Balanced Trees
Part Two
Outline for Today

- **Red/Black Trees**
  - Using our isometry!

- **Tree Rotations**
  - A key primitive in restructuring trees.

- **Augmented Binary Search Trees**
  - Leveraging red/black trees.
Recap from Last Time
2-3-4 Trees

- A **2-3-4 tree** is a multiway search tree where
  - every node has 1, 2, or 3 keys,
  - any non-leaf node with \( k \) keys has exactly \( k+1 \) children, and
  - all leaves are at the same depth.
- To insert a key, place it in a leaf. If out of space, split the leaf and kick the median key one level higher, repeating this process.
Red/Black Trees

- A **red/black tree** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.

![Red/Black Tree Diagram]

After we hoist red nodes into their parents:
- Each "meta node" has 1, 2, or 3 keys in it. (No red node has a red child.)
- Each "meta node" is either a leaf or has one more key than its parent. (Root-null path property.)
- Each "meta leaf" is at the same depth. (Root-null path property.)
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*This is a 2-3-4 tree!*
New Stuff!
Data Structure Isometries

- Red/black trees are an isometry of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- That gives us some really easy theorems basically for free.
- **Theorem:** The maximum height of a red/black tree with \( n \) nodes is \( O(\log n) \).
- **Proof idea:** Pulling red nodes into their parents forms a 2-3-4 tree with \( n \) keys, which has height \( O(\log n) \). Undoing this at most doubles the height of the tree. ■-ish
Exploring the Isometry

- Nodes in a 2-3-4 tree are classified into types based on the number of children they can have.
  - **2-nodes** have one key (two children).
  - **3-nodes** have two keys (three children).
  - **4-nodes** have three keys (four children).
- How might these nodes be represented?
Exploring the Isometry
Goal
Red/Black Tree Insertion

• **Rule #1**: When inserting a node, if its parent is black, make the node red and stop.

• **Justification**: This simulates inserting a key into an existing 2-node or 3-node.
Goal
Tree Rotations
This applies any time we're inserting a new node into the middle of a “3-node” in this pattern. By making observations like these, we can determine how to update a red/black tree after an insertion.
Goal
Goal
add
16

add
16

add
15

add
16
Building Up Rules

• The complex rules on red/black trees make perfect sense if you connect it back to 2-3-4 trees.

• There are lots of cases to consider because there are many different ways you can insert into a red/black tree.

• **Main point:** Simulating the insertion of a key into a node takes time $O(1)$ in all cases. Therefore, since 2-3-4 trees support $O(\log n)$ insertions, red/black trees support $O(\log n)$ insertions.

• The same is true of deletions.
My Advice

- **Do** know how to do B-tree insertions and searches.
  - You can derive these easily if you remember to split nodes.
- **Do** remember the rules for red/black trees and B-trees.
  - These are useful for proving bounds and deriving results.
- **Do** remember the isometry between red/black trees and 2-3-4 trees.
  - Gives immediate intuition for all the red/black tree operations.
- **Don't** memorize the red/black rotations and color flips.
  - This is rarely useful. If you're coding up a red/black tree, just flip open CLRS and translate the pseudocode. 😊
Dynamic Problems
Classical Algorithms

- The “classical” algorithms model goes something like this:

\[ \text{Given some input } X, \text{ compute some interesting function } f(X). \]

- The input \( X \) is provided up front, and only a single answer is produced.
Dynamic Problems

- *Dynamic versions* of problems are framed like this: 

  *Given an input $X$ that can change in fixed ways, maintain $X$ while being able to compute $f(X)$ efficiently at any point in time.*

- These problems are typically harder to solve efficiently than the “classical” static versions.
Dynamic Selection

- The *selection* problem is the following:

  *Given a list of distinct values and a number k, return the kth-smallest value.*

- In the static case, where the data set is fixed in advance and k is known, we can solve this in time O(n) using quickselect or the median-of-medians algorithm.

- **Goal:** Solve this problem efficiently when the data set is changing – that is, the underlying set of elements can have insertions and deletions intermixed with queries.
Dynamic Selection
Dynamic Selection
Dynamic Selection

**Problem:** After inserting a new value, we may have to update $\Theta(n)$ values.

This is inherent in this solution route. These numbers track *global* properties of the tree.
If new nodes are added to the left subtree, the numbers on the right don’t need to update.
Mechanically: Number each key so that it only stores its order statistic in the subtree rooted at itself.

Operationally: Annotate each key with the number of keys in its left subtree.
Dynamic Selection
We only update values on nodes that gained a new key in their left subtree. And there are only $O(\log n)$ of these!
We only update values on nodes that gained a new key in their left subtree. And there are only $O(\log n)$ of these!
How do we update the numbers after the rotation?
How do we update the numbers after the rotation?
Dynamic Selection
Dynamic Selection

Diagram of a data structure used in dynamic selection.
Dynamic Selection

```
    6 14
   /  \
  3   19
 / \
7   3
 /   \
1   1
 / \
4   1
 /
3
```

```
    1
   /  \
  1   1
 / \
3   1
 /
9
```

```
    1
   /  \
  1   1
 / \
15   1
 /
0
```

```
    1
   /  \
  1   1
 / \
17   0
 /
0
```

```
    1
   /  \
  1   1
 / \
23   0
```
Order Statistic Trees

- This modified red/black tree is called an order statistics tree.
  - Start with a red/black tree.
  - Tag each node with the number of nodes in its left subtree.
  - Use the preceding update rules to preserve values during rotations.
  - Propagate other changes up to the root of the tree.
- Only $O(\log n)$ values must be updated on an insertion or deletion and each can be updated in time $O(1)$.
- Supports all BST operations plus select (find kth order statistic) and rank (given a key, report its order statistic) in time $O(\log n)$.
Generalizing our Idea
Edits to values are localized along the access path.
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Imagine we cache some value in each node that can be computed just from (1) the node itself and (2) its children’s values.
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Recompute values on this access path, bottom-up.
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Recompute the values in these nodes.
Imagine we cache some value in each node that can be computed just from (1) the node itself and (2) its children’s values.
Theorem: Suppose we want to cache some computed value in each node of a red/black tree. Provided that the value can be recomputed purely from the node’s value and from it’s children’s values, and provided that each value can be computed in time $O(1)$, then these values can be cached in each node with insertions, lookups, and deletions still taking time $O(\log n)$. 
Example: *Hierarchical Clustering*
1D Hierarchical Clustering

This tree is called a dendrogram.
Analyzing the Runtime

• How efficient is this algorithm?
  • Number of rounds: $\Theta(n)$.
  • Work to find closest pair: $O(n)$.
  • Total runtime: $\Theta(n^2)$.

• Can we do better?
Dynamic 1D Closest Points

• The *dynamic 1D closest points problem* is the following:

  Maintain a set of real numbers undergoing insertion and deletion while efficiently supporting queries of the form “what is the closest pair of points?”

• Can we build a better data structure for this?
Dynamic 1D Closest Points

\[ k \]

\[ \text{max} \]

\[ \text{min} \]
A Tree Augmentation

• Augment each node to store the following:
  • The maximum value in the tree.
  • The minimum value in the tree.
  • The closest pair of points in the tree.

• **Claim:** Each of these properties can be computed in time $O(1)$ from the left and right subtrees.

• These properties can be augmented into a red/black tree so that insertions and deletions take time $O(\log n)$ and “what is the closest pair of points?” can be answered in time $O(1)$. 
Dynamic 1D Closest Points

137
Min: -17
Max: 415
Closest: 137, 142

42
Min: -17
Max: 67
Closest: 15, 21

271
Min: 142
Max: 415
Closest: 300, 310
Some Other Questions

- How would you augment this tree so that you can efficiently (in time $O(1)$) compute the appropriate weighted averages?

- **Trickier**: Is this the fastest possible algorithm for this problem?
  - What if you’re guaranteed that the keys are all integers in some nice range?
A Helpful Intuition
Divide-and-Conquer

• Initially, it can be tricky to come up with the right tree augmentations.

• **Useful intuition:** Imagine you're writing a divide-and-conquer algorithm over the elements and have $O(1)$ time per “conquer” step.

```
< k

k

> k
```
Next Time

- **Randomized Data Structures**
  - Harnessing randomness to seemingly do the impossible.

- **Families of Hash Functions**
  - What do we want from our hash functions?

- **Count-Min Sketches**
  - Counting without counting.