1 Find the Universal Hash Families

Let \( U = \{000, 001, 002, \ldots, 999\} \) (aka, all of the numbers between 0 and 999, padded so that they are three digits long) and let \( n = 10 \). For each of the following hash families \( H \) consisting of functions \( h : U \to \{0, \ldots, n - 1\} \), decide whether \( H \) is universal or not, and justify your result with a formal proof.

(a) For \( i = 1, 2, 3 \), let \( h_i(x) \) be the \( i \)th least-significant digit of \( x \). (For example, \( h_2(456) = 5 \)). Define \( H = \{h_1, h_2, h_3\} \). Is \( H \) a universal hash family?

(b) For \( a \in \{1, \ldots, 9\} \), let \( h_a(x) \) be the least-significant digit of \( ax \). (For example, \( h_2(123) \) is the least-significant digit of \( 2 \times 123 = 246 \), which is 6). Define \( H = \{h_i : i = 1, \ldots, 9\} \). Is \( H \) a universal hash family?

2 Zero Sum Subarrays

Given an array \( A \) of positive and negative integers, determine if there is a subarray with zero sum. A subarray is a contiguous chunk of the original array and has at least one element.

(a) What is a brute force solution, and what is its runtime?

(b) Design an algorithm that uses hashing to solve this problem in \( O(n) \) time.

3 Strongly Connected Components

Ten “friends” decided to make a Valentine’s Day graph in which each person is a node, and an edge points from \( v_i \) to \( v_j \) if Person \( j \) has a crush on Person \( i \). Below is their graph:
What are all the strongly connected components of this graph? (i.e., groups of vertices such that there exists a path between any two vertices in the group)

4 Bipartite Graphs

A Bipartite Graph is a graph whose vertices can be divided into two independent sets, \(U\) and \(V\) such that every edge \((u, v)\) either connects a vertex from \(U\) to \(V\) or a vertex from \(V\) to \(U\). A bipartite graph is possible if the graph can be colored using two colors such that vertices in the same set are colored with the same color. Design an algorithm using DFS to determine whether or not a graph is bipartite.

5 Russian Boxes

Through hard work and a small stroke of good luck you have glowed up over the past year and are suddenly inundated with admirers for Valentine’s Day, who have each sent you a box of chocolates (if you are allergic to chocolates, you may assume they sent you strawberries or something else that is nice and comes in a box). After you finish consuming the spoils of your attractiveness, you are left with \(n\) empty rectangular boxes (you may assume \(n\) VERY large), and you decide to nest some of them within each other for easy storage. The \(i\)-th box has dimensions \(w_i \times h_i\). Box \(i\) can fit inside box \(j\) if and only if \(w_i < w_j\) and \(h_i < h_j\).

A sequence of boxes \(b_1, b_2, ..., b_k\) form a chain if box \(b_i\) fits inside box \(b_{i+1}\) for each \(1 \leq i < k\). Design an algorithm which takes as input a list of dimensions \(w_i \times h_i\) and returns the length of the longest possible chain of boxes. You must construct a directed graph as part of your solution.

BONUS: Having found the length of the longest possible chain, how can you use your directed graph to return the chain itself? (If there is more than one longest chain, you may return any of them)