists an optimal solution consistent with the choices we’ve made so far.
• At the end of the day:
  • you’ve built only one solution,
  • never having ruled out success,
  • so your solution must be correct.
Today

• Greedy algorithms for Minimum Spanning Tree.

• Agenda:
  1. What is a Minimum Spanning Tree?
  2. Short break to introduce some graph theory tools
  3. Prim’s algorithm
  4. Kruskal’s algorithm
Minimum Spanning Tree
Say we have an undirected weighted graph

A spanning tree is a tree that connects all of the vertices.
Minimum Spanning Tree

Say we have an undirected weighted graph

The **cost** of a spanning tree is the sum of the weights on the edges.

A spanning tree is a tree that connects all of the vertices.

This is a spanning tree. It has cost 67.
Minimum Spanning Tree
Say we have an undirected weighted graph

A spanning tree is a tree that connects all of the vertices.

This is also a spanning tree.
It has cost 37

A tree is a connected graph with no cycles!
Minimum Spanning Tree
Say we have an undirected weighted graph

A spanning tree is a tree that connects all of the vertices.
Minimum Spanning Tree

Say we have an undirected weighted graph. A spanning tree is a tree that connects all of the vertices. This is a minimum spanning tree. It has cost 37, which is the minimum of minimal cost.
Why MSTs?

- Network design
- Connecting cities with roads/electricity/telephone/...
- Cluster analysis
  - eg, genetic distance
- Image processing
  - eg, image segmentation
- Useful primitive
  - for other graph algs

Figure 2: Fully parsimonious minimal spanning tree of 933 SNPs for 282 isolates of *Y. pestis* colored by location.
Morelli et al. Nature genetics 2010
How to find an MST?

• Today we’ll see two greedy algorithms.
• In order to prove that these greedy algorithms work, we’ll need to show something like:

  Suppose that our choices so far haven’t ruled out success.

  Then the next greedy choice that we make also won’t rule out success.

• Here, success means finding an MST.
From your pre-lecture exercise

- How would we design a greedy algorithm?
Brief aside

for a discussion of cuts in graphs!
Cuts in graphs

• A cut is a partition of the vertices into two parts:

This is the cut “\{A,B,D,E\} and \{C,I,H,G,F\}”
Cuts in graphs

- It could be any partition

This is the cut “\{B,C,E,G,H\} and \{A,D,I,F\}”
Let $S$ be a set of edges in $G$

- We say a cut respects $S$ if no edges in $A$ cross the cut.
- An edge crossing a cut is called light if it has the smallest weight of any edge crossing the cut.

$S$ is the set of thick orange edges.
Let $S$ be a set of edges in $G$

- We say a cut respects $S$ if no edges in $A$ cross the cut.
- An edge crossing a cut is called light if it has the smallest weight of any edge crossing the cut.

$S$ is the set of thick orange edges
Lemma

• Let $S$ be a set of edges, and consider a cut that respects $S$.
• Suppose there is an MST containing $S$.
• Let $(u,v)$ be a light edge.
• Then there is an MST containing $S \cup \{(u,v)\}$
Lemma

- Let $S$ be a set of edges, and consider a cut that respects $S$.
- Suppose there is an MST containing $S$.
- Let $(u,v)$ be a light edge.
- Then there is an MST containing $S \cup \{(u,v)\}$

Aka:

If we haven’t ruled out the possibility of success so far, then adding a light edge still won’t rule it out.

$S$ is the set of thick orange edges
Proof of Lemma

• Assume that we have:
  • a cut that respects $S$
Proof of Lemma

• Assume that we have:
  • a **cut** that respects **A**
  • **A** is part of some **MST T**.

• Say that \((u,v)\) is light.
  • lowest cost crossing the cut
Proof of Lemma

• Assume that we have:
  • a cut that respects $A$
  • $A$ is part of some MST $T$.

• Say that $(u,v)$ is light.
  • lowest cost crossing the cut

• But say $(u,v)$ is not in $T$.
  • So adding $(u,v)$ to $T$ will make a cycle.

Claim: Adding any additional edge to a spanning tree will create a cycle.

Proof: Both endpoints are already in the tree and connected to each other.
Proof of Lemma

• Assume that we have:
  • a cut that respects $A$
  • $A$ is part of some MST $T$.

• Say that $(u,v)$ is light.
  • lowest cost crossing the cut

• But say $(u,v)$ is not in $T$.
  • So adding $(u,v)$ to $T$ will make a cycle.

• So there is at least one other edge in this cycle crossing the cut.
  • call it $(x,y)$

Claim: Adding any additional edge to a spanning tree will create a cycle.

Proof: Both endpoints are already in the tree and connected to each other.
Proof of Lemma ctd.

• Consider swapping \((u,v)\) for \((x,y)\) in \(T\).
  • Call the resulting tree \(T'\).
Proof of Lemma ctd.

• Consider swapping \((u,v)\) for \((x,y)\) in \(T\).
  • Call the resulting tree \(T'\).

• **Claim:** \(T'\) is still an MST.
  • It is still a tree:
    • we deleted \((x,y)\)
  • It has cost at most that of \(T\)
    • because \((u,v)\) was light.
  • \(T\) had minimal cost.
  • So \(T'\) does too.

• So \(T'\) is an MST containing \((u,v)\).
  • This is what we wanted.
Lemma

• Let $S$ be a set of edges, and consider a cut that respects $S$.
• Suppose there is an MST containing $S$.
• Let $(u,v)$ be a light edge.
• Then there is an MST containing $S \cup \{(u,v)\}$.
End aside

Back to MSTs!
Back to MSTs

• How do we find one?
• Today we’ll see two greedy algorithms.

• The strategy:
  • Make a *series of choices*, adding edges to the tree.
  • Show that each edge we add is **safe to add**:
    • we do not rule out the possibility of success
    • we will choose *light edges* crossing *cuts* and use the Lemma.
  • **Keep going** until we have an MST.
Idea 1
Start growing a tree, greedily add the shortest edge we can to grow the tree.
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Start growing a tree, greedily add the shortest edge we can to grow the tree.
We’ve discovered
Prim’s algorithm!

- slowPrim( G = (V,E), starting vertex s ):
  - Let (s,u) be the lightest edge coming out of s.
  - MST = \{ (s,u) \}
  - verticesVisited = \{ s, u \}
  - while |verticesVisited| < |V|:
    - find the lightest edge (x,v) in E so that:
      - x is in verticesVisited
      - v is not in verticesVisited
    - add (x,v) to MST
    - add v to verticesVisited
  - return MST

Naively, the running time is O(nm):
- For each of n-1 iterations of the while loop:
  - Maybe go through all the edges.
Two questions

1. Does it work?
   • That is, does it actually return a MST?

2. How do we actually implement this?
   • the pseudocode above says “slowPrim”...
Does it work?

• We need to show that our greedy choices don’t rule out success.

• That is, at every step:
  • There exists an MST that contains all of the edges we have added so far.

• Now it is time to use our lemma!
Lemma

- Let $S$ be a set of edges, and consider a cut that respects $S$.
- Suppose there is an MST containing $S$.
- Let $(u,v)$ be a light edge.
- Then there is an MST containing $S \cup \{(u,v)\}$
Partway through Prim

- Assume that our choices $S$ so far are safe.
  - they don’t rule out success
- Consider the cut \{visited, unvisited\}
  - $S$ respects this cut.

$S$ is the set of edges selected so far.
Partway through Prim

• Assume that our choices \( S \) so far are **safe**.
  • they don’t rule out success

• Consider the cut \{**visited**, **unvisited**\}
  • \( S \) respects this cut.

• The edge we add next is a **light edge**.
  • Least weight of any edge crossing the cut.

• By the Lemma, **that edge** is **safe**.
  • it also doesn’t rule out success.

S is the set of edges selected so far.
Hooray!

- Our greedy choices don’t rule out success.

- This is enough (along with an argument by induction) to guarantee correctness of Prim’s algorithm.
Formally(ish)

• Inductive hypothesis:
  • After adding the $t$’th edge, there exists an MST with the edges added so far.

• Base case:
  • After adding the 0’th edge, there exists an MST with the edges added so far. **YEP.**

• Inductive step:
  • If the inductive hypothesis holds for $t$ (aka, the choices so far are safe), then it holds for $t+1$ (aka, the next edge we add is safe).
  • **That’s what we just showed.**

• Conclusion:
  • After adding the $n-1$’st edge, there exists an MST with the edges added so far.
  • At this point we have a spanning tree, so it better be minimal.
Two questions

1. Does it work?
   • That is, does it actually return a MST?
     • Yes!

2. How do we actually implement this?
   • the pseudocode above says “slowPrim”...
How do we actually implement this?

• Each vertex keeps:
  • the **distance** from itself to the **growing spanning tree**
  • **how to get there**.

I’m 7 away.
C is the closest.

I can’t get to the tree in one edge.
How do we actually implement this?

• Each vertex keeps:
  • the distance from itself to the **growing spanning tree**
  • how to get there.

• Choose the closest vertex, add it.
How do we actually implement this?

- Each vertex keeps:
  - the distance from itself to the growing spanning tree
  - how to get there.

- Choose the closest vertex, add it.
How do we actually implement this?

• Each vertex keeps:
  • the **distance** from itself to the **growing spanning tree**
  • **how to get there**.

• Choose the closest vertex, add it.
• Update stored info.

![Graph diagram](image-url)

- I’m 7 away. C is the closest.
- I’m 10 away. F is the closest.
Efficient implementation

Every vertex has a key and a parent

Until all the vertices are reached:

- Can’t reach x yet
- x is “active”
- Can reach x
- $k[x]$ is the distance of x from the growing tree
- $p[b] = a$, meaning that a was the vertex that $k[b]$ comes from.
Efficient implementation

Every vertex has a key and a parent

**Until** all the vertices are **reached**:

- Activate the **unreached** vertex \( u \) with the **smallest key**.

\[ k[x] \] is the distance of \( x \) from the growing tree

Can’t reach \( x \) yet

\( x \) is “active”

Can reach \( x \)

\( p[b] = a \), meaning that \( a \) was the vertex that \( k[b] \) comes from.
Efficient implementation

Every vertex has a key and a parent

Until all the vertices are reached:

- Activate the unreached vertex \( u \) with the smallest key.
- for each of \( u \)'s neighbors \( v \):
  - \( k[v] = \min(k[v], \text{weight}(u,v)) \)
  - if \( k[v] \) updated, \( p[v] = u \)

Can’t reach \( x \) yet
\( x \) is “active”
Can reach \( x \)

\( k[x] \) is the distance of \( x \) from the growing tree
\( p[b] = a \), meaning that \( a \) was the vertex that \( k[b] \) comes from.

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**Diagram:**

- **Vertices:** A, B, C, D, E, F, G, H, I
- **Edges:**
  - A to B: 4
  - A to I: 11
  - B to C: 8
  - B to H: 8
  - C to G: 7
  - C to I: ∞
  - D to F: 9
  - D to G: 14
  - E to F: 10
  - I to C: 2
  - I to H: 7
  - I to G: ∞
  - H to G: 6
  - H to B: 4
  - G to F: 2
  - G to H: 1

**Keys and Parents:**

- \( k[A] = 0 \)
- \( k[B] = 4 \)
- \( k[C] = ∞ \)
- \( k[D] = ∞ \)
- \( k[E] = ∞ \)
- \( k[F] = ∞ \)
- \( k[G] = ∞ \)
- \( k[H] = 8 \)
- \( k[I] = ∞ \)

**Parents:**

- \( p[A] = \) (unreached)
- \( p[B] = A \)
- \( p[C] = \) (unreached)
- \( p[D] = \) (unreached)
- \( p[E] = \) (unreached)
- \( p[F] = \) (unreached)
- \( p[G] = \) (unreached)
- \( p[H] = \) (unreached)
- \( p[I] = \) (unreached)
Efficient implementation

Every vertex has a key and a parent

Until all the vertices are reached:

- Activate the unreached vertex u with the smallest key.
- for each of u’s neighbors v:
  - k[v] = min( k[v], weight(u,v) )
  - if k[v] updated, p[v] = u
- Mark u as reached, and add (p[u],u) to MST.
Efficient implementation
Every vertex has a key and a parent

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$k[x]$ is the distance of $x$ from the growing tree
Can’t reach $x$ yet $x$ is “active”
Can reach $x$
p[$b$] = $a$, meaning that $a$ was the vertex that $k[b]$ comes from.
Efficient implementation
Every vertex has a key and a parent

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Efficient implementation
Every vertex has a key and a parent

Until all the vertices are reached:

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$k[x]$ is the distance of $x$ from the growing tree
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\( k[x] \) is the distance of \( x \) from the growing tree

Can’t reach \( x \) yet
\( x \) is “active”
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\( p[b] = a \), meaning that \( a \) was the vertex that \( k[b] \) comes from.
Efficient implementation

Every vertex has a key and a parent

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$k[x]$ is the distance of $x$ from the growing tree

Can’t reach $x$ yet
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  • If \( k[v] \) updated, \( p[v] = u \)
• Mark \( u \) as reached, and add \((p[u], u)\) to MST.
Efficient implementation

Every vertex has a key and a parent

Until all the vertices are reached:

- Activate the **unreached** vertex \( u \) with the **smallest key**.
- for each of \( u \)'s neighbors \( v \):
  - \( k[v] = \min( k[v], \text{weight}(u,v) ) \)
  - if \( k[v] \) updated, \( p[v] = u \)
- Mark \( u \) as **reached**, and add \( (p[u],u) \) to MST.

Can’t reach \( x \) yet
\( x \) is “active”
Can reach \( x \)

\( k[x] \) is the distance of \( x \) from the growing tree

\( p[b] = a \), meaning that \( a \) was the vertex that \( k[b] \) comes from.

Etc.
This should look pretty familiar

• Very similar to Dijkstra’s algorithm!
  • For the IPython notebook, I actually copied and pasted from the Lecture 11 IPython notebook...

• Differences:
  1. Keep track of \( p[v] \) in order to return a tree at the end
     • But Dijkstra’s can do that too, that’s not a big difference.

  2. Instead of \( d[v] \) which we update by
     • \( d[v] = \min( d[v], d[u] + w(u,v) ) \)
     we keep \( k[v] \) which we update by
     • \( k[v] = \min( k[v], w(u,v) ) \)

• To see the difference, consider: 

[Diagram of a network with nodes S, U, and T and edges labeled with weights.]

Thing 2 is the big difference.
One thing that is similar: Running time

- Exactly the same as Dijkstra:
  - $O(m \log(n))$ using a Red-Black tree as a priority queue.
  - $O(m + n \log(n))$ amortized time if we use a Fibonacci Heap*.

*See CS166
Two questions

1. Does it work?
   • That is, does it actually return a MST?
     • Yes!

2. How do we actually implement this?
   • the pseudocode above says “slowPrim”...
     • **Implement it basically the same way we’d implement Dijkstra!**
What have we learned?

• Prim’s algorithm greedily grows a tree
  • smells a lot like Dijkstra’s algorithm

• It finds a Minimum Spanning Tree in time $O(m\log(n))$
  • if we implement it with a Red-Black Tree

• To prove it worked, we followed the same recipe for greedy algorithms we saw last time.
  • Show that, at every step, we don’t rule out success.
That’s not the only greedy algorithm
what if we just always take the cheapest edge?
whether or not it’s connected to what we have so far?
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That won’t cause a cycle
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That won’t cause a cycle
That’s not the only greedy algorithm.

What if we just always take the cheapest edge?

Whether or not it’s connected to what we have so far?

That won’t cause a cycle.
We’ve discovered Kruskal’s algorithm!

- **slowKruskal**(G = (V,E)):
  - Sort the edges in E by non-decreasing weight.
  - MST = {}
  - **for** e in E (in sorted order):
    - **if** adding e to MST won’t cause a cycle:
      - add e to MST.
  - **return** MST

Naively, the running time is ???:
- For each of m iterations of the for loop:
  - Check if adding e would cause a cycle...
Two questions

1. Does it work?
   • That is, does it actually return a MST?

2. How do we actually implement this?
   • the pseudocode above says “slowKruskal”...
At each step of Kruskal’s, we are maintaining a forest.
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When we add an edge, we merge two trees:
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When we add an edge, we merge two trees:
At each step of Kruskal’s, we are maintaining a **forest**.

When we add an edge, we merge two trees:

We never add an edge within a tree since that would create a cycle.
Keep the trees in a special data structure

“treehouse”? 
Union-find data structure
also called disjoint-set data structure

• Used for storing collections of sets

• Supports:
  • \texttt{makeSet(u)}: create a set \{u\}
  • \texttt{find(u)}: return the set that \( u \) is in
  • \texttt{union(u,v)}: merge the set that \( u \) is in with the set that \( v \) is in.

\begin{align*}
\text{makeSet(x)}
\text{makeSet(y)}
\text{makeSet(z)}
\text{union(x,y)}
\end{align*}
Union-find data structure
also called disjoint-set data structure

• Used for storing collections of sets

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  • \texttt{makeSet(u)}: create a set \{u\}
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\texttt{makeSet(x)}
\texttt{makeSet(y)}
\texttt{makeSet(z)}
\texttt{union(x,y)}
Union-find data structure
also called disjoint-set data structure

• Used for storing collections of sets

• Supports:
  • `makeSet(u)`: create a set \{u\}
  • `find(u)`: return the set that u is in
  • `union(u,v)`: merge the set that u is in with the set that v is in.

makeSet(x)
makeSet(y)
makeSet(z)

union(x,y)

find(x)
Kruskal pseudo-code

• **kruskal**\(^{(G = (V,E))}\):
  • Sort E by weight in non-decreasing order
  • \(\text{MST} = \{\}\) \hspace{1cm} // initialize an empty tree
  • **for** \(v\) in \(V\):
    • \text{makeSet}(v) \hspace{1cm} // put each vertex in its own tree in the forest
  • **for** \((u,v)\) in \(E\):
    • **if** \text{find}(u) \(!=\) \text{find}(v):
      • add \((u,v)\) to \text{MST} \hspace{1cm} // if \(u\) and \(v\) are not in the same tree
      • \text{union}(u,v) \hspace{1cm} // merge \(u\)’s tree with \(v\)’s tree
  • **return** \text{MST}
Once more...

To start, every vertex is in it’s own tree.
Once more...

Then start merging.
Once more...

Then start merging.
Once more...

Then start merging.
Once more...

Then start merging.
Once more...

Then start merging.
Once more...

Then start merging.
Once more...

Then start merging.
Once more...

Then start merging.

Stop when we have one big tree!
Running time

• Sorting the edges takes $O(m \log(n))$
  • In practice, if the weights are small integers we can use radixSort and take time $O(m)$

• For the rest:
  • $n$ calls to `makeSet`
    • put each vertex in its own set
  • $2m$ calls to `find`
    • for each edge, `find` its endpoints
  • $n$ calls to `union`
    • we will never add more than $n-1$ edges to the tree,
    • so we will never call `union` more than $n-1$ times.

• Total running time:
  • Worst-case $O(m \log(n))$, just like Prim.
  • Closer to $O(m)$ if you can do radixSort

*technically, they run in *amortized time* $O(\alpha(n))$, where $\alpha(n)$ is the *inverse Ackerman function*. $\alpha(n) \leq 4$ provided that $n$ is smaller than the number of atoms in the universe.
Two questions

1. Does it work?
   • That is, does it actually return a MST?

2. How do we actually implement this?
   • the pseudocode above says “slowKruskal”...
     • **Worst-case running time $O(m\log(n))$** using a union-find data structure.
Does it work?

• We need to show that our greedy choices don’t rule out success.

• That is, at every step:
  • There exists an MST that contains all of the edges we have added so far.

• Now it is time to use our lemma! again!
Lemma

- Let $S$ be a set of edges, and consider a cut that respects $S$.
- Suppose there is an MST containing $S$.
- Let $(u,v)$ be a light edge.
- Then there is an MST containing $S \cup \{(u,v)\}$.
Partway through Kruskal

• Assume that our choices $S$ so far are **safe**.
  • they don’t rule out success

• The **next edge** we add will merge two trees, $T_1$, $T_2$

$S$ is the set of edges selected so far.
Partway through Kruskal

• Assume that our choices $S$ so far are safe.
  • they don’t rule out success

• The next edge we add will merge two trees, $T_1, T_2$

• Consider the cut $\{T_1, V - T_1\}$.
  • A respects this cut.
  • Our new edge is light for the cut
Partway through Kruskal

- Assume that our choices $S$ so far are **safe**.
  - they don’t rule out success
- The **next edge** we add will merge two trees, $T_1, T_2$
- Consider the cut $\{T_1, V - T_1\}$.
  - A respects this cut.
  - Our **new edge is light** for the cut
- By the Lemma, **that edge is safe**.
  - it also doesn’t rule out success.

$S$ is the set of edges selected so far.
Hooray!

• Our greedy choices don’t rule out success.

• This is enough (along with an argument by induction) to guarantee correctness of Kruskal’s algorithm.
Formally(ish)

- Inductive hypothesis:
  - After adding the t’th edge, there exists an MST with the edges added so far.

- Base case:
  - After adding the 0’th edge, there exists an MST with the edges added so far. **YEP.**

- Inductive step:
  - If the inductive hypothesis holds for t (aka, the choices so far are safe), then it holds for t+1 (aka, the next edge we add is safe).
  - **That’s what we just showed.**

- Conclusion:
  - After adding the n-1’st edge, there exists an MST with the edges added so far.
  - At this point we have a spanning tree, so it better be minimal.
Two questions

1. Does it work?
   • That is, does it actually return a MST?
     • Yes

2. How do we actually implement this?
   • the pseudocode above says “slowKruskal”...
     • Using a union-find data structure!
What have we learned?

• Kruskal’s algorithm greedily grows a forest
• It finds a Minimum Spanning Tree in time $O(m\log(n))$
  • if we implement it with a Union-Find data structure
  • if the edge weights are reasonably-sized integers and we ignore the inverse Ackerman function, basically $O(m)$ in practice.

• To prove it worked, we followed the same recipe for greedy algorithms we saw last time.
  • Show that, at every step, we don’t rule out success.
Compare and contrast

• Prim:
  • Grows a tree.
  • Time $O(m \log(n))$ with a red-black tree
  • Time $O(m + n \log(n))$ with a Fibonacci heap

• Kruskal:
  • Grows a forest.
  • Time $O(m \log(n))$ with a union-find data structure
  • If you can do radixSort on the edge weights, morally $O(m)$
Both Prim and Kruskal

- Greedy algorithms for MST.
- Similar reasoning:
  - Optimal substructure: subgraphs generated by cuts.
  - The way to make safe choices is to choose light edges crossing the cut.

S is the set of **thick orange** edges

This edge is light
Can we do better?
State-of-the-art MST on connected undirected graphs

• Karger-Klein-Tarjan 1995:
  • $O(m)$ time randomized algorithm

• Chazelle 2000:
  • $O(m \cdot \alpha(n))$ time deterministic algorithm

• Pettie-Ramachandran 2002:
  • $O\left(\frac{\text{The optimal number of comparisons}}{N^*(n,m) \text{ you need to solve the problem, whatever that is...}}\right)$ time deterministic algorithm

What this number is still open!
Do we need that silly $\alpha(n)$?
Recap

• Two algorithms for Minimum Spanning Tree
  • Prim’s algorithm
  • Kruskal’s algorithm

• Both are (more) examples of greedy algorithms!
  • Make a series of choices.
  • Show that at each step, your choice does not rule out success.
  • At the end of the day, you haven’t ruled out success, so you must be successful.
Next time

- Cuts and flows!
- In the meantime,

Happy Thanksgiving, enjoy the break!