Balanced Trees

Part One

Balanced Trees

- Balanced search trees are among the most useful and versatile data structures.
- Many programming languages ship with a balanced tree library.
  - C++: `std::map` / `std::set`
  - Java: `TreeMap` / `TreeSet`
  - Haskell: `Data.Map`
- Many advanced data structures are layered on top of balanced trees.
  - We'll see them used to build *y*-Fast Tries and (one of) Euler tour trees and link/cut trees.

Where We're Going

- ***B-Trees***
  - A simple type of balanced tree developed for block storage.
- ***Red/Black Trees***
  - The canonical balanced binary search tree.
- ***Augmented Search Trees***
  - Adding extra information to balanced trees to supercharge the data structure.
- ***Two Advanced Operations***
  - The split and join operations.

Outline for Today

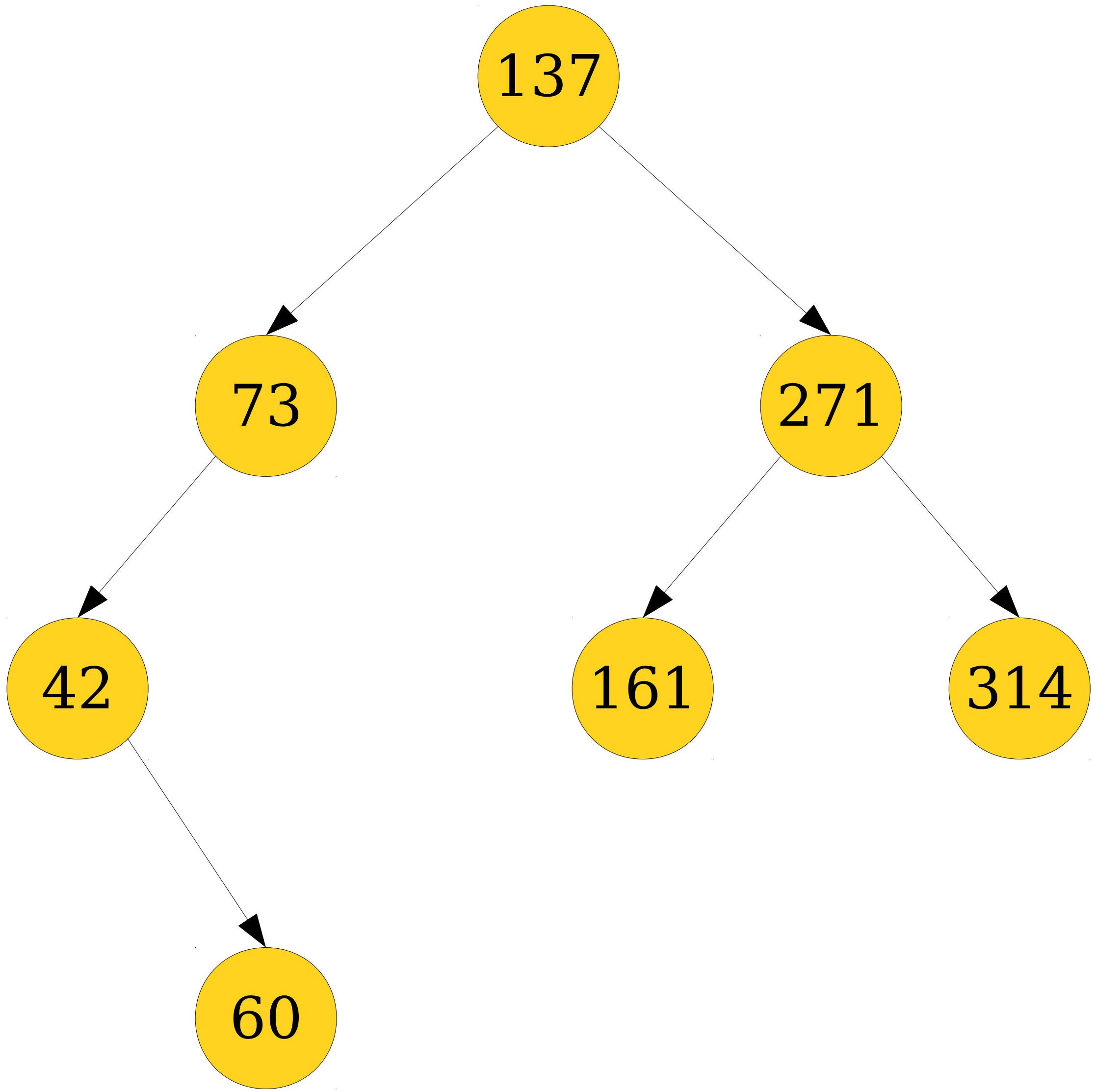
- ***BST Review***
  - Refresher on basic BST concepts and runtimes.
- ***Overview of Red/Black Trees***
  - What we're building toward.
- ***B-Trees and 2-3-4- Trees***
  - A simple balanced tree in depth.
- ***Intuiting Red/Black Trees***
  - A much better feel for red/black trees.

A Quick BST Review

Binary Search Trees

- A ***binary search tree*** is a binary tree with the following properties:
  - Each node in the BST stores a ***key***, and optionally, some auxiliary information.
  - The key of every node in a BST is strictly greater than all keys to its left and strictly smaller than all keys to its right.
- The ***height*** of a binary search tree is the length of the longest path from the root to a leaf, measured in the number of *edges*.
  - A tree with one node has height 0.
  - A tree with no nodes has height -1, by convention.

Searching a BST



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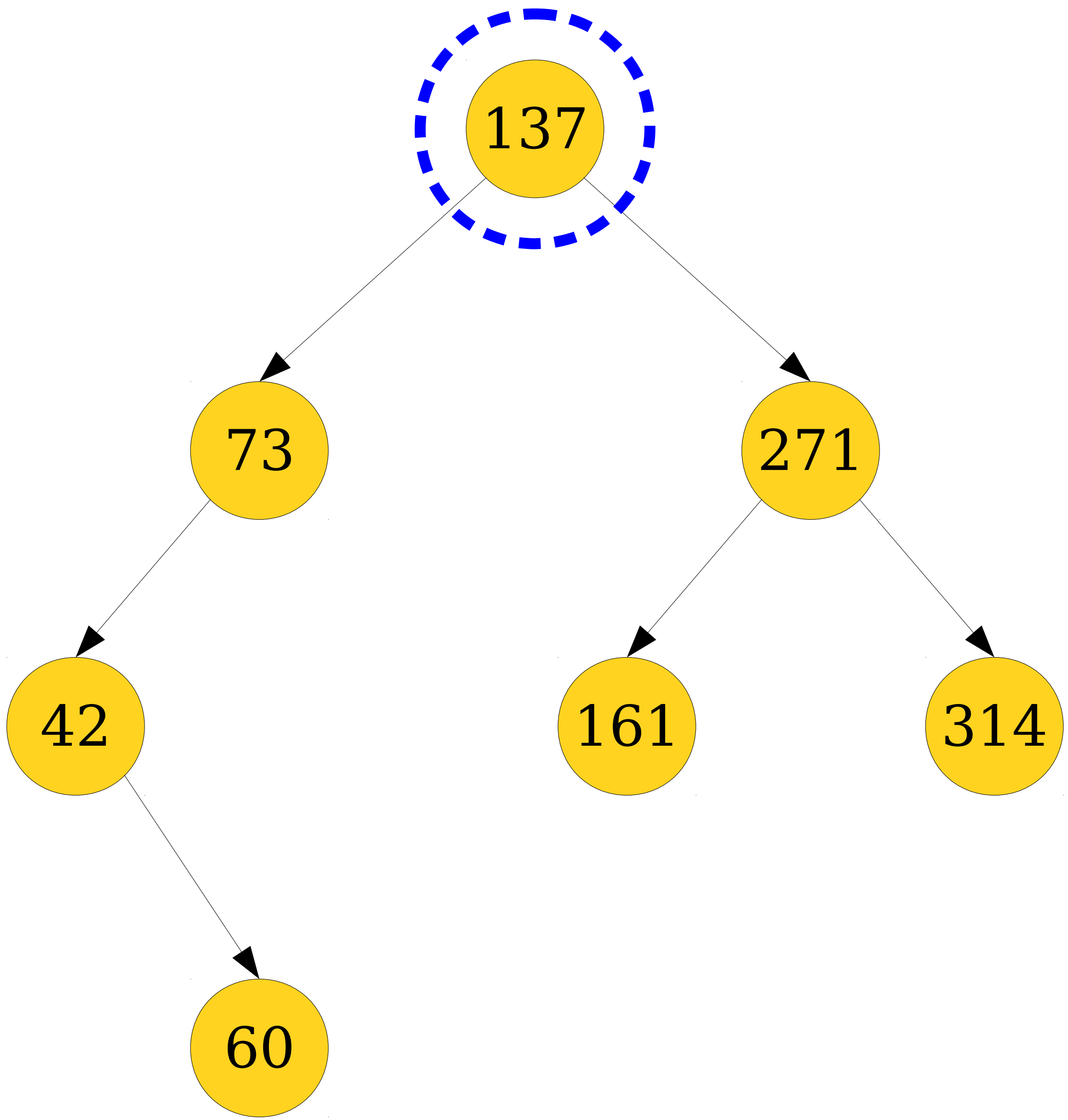

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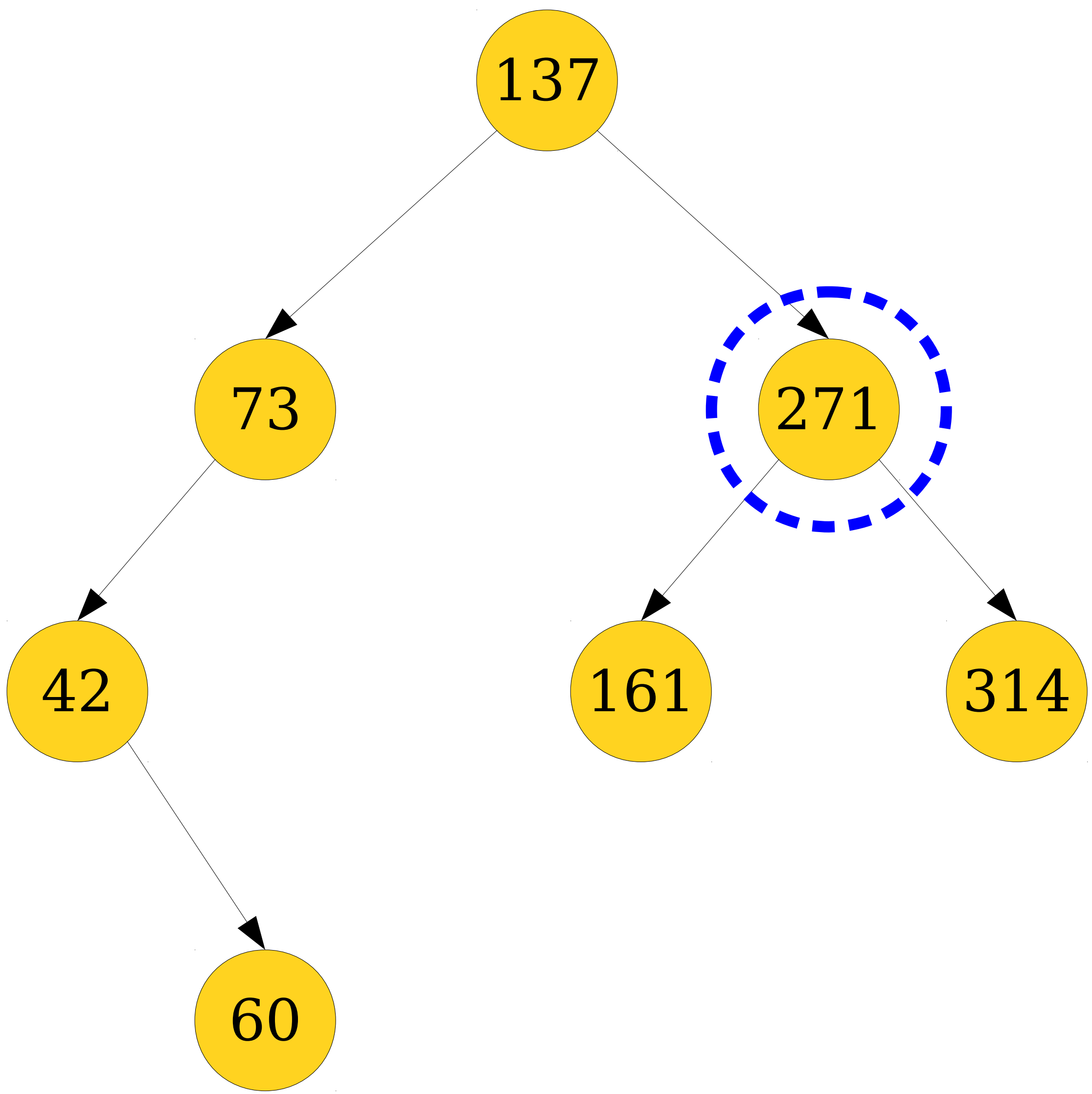

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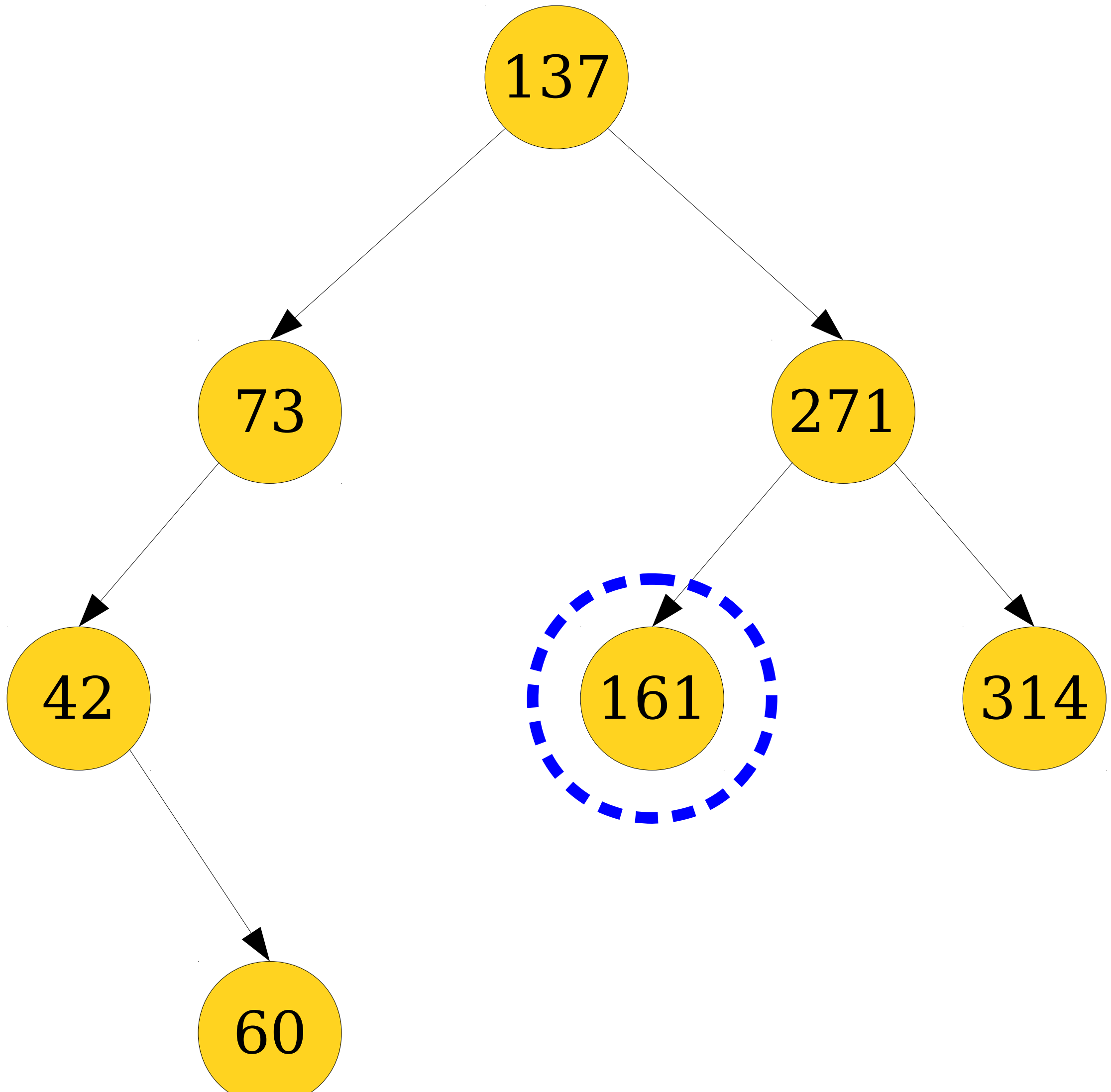

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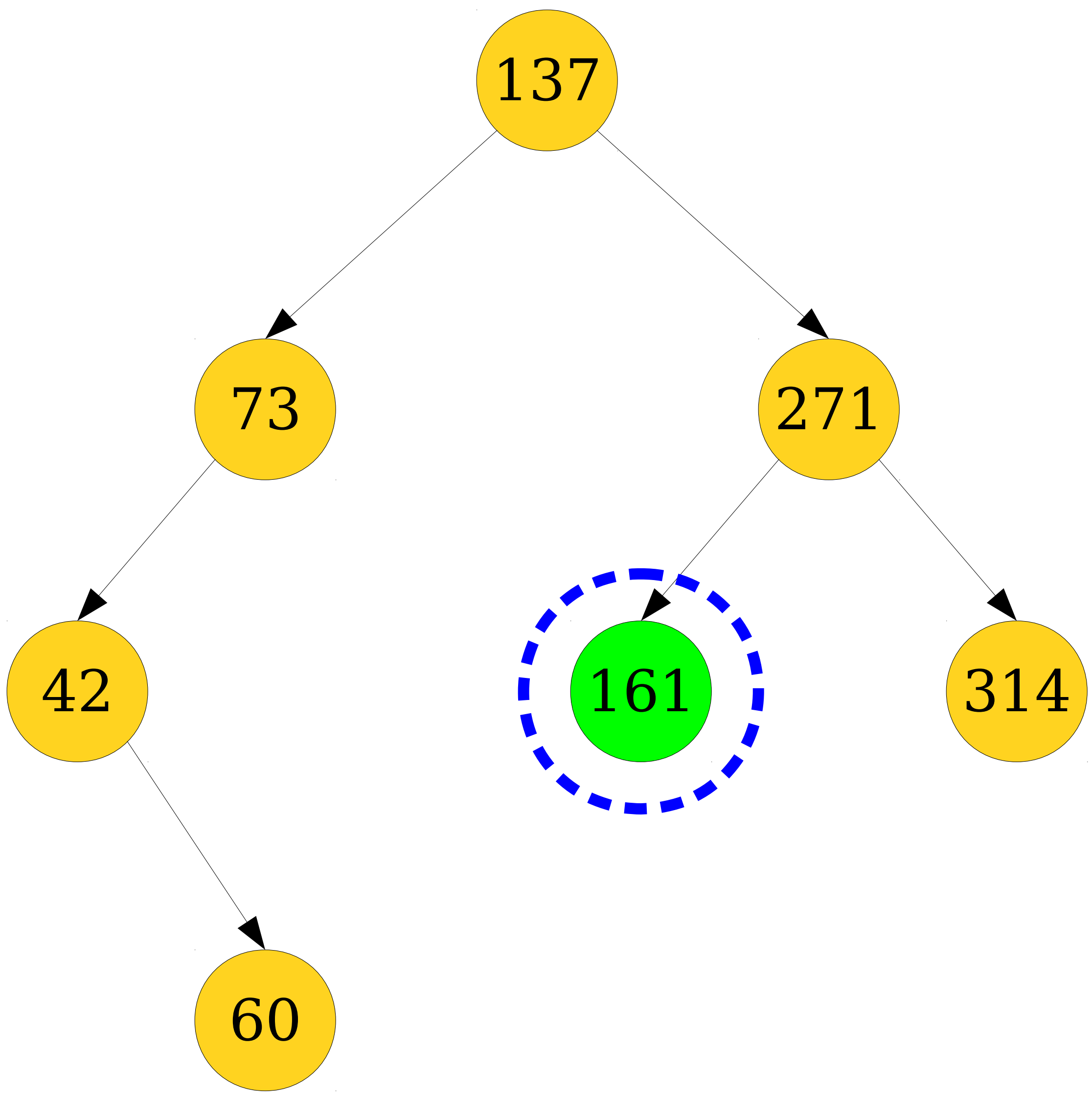

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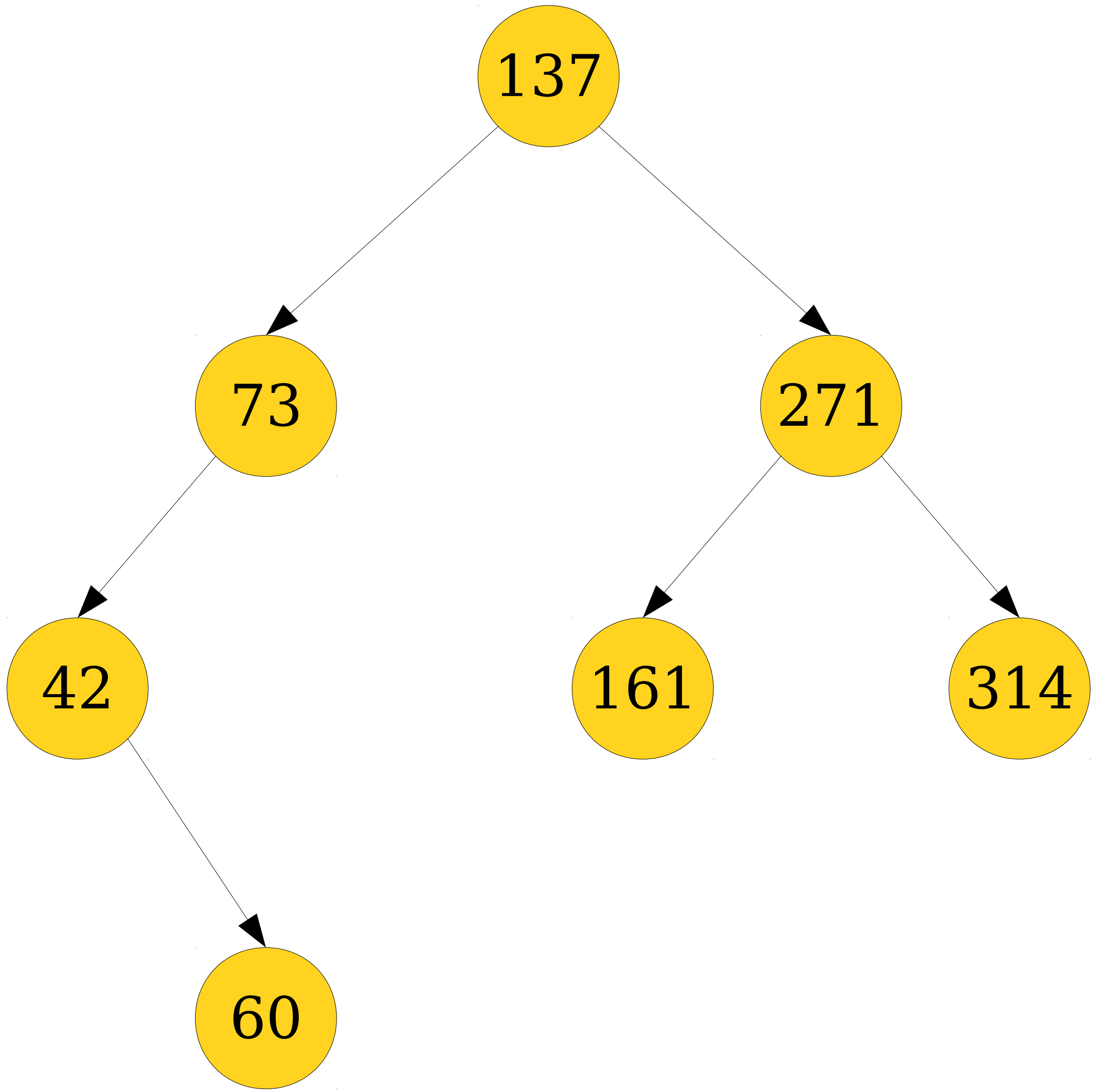

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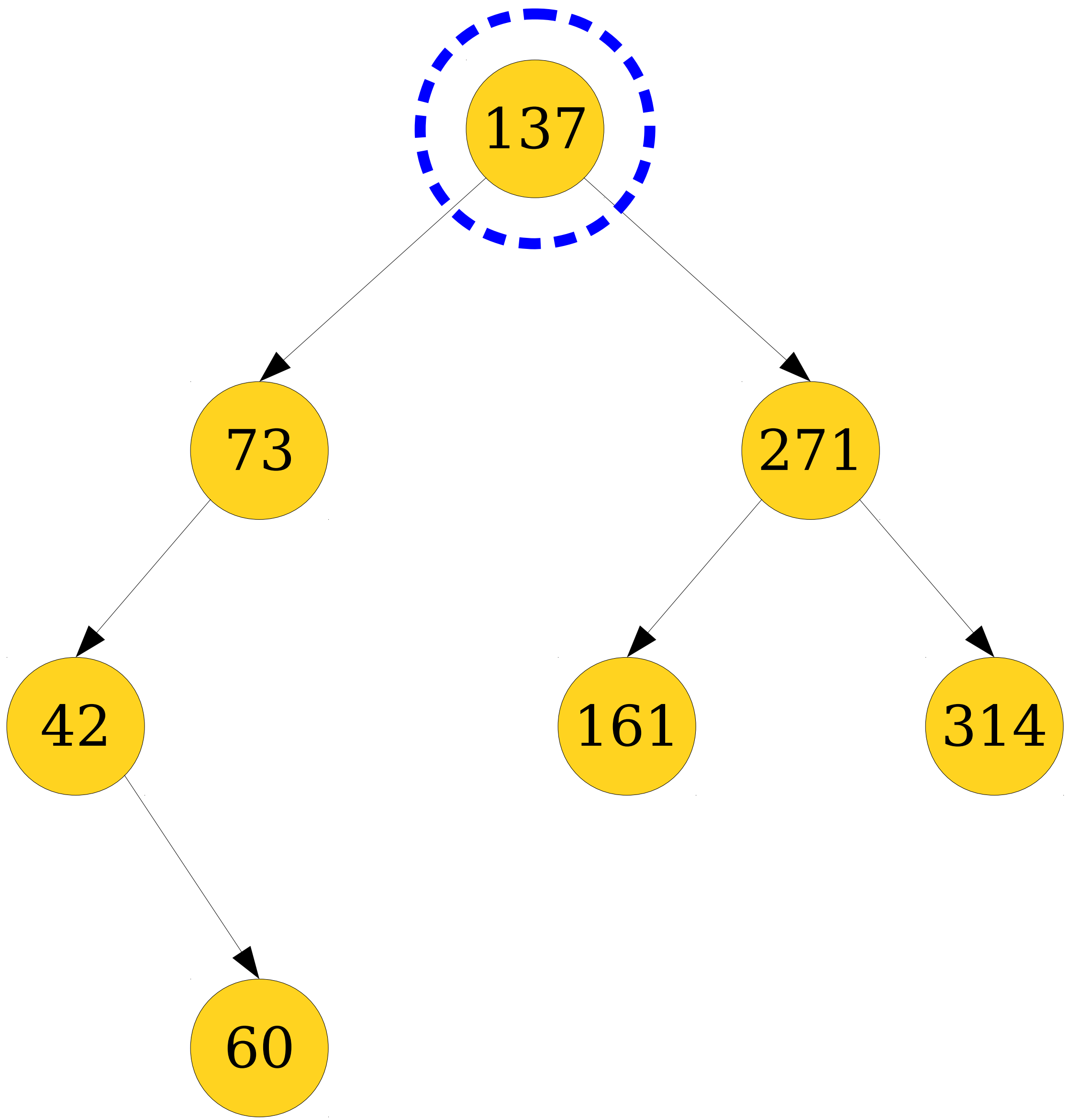

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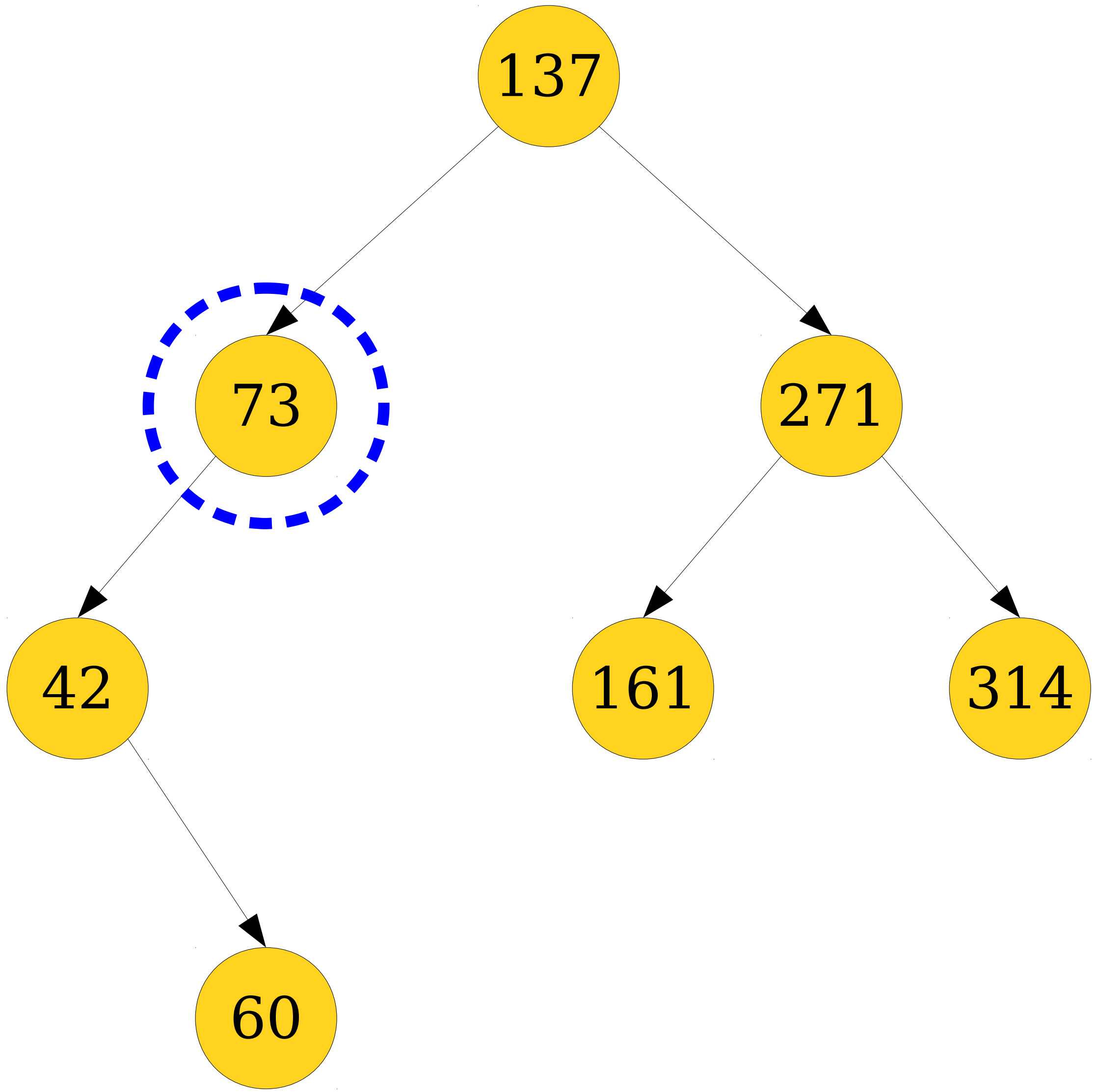

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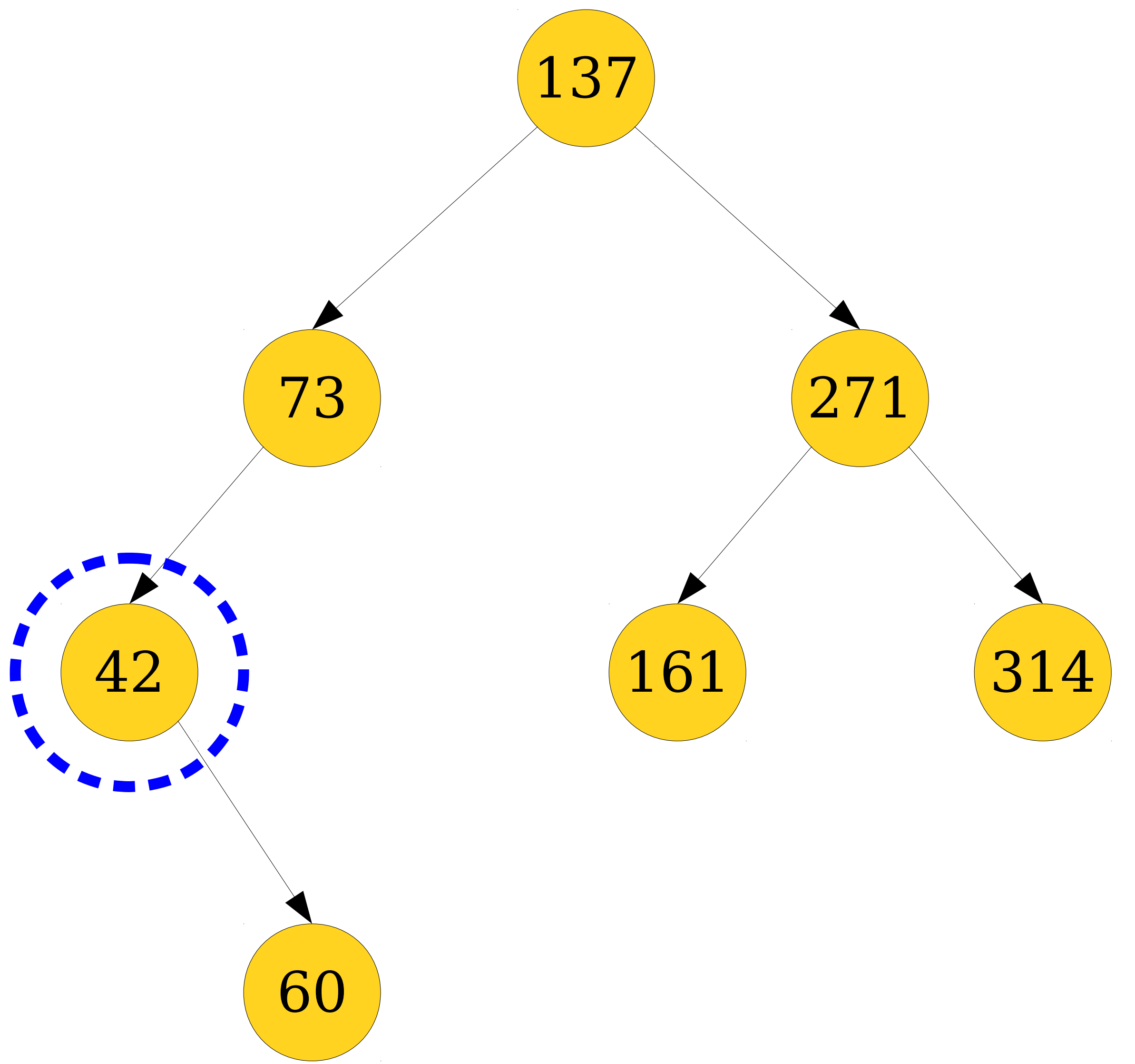

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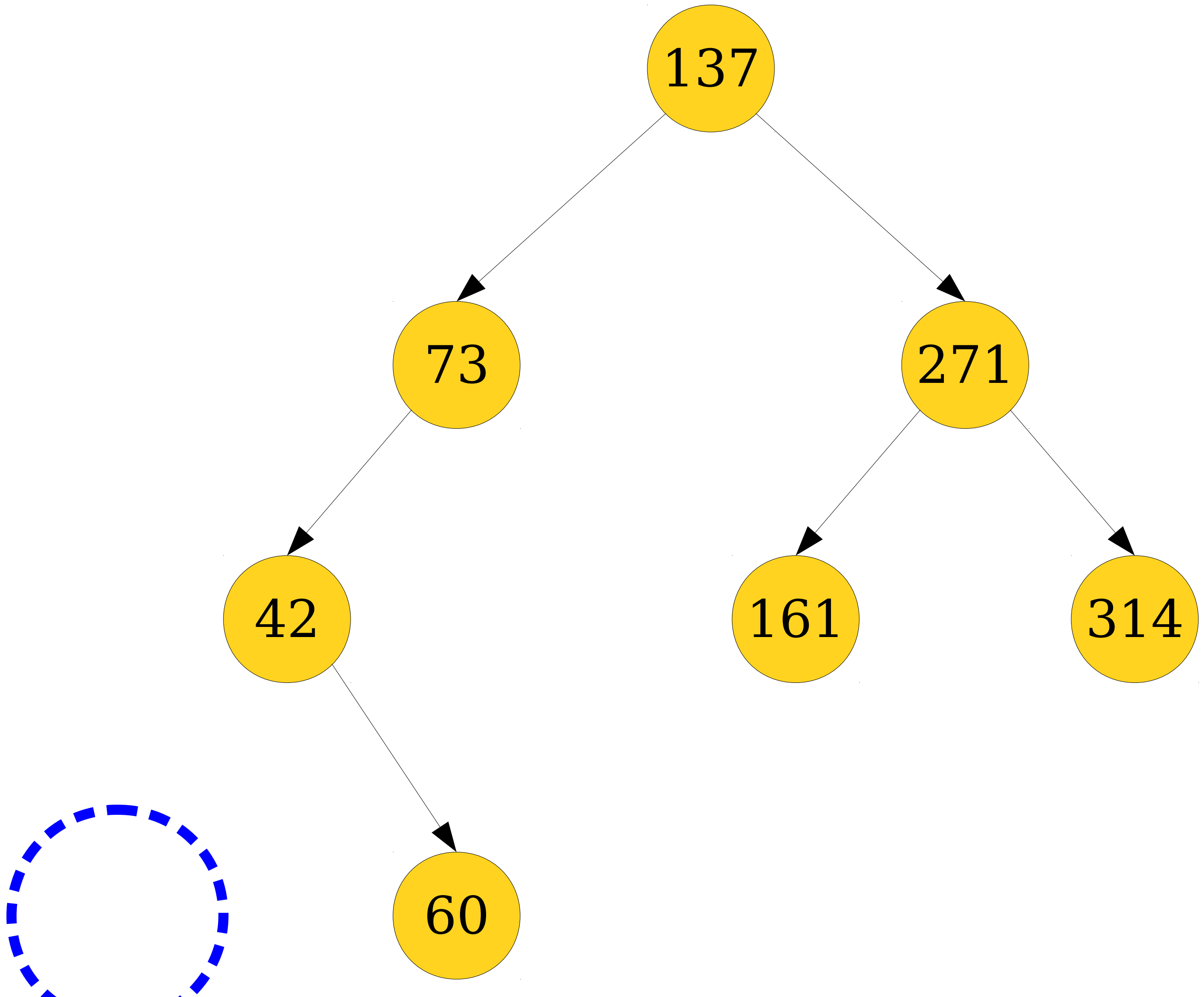

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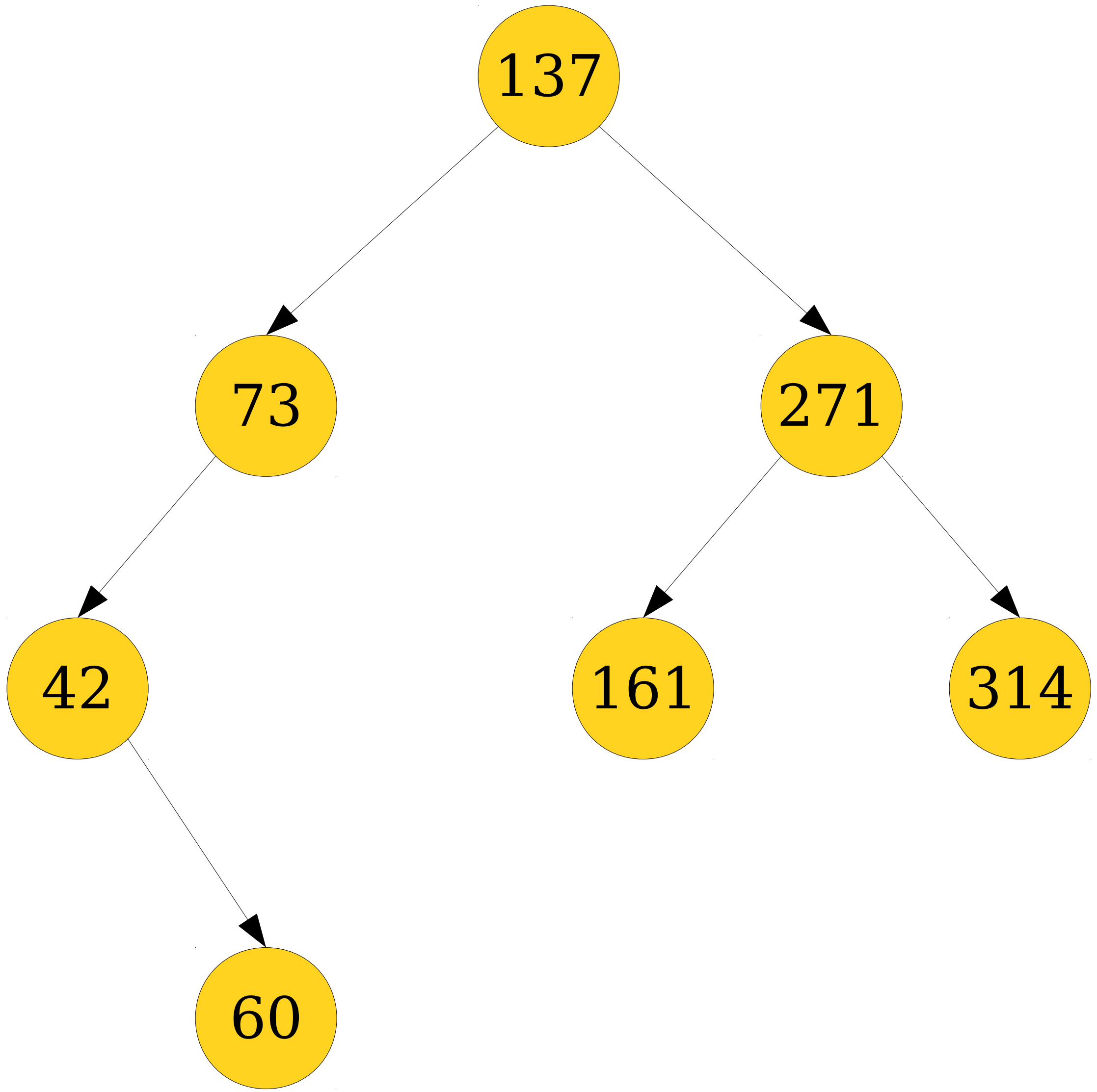

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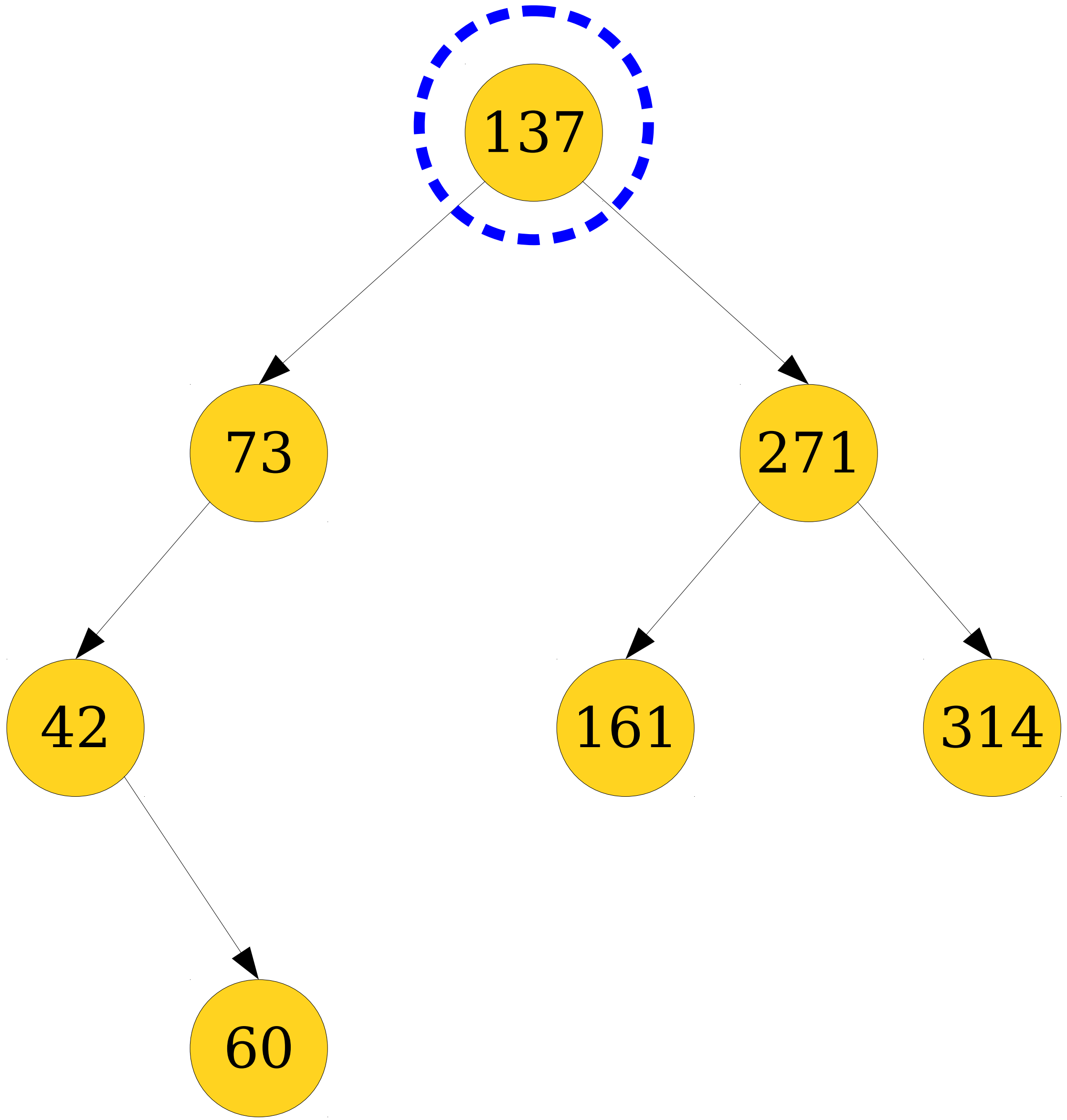

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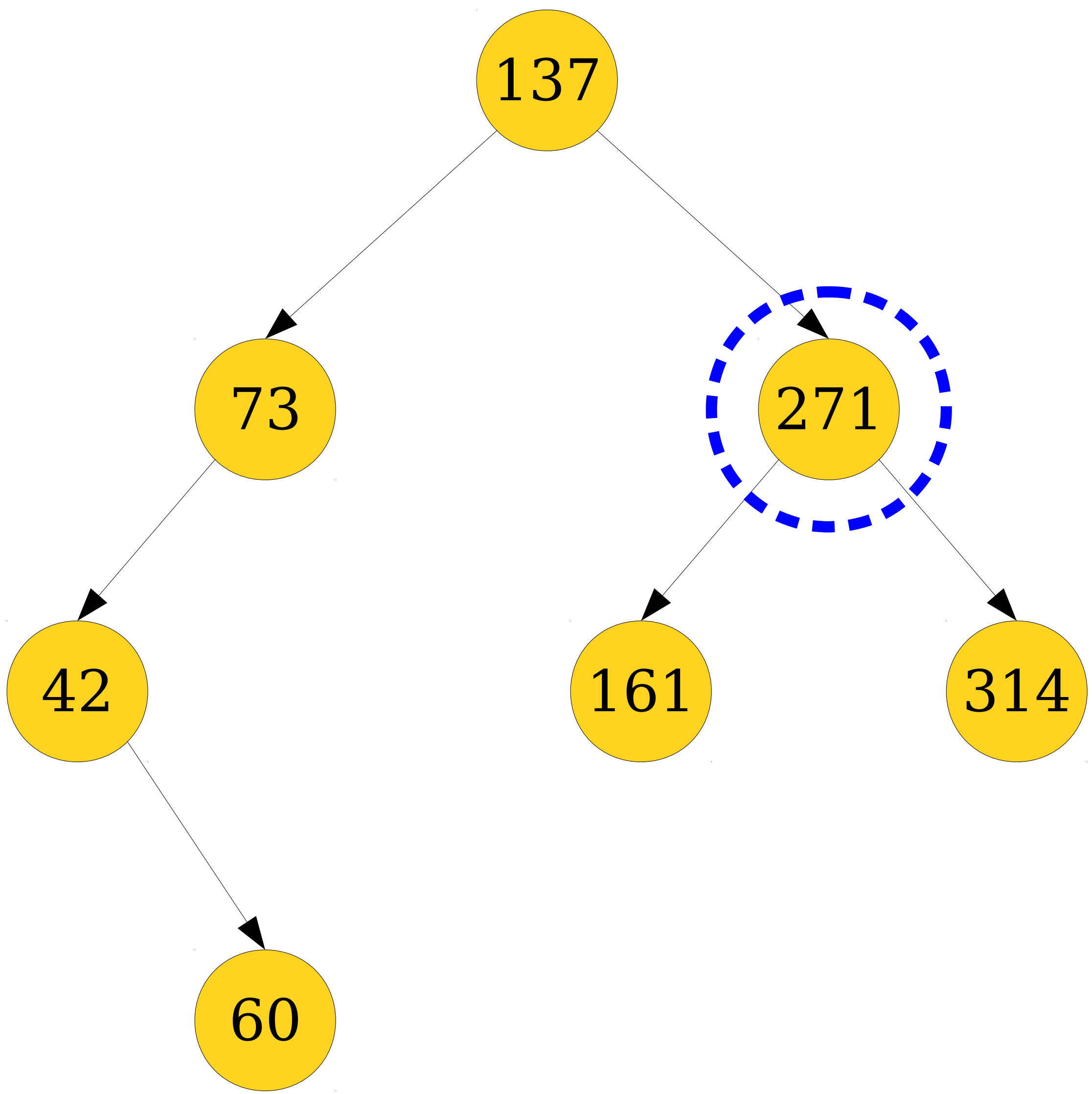

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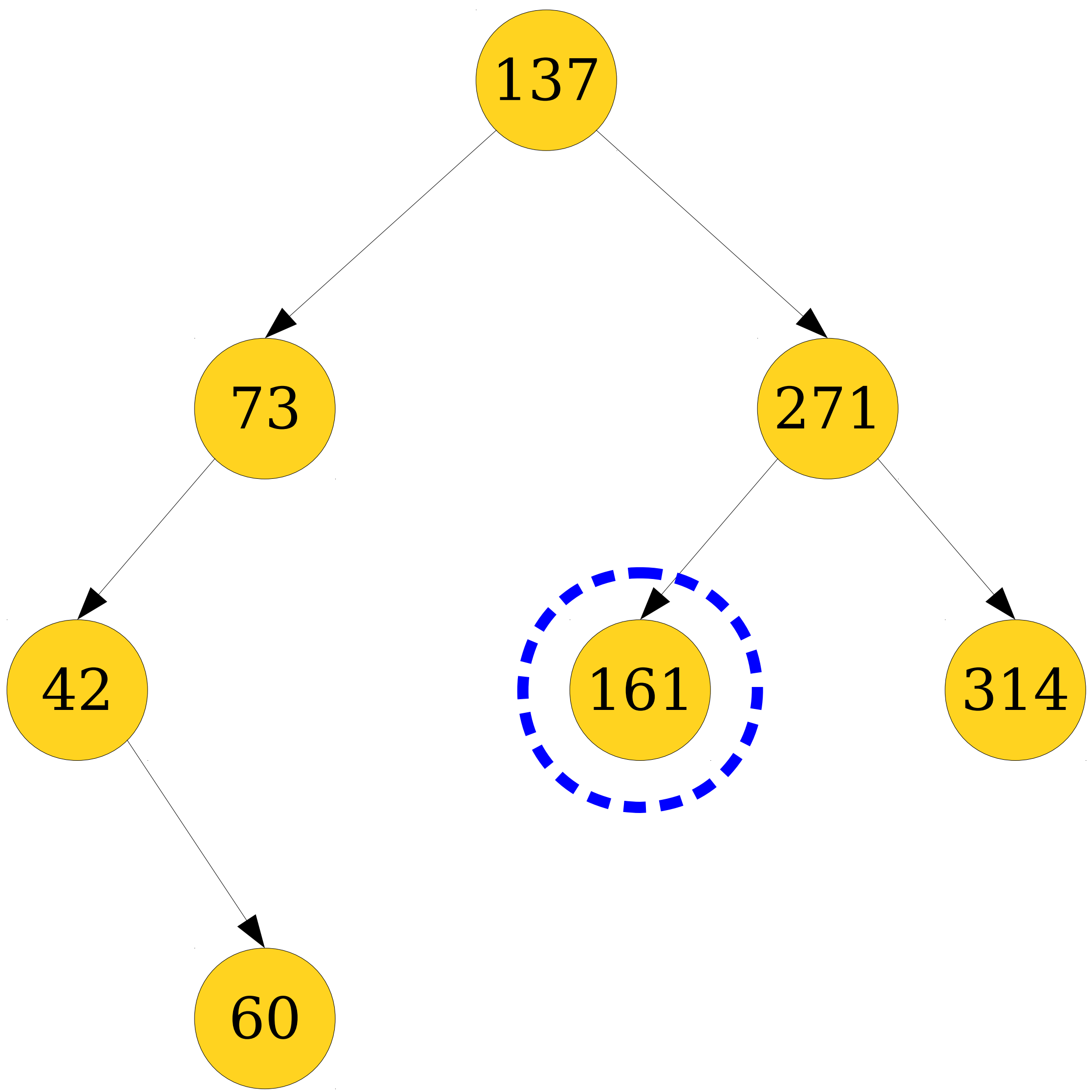

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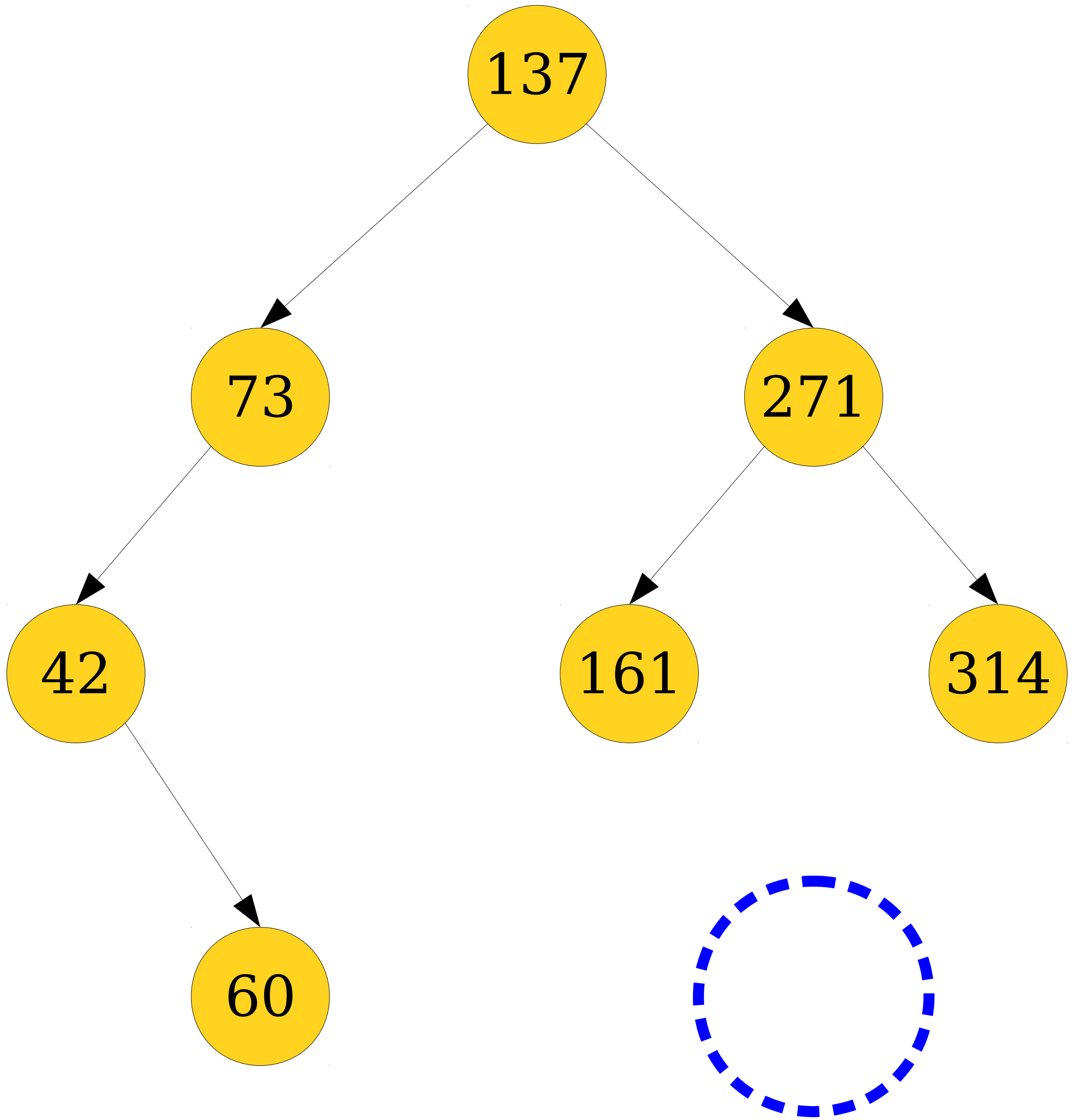

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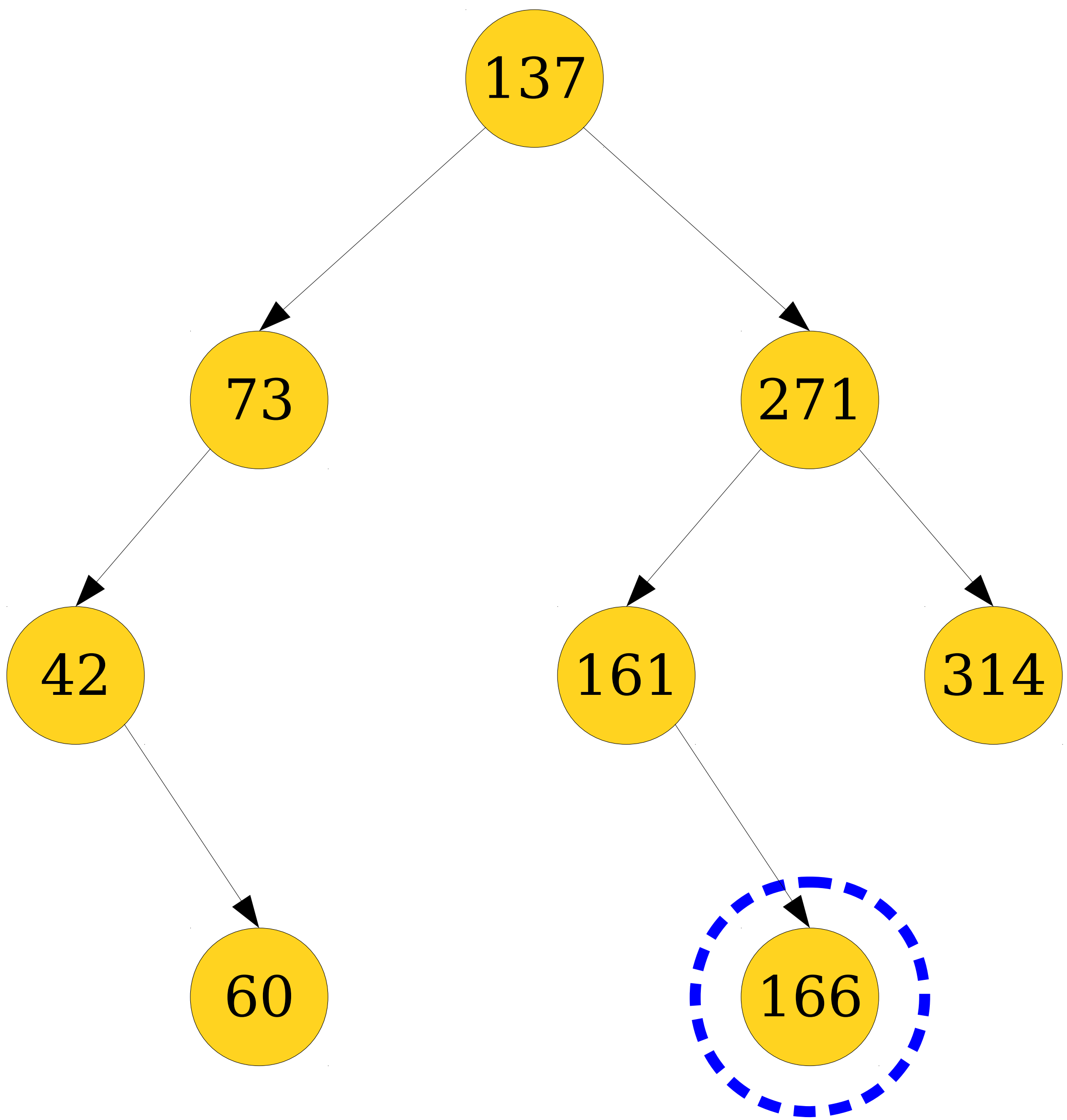

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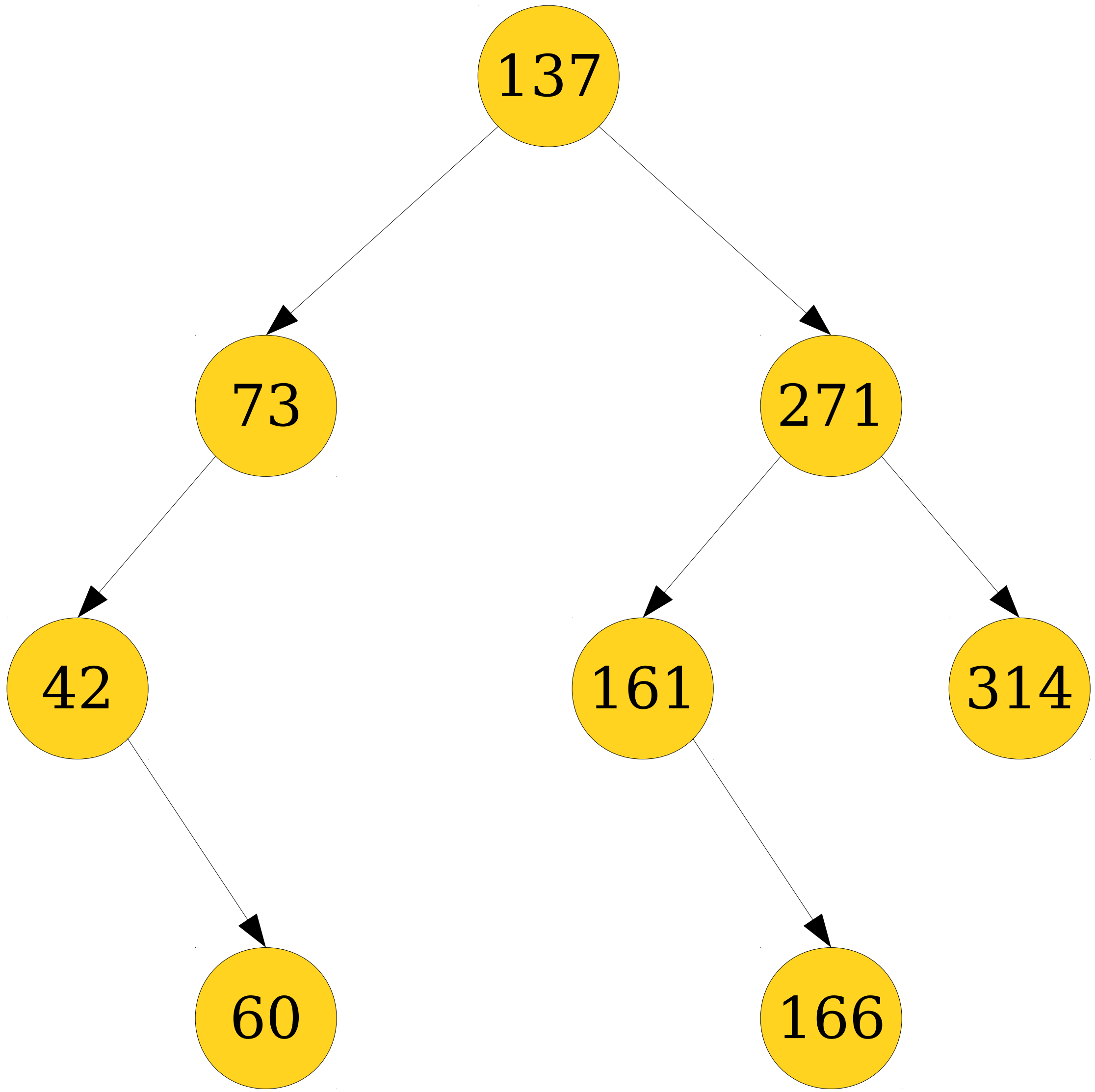

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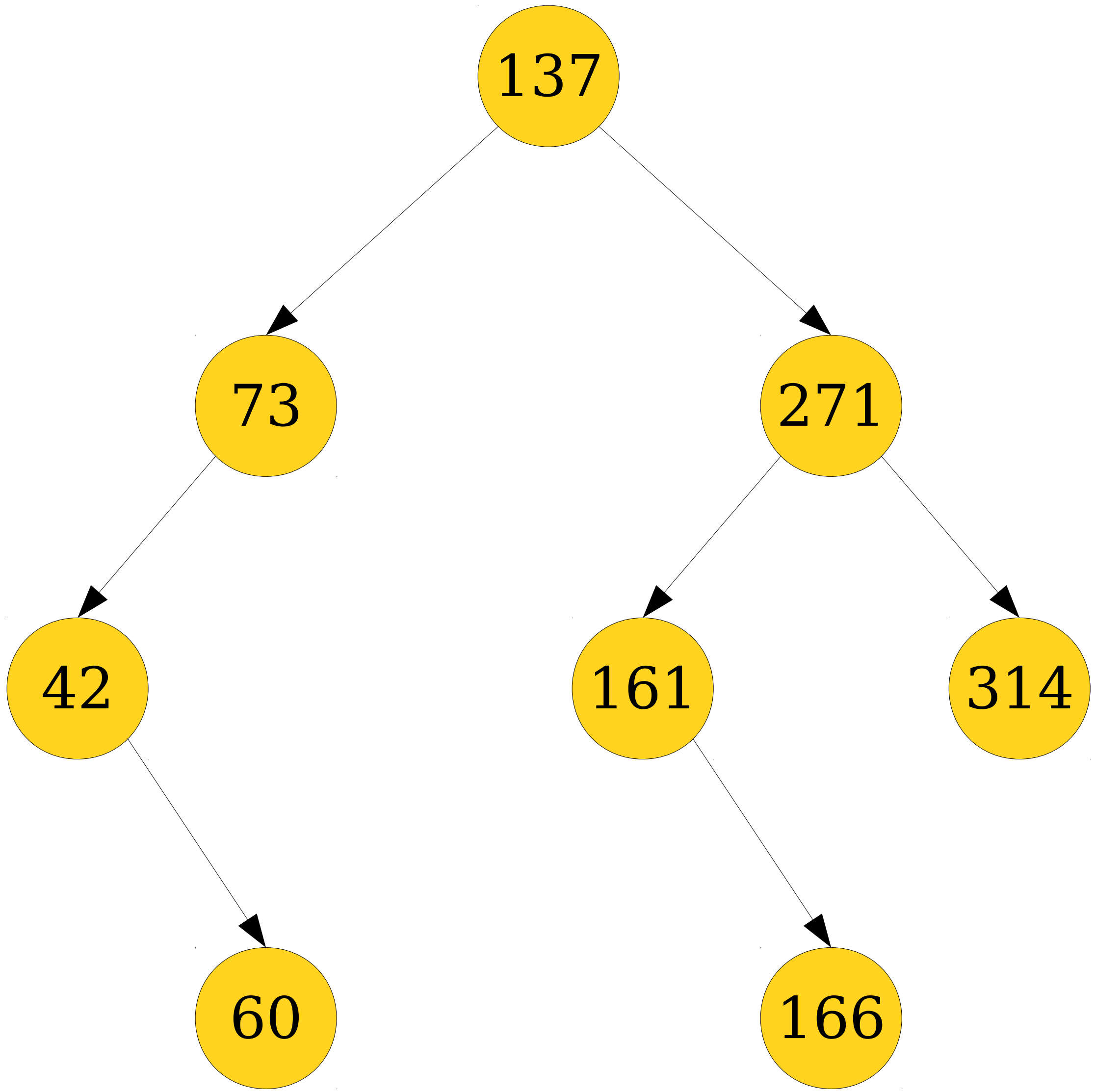

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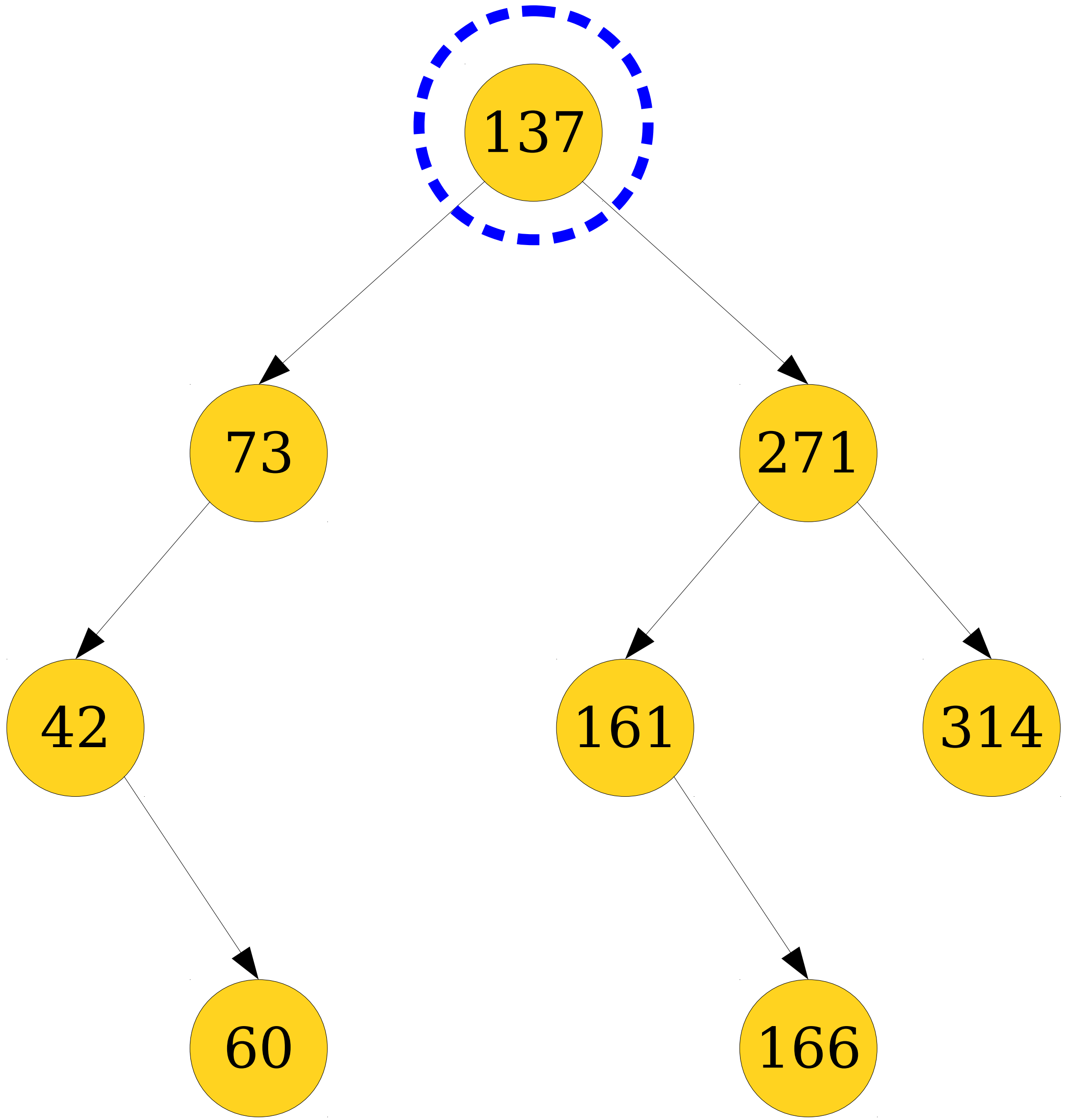

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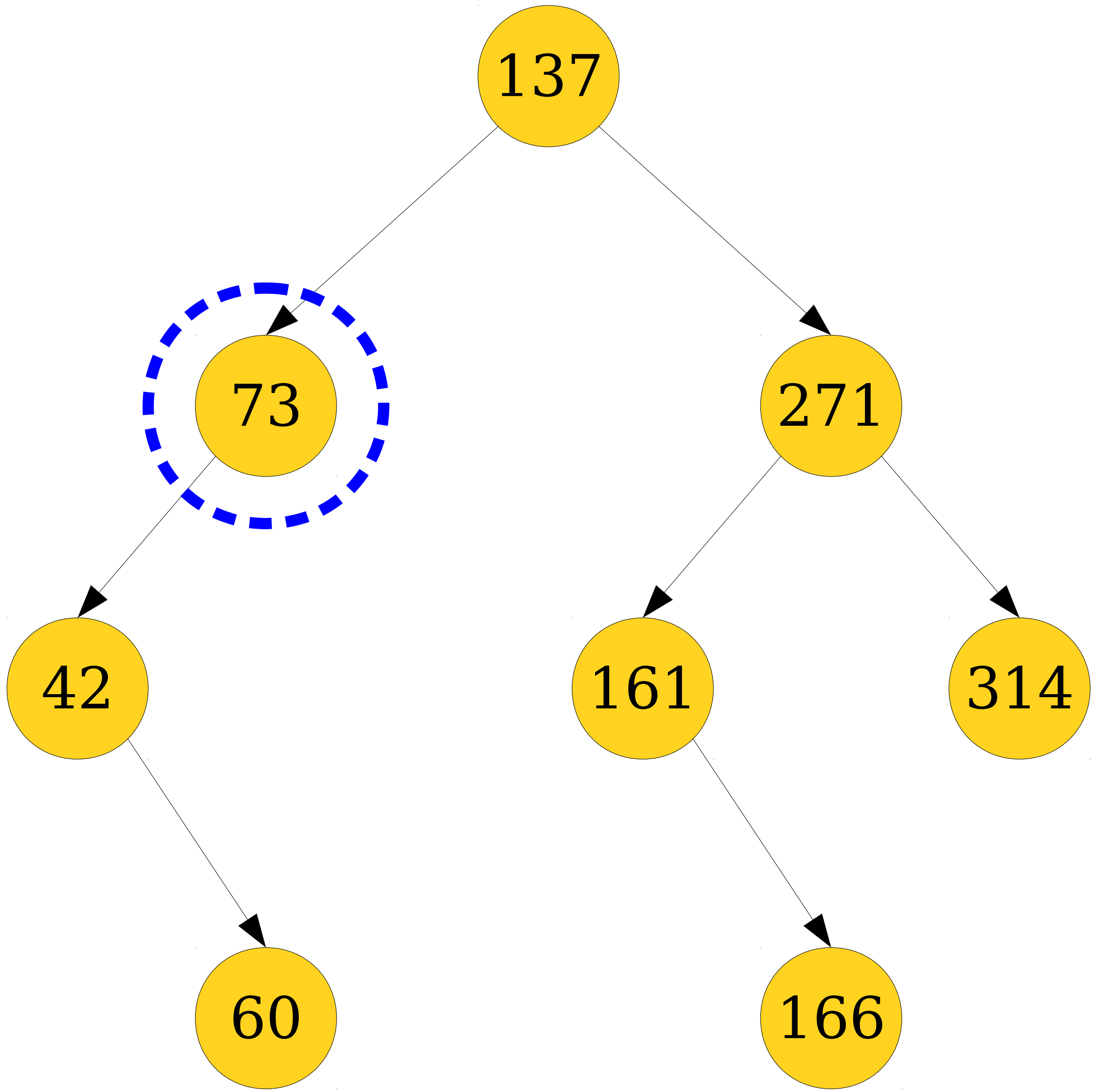

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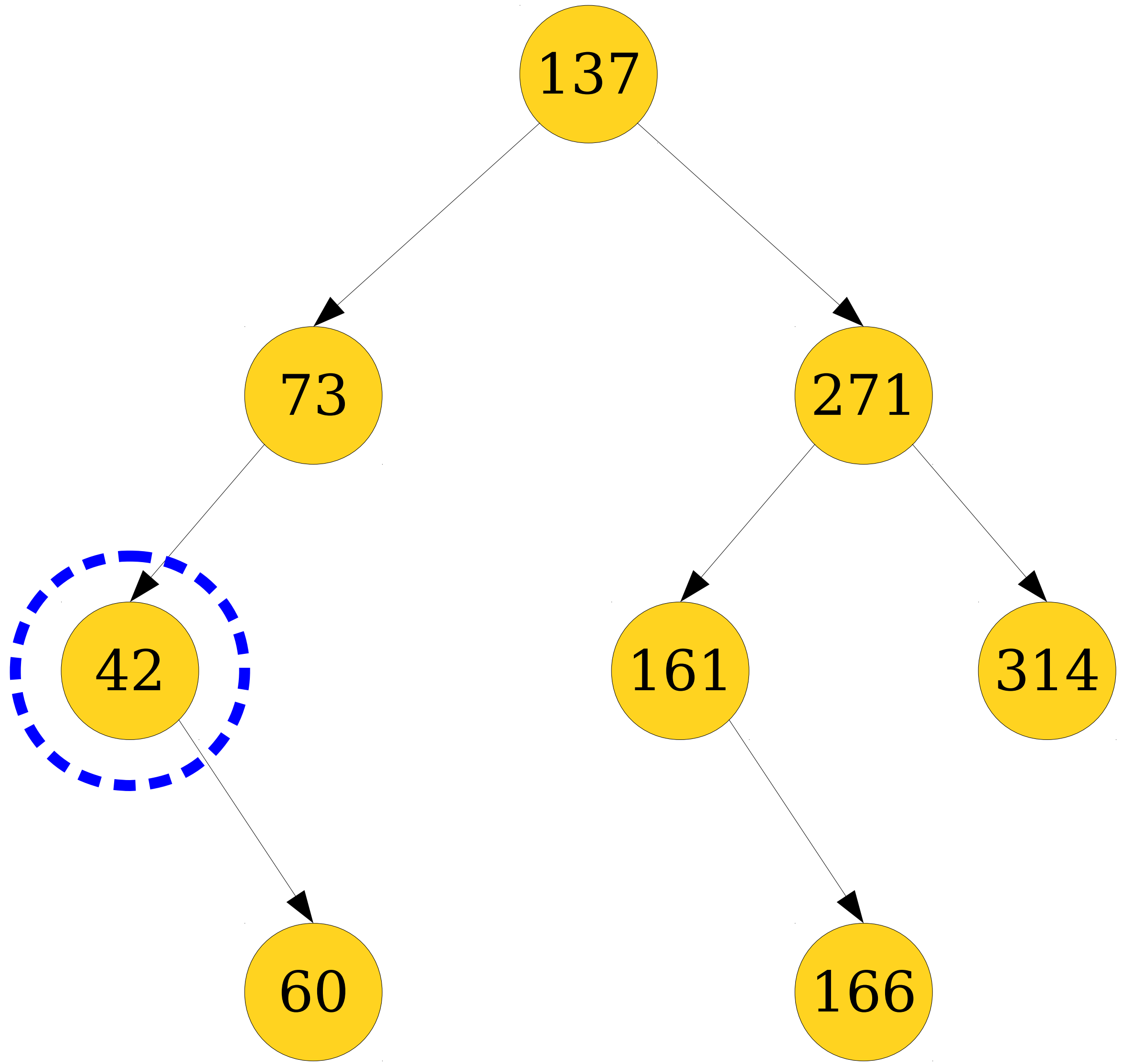

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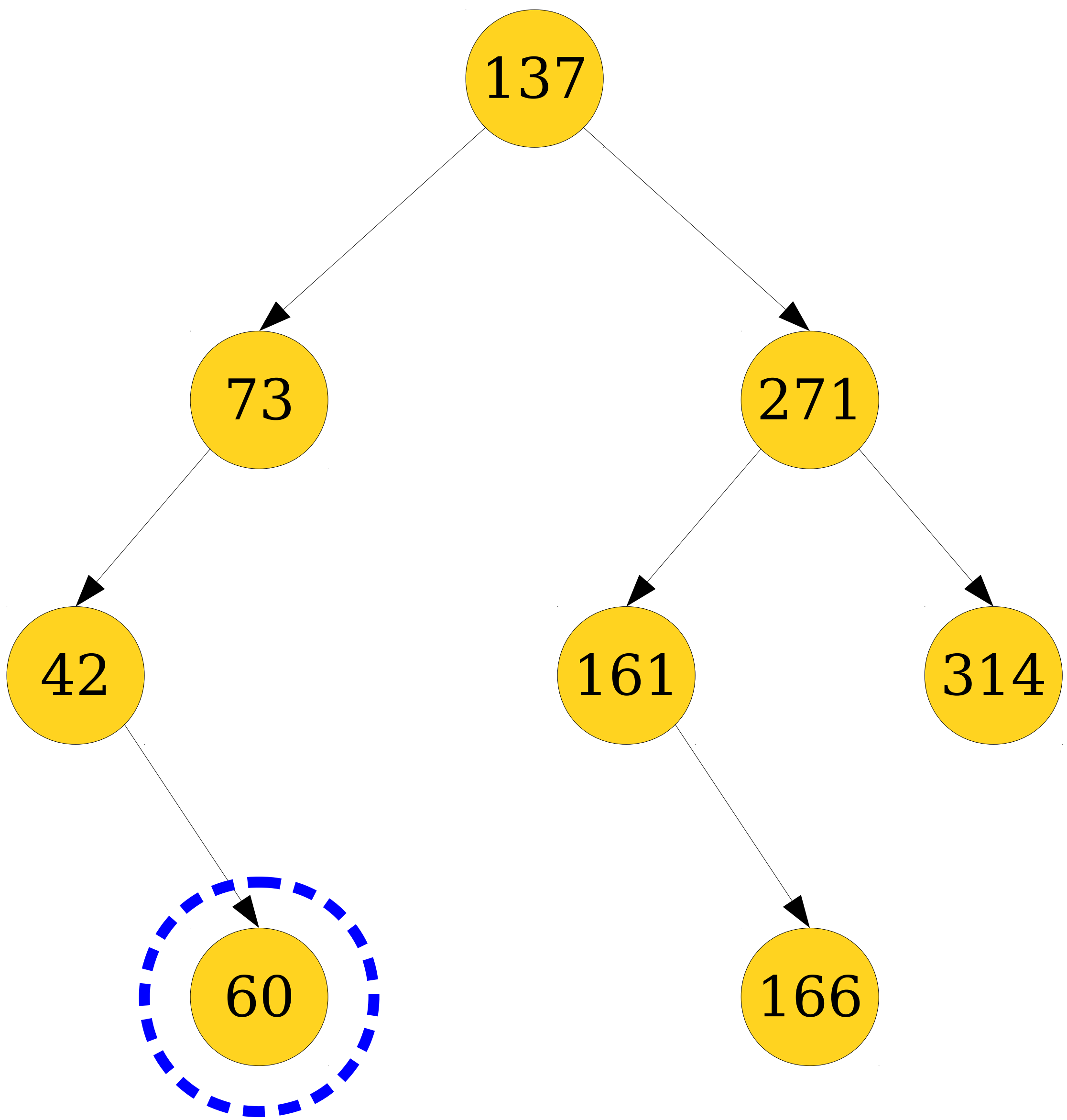

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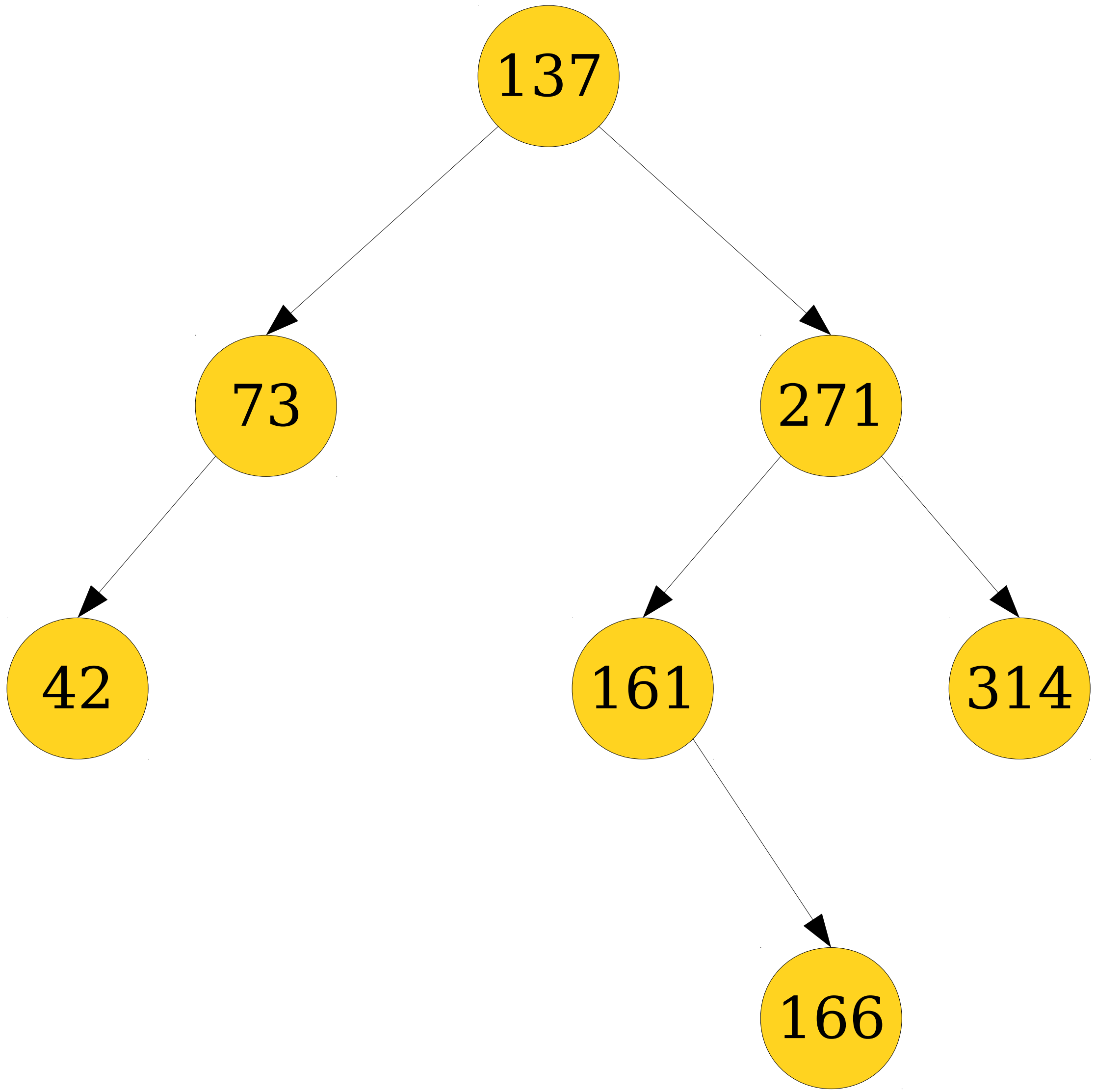

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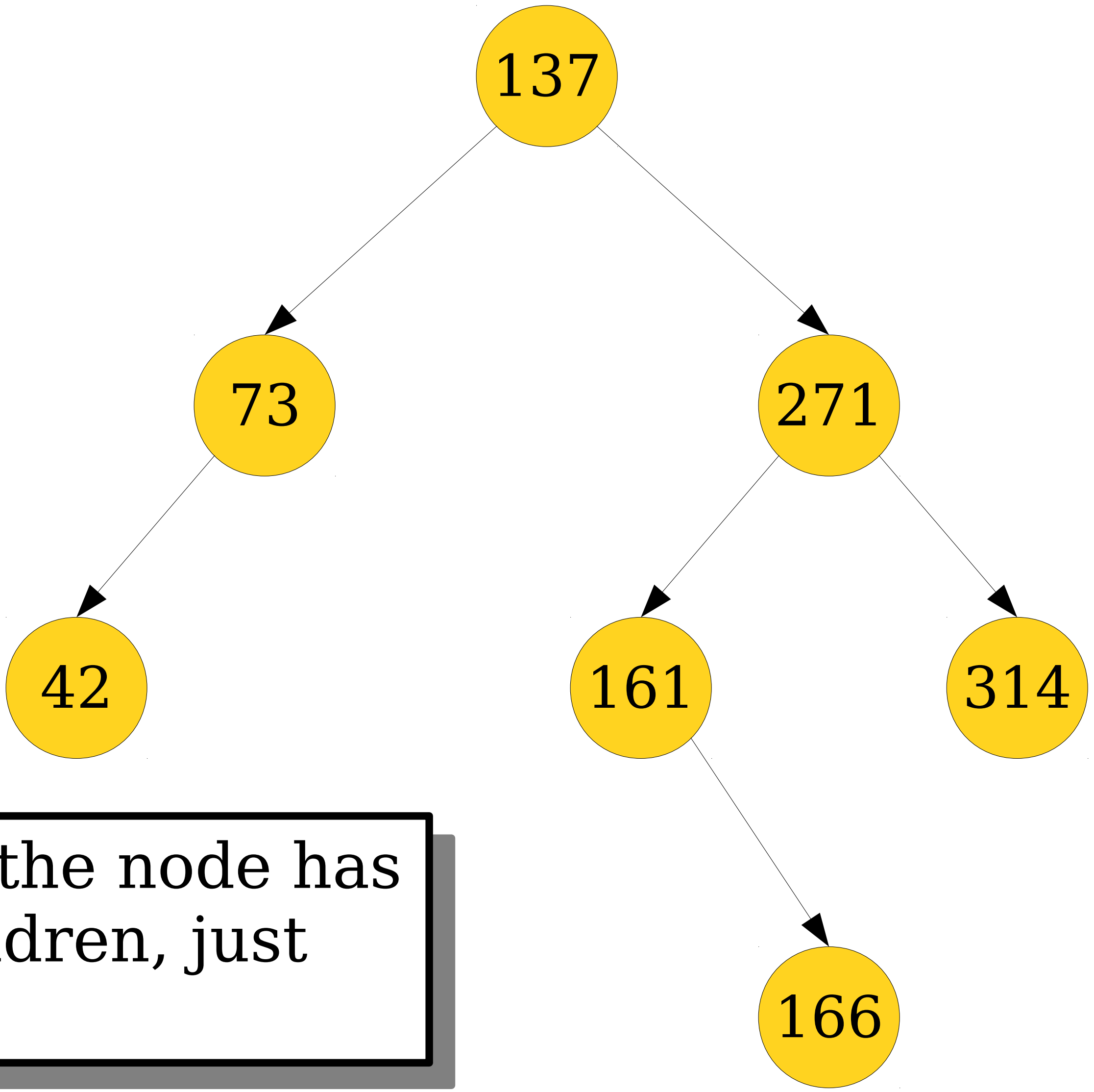
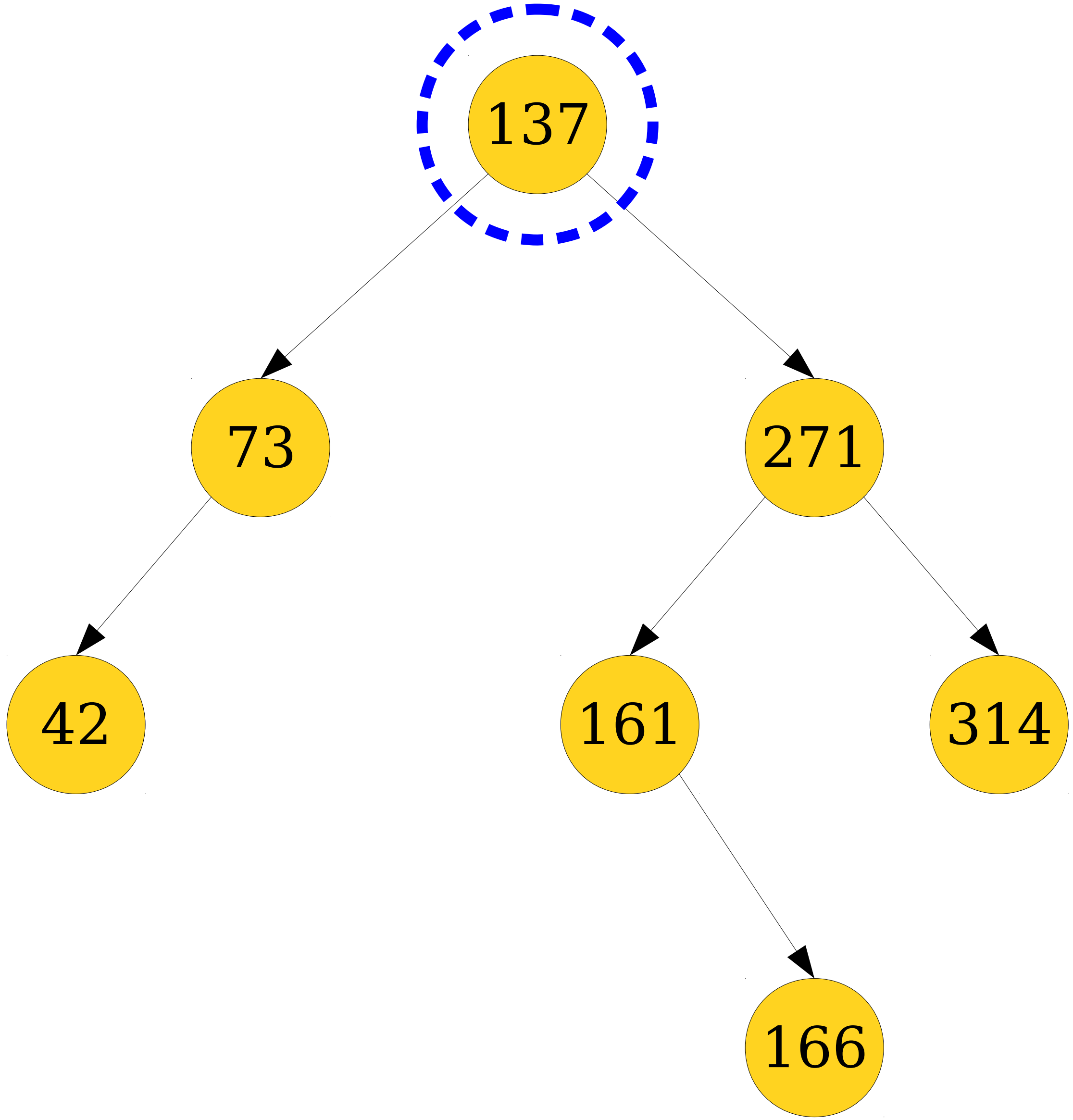


Case 1: If the node has just no children, just remove it.

Deleting from a BST



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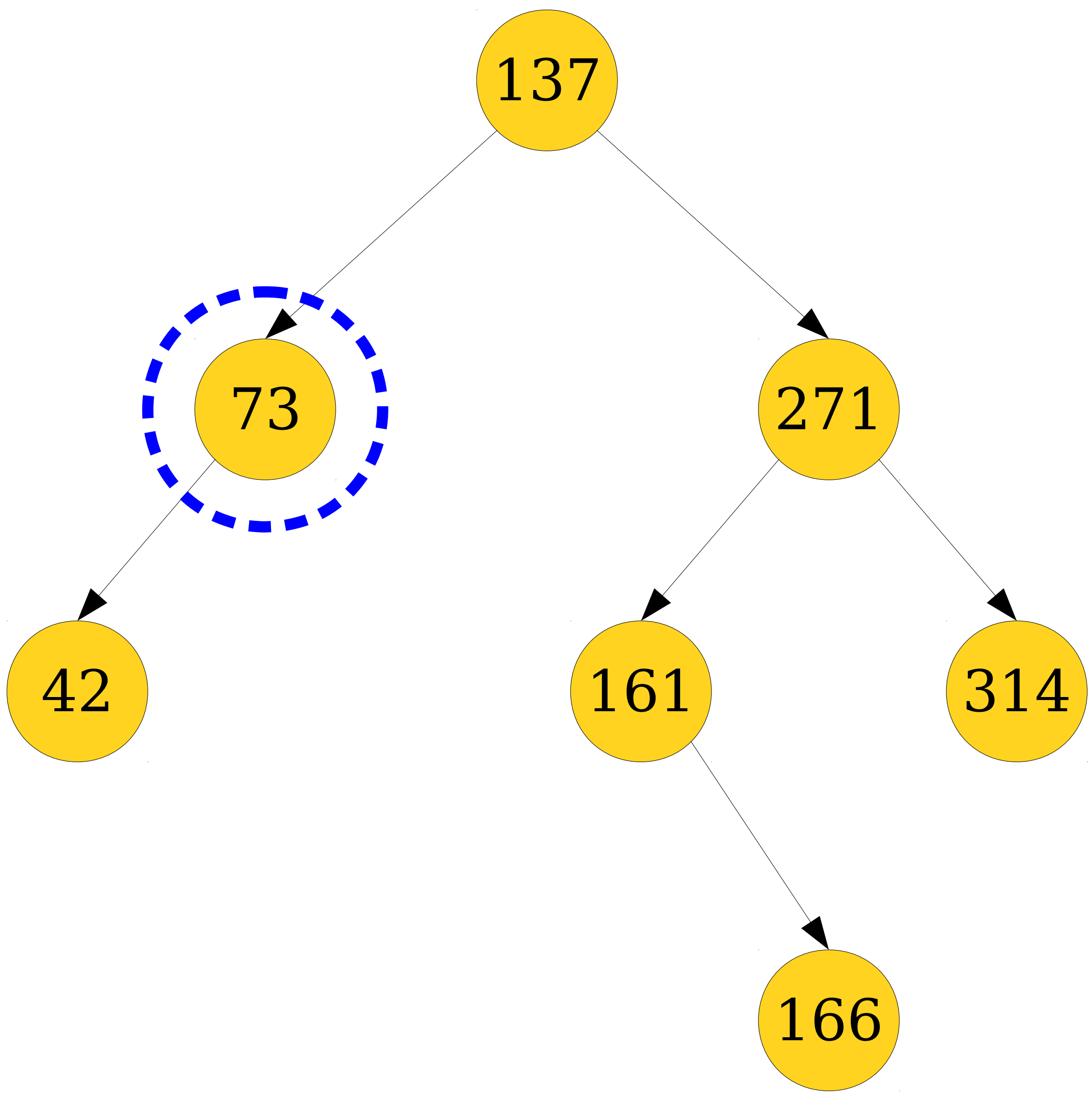

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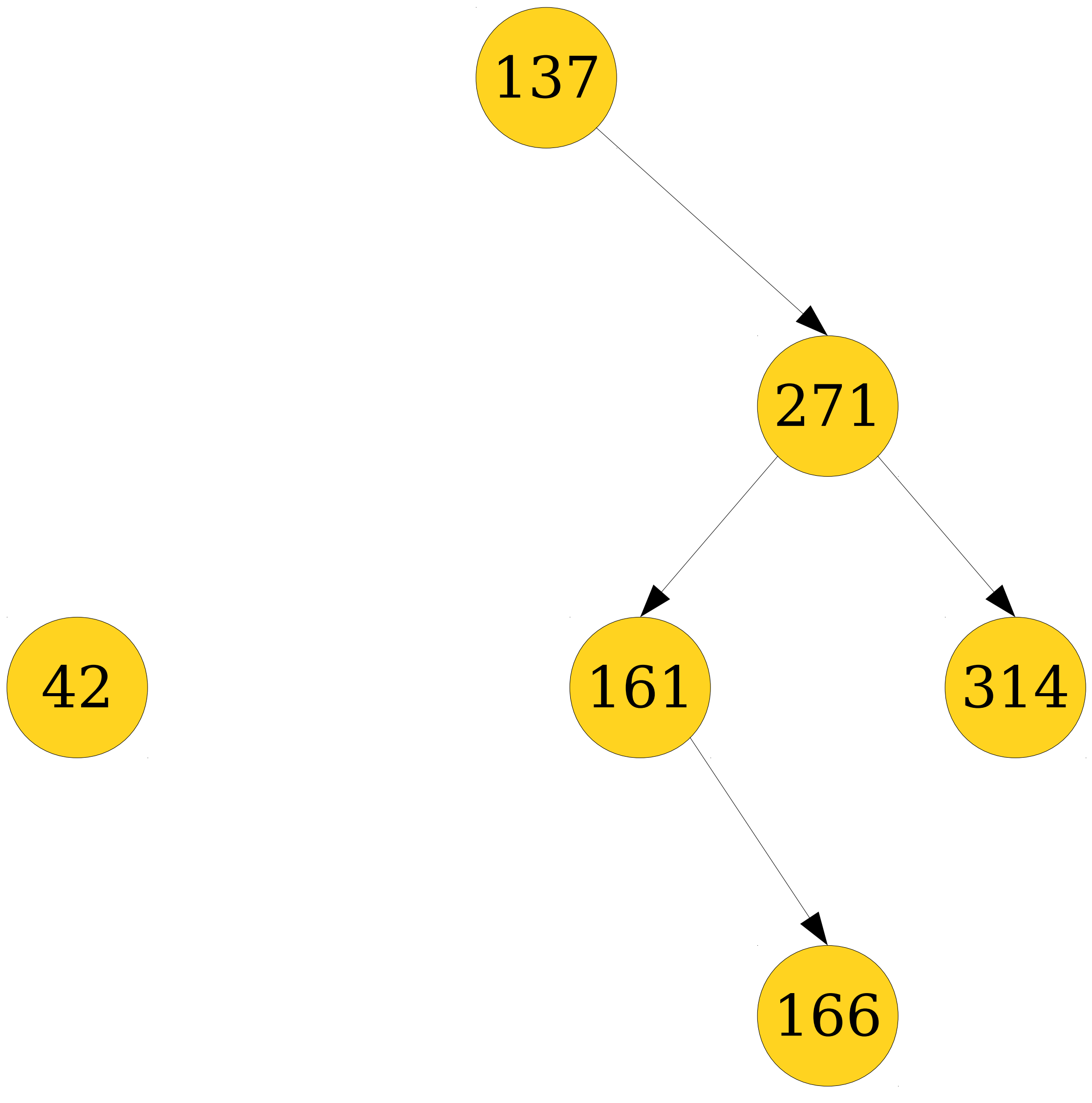

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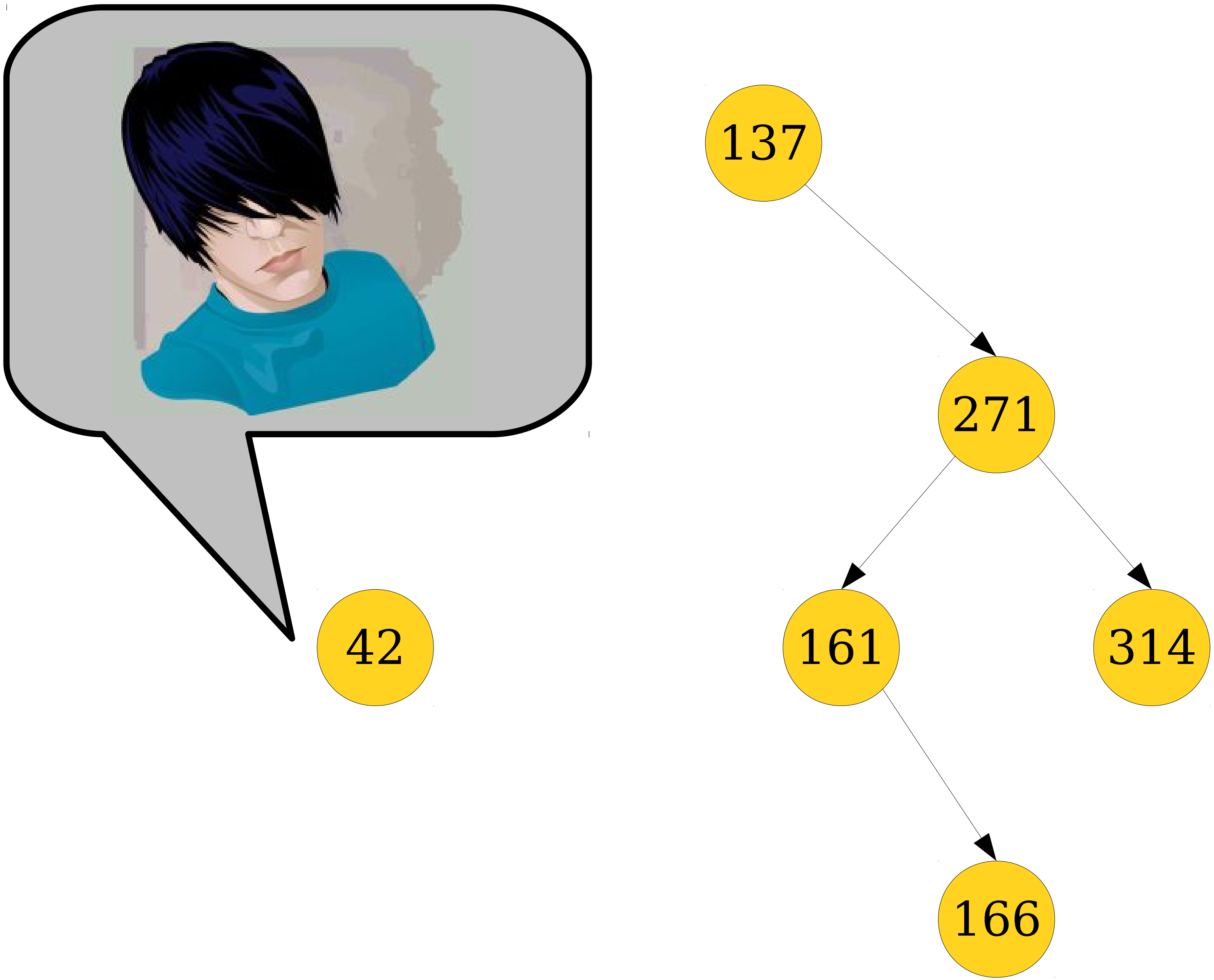

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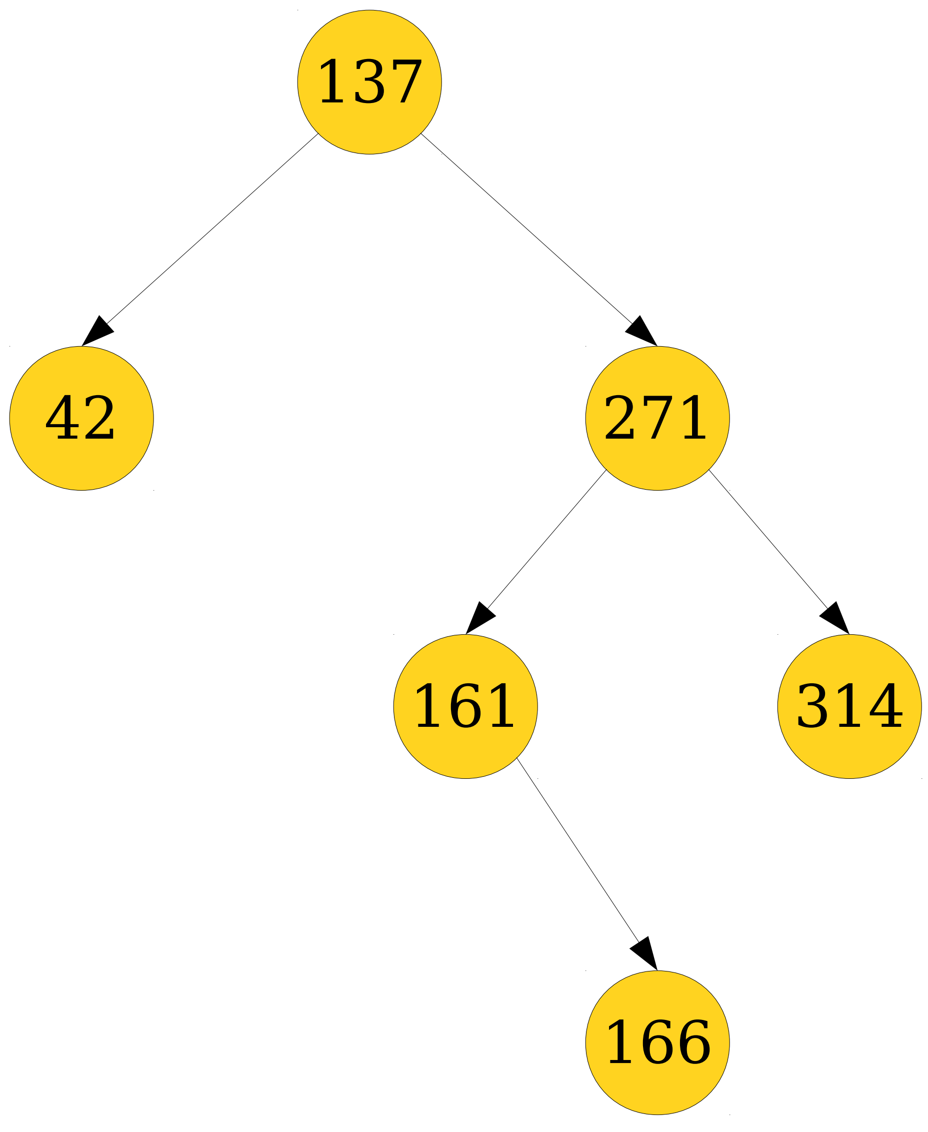

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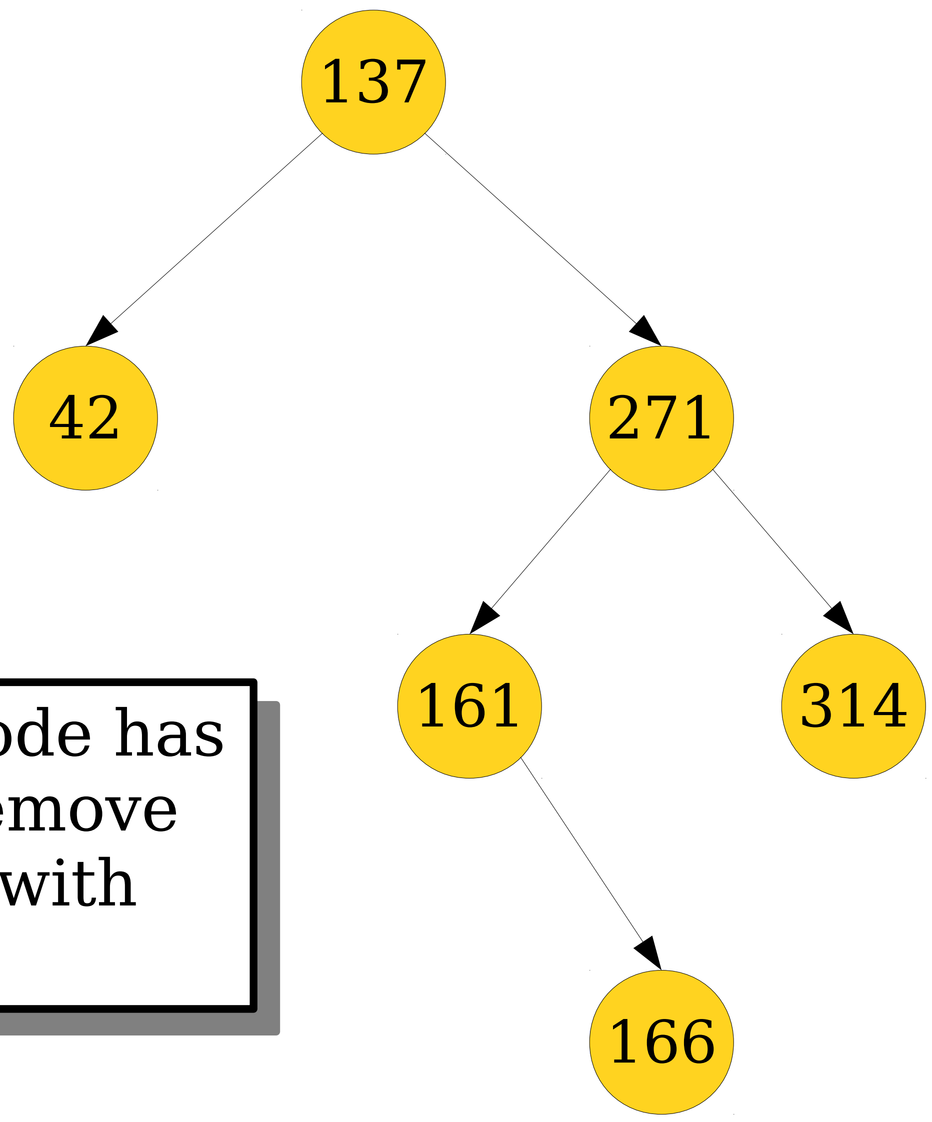

Case 2: If the node has just one child, remove it and replace it with its child.

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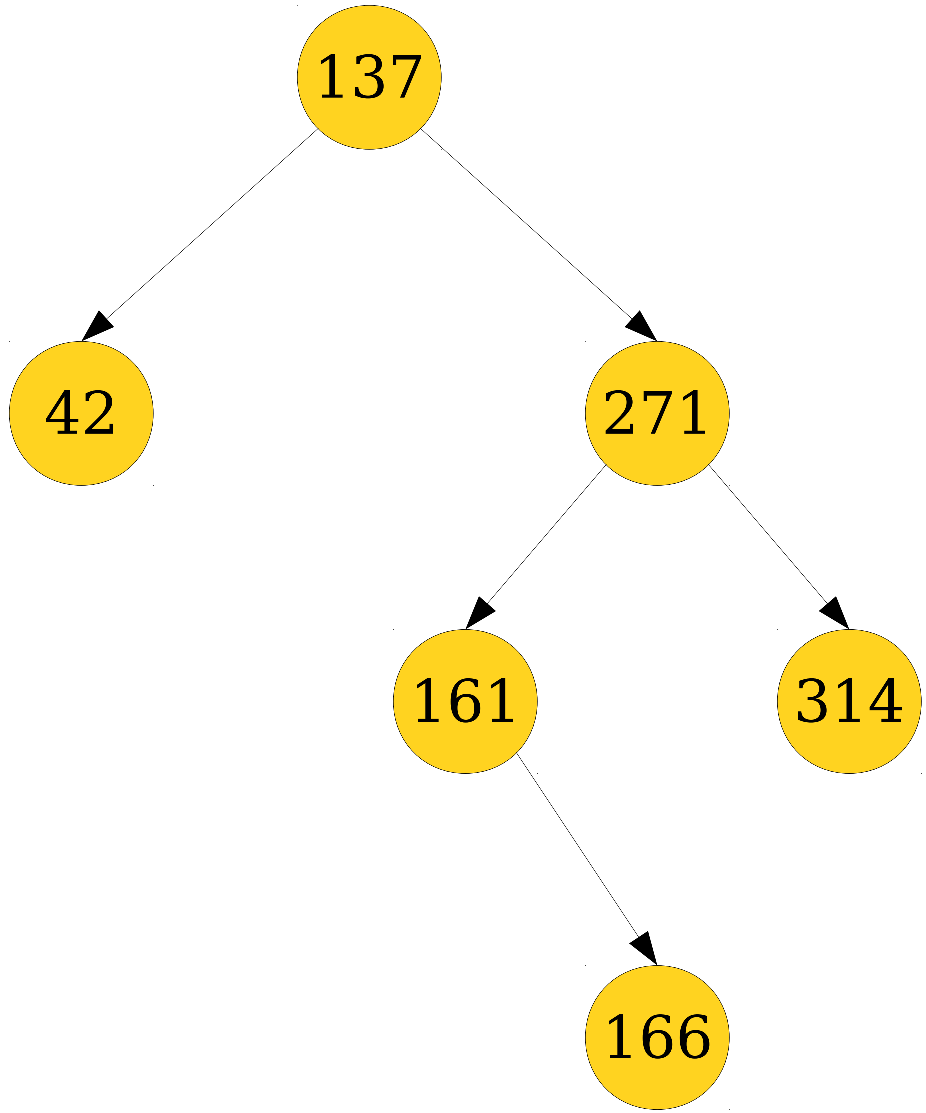

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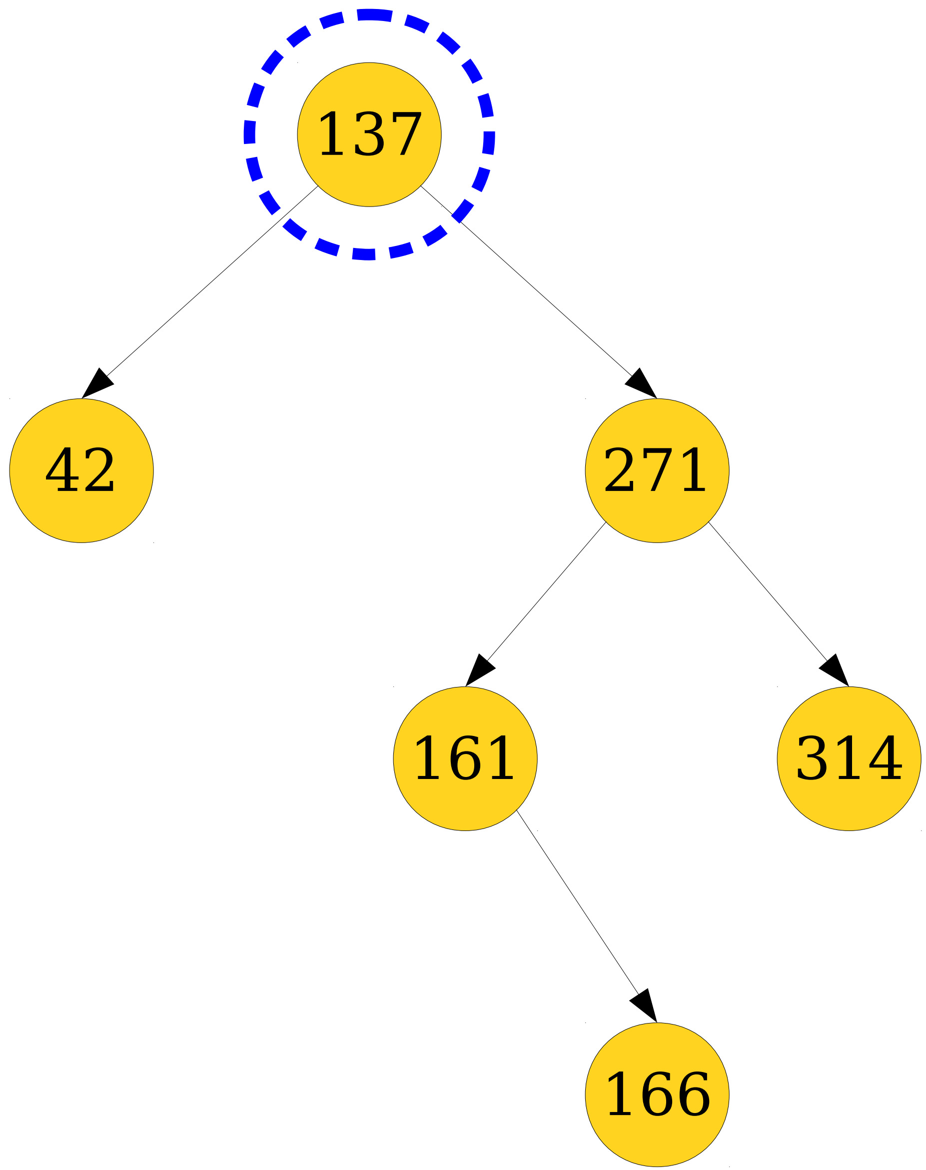

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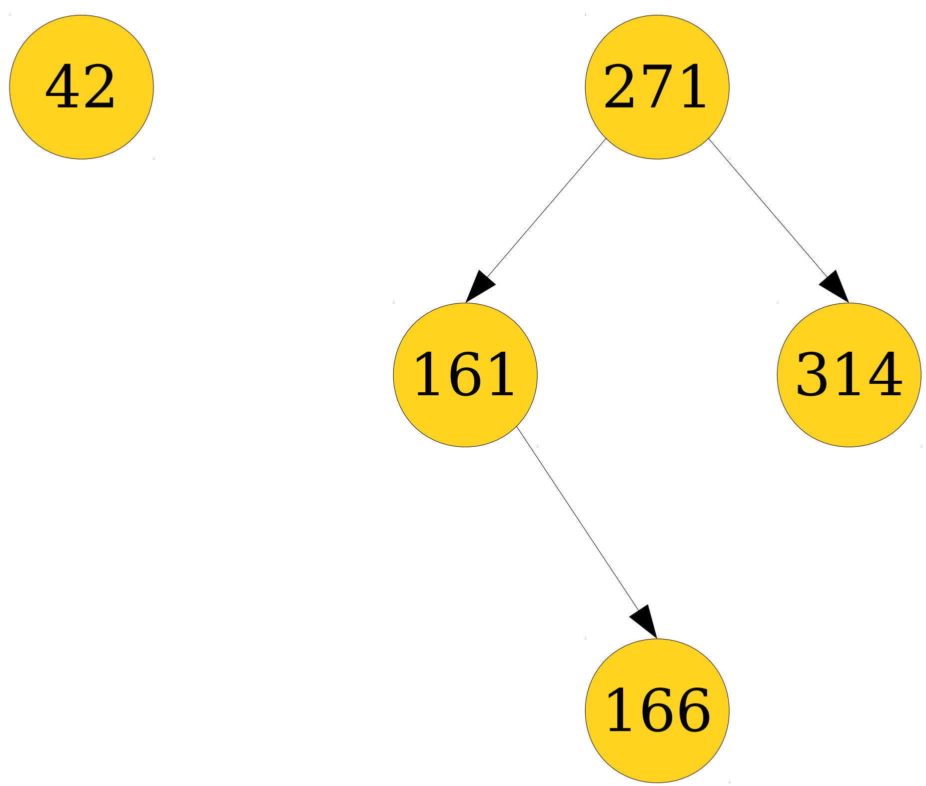

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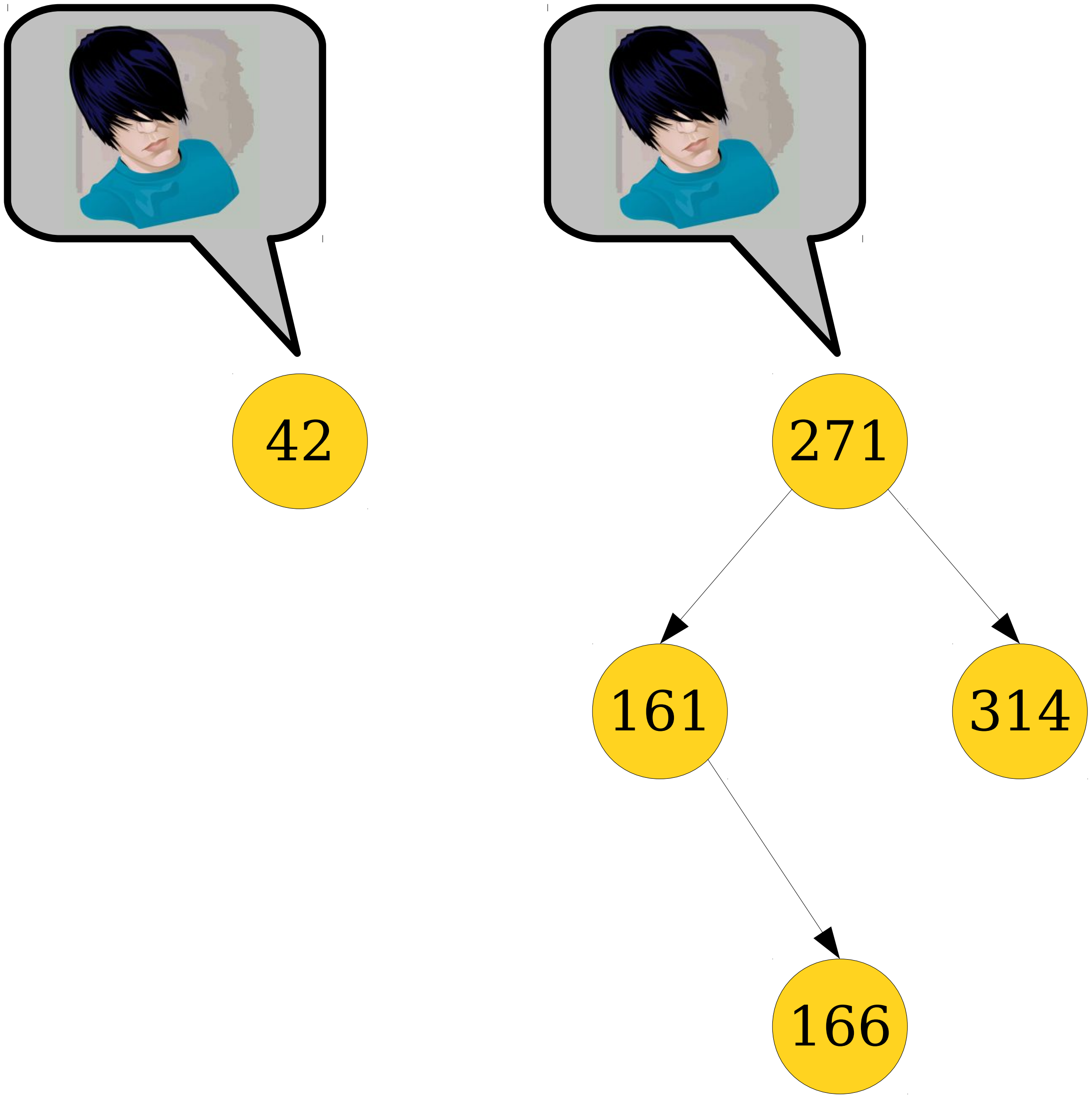

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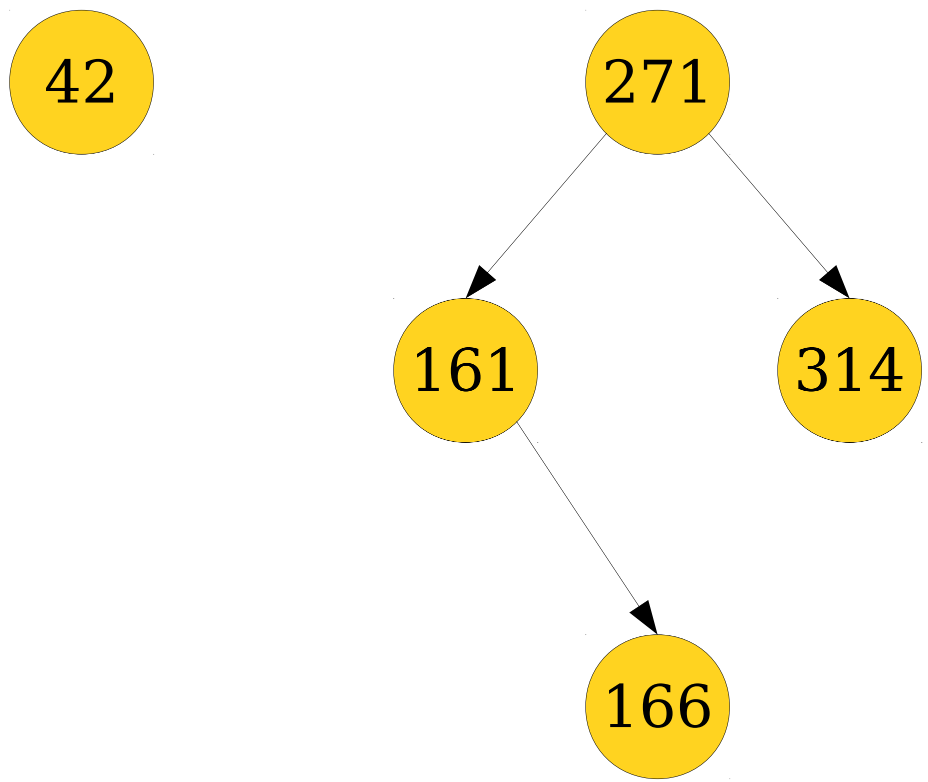

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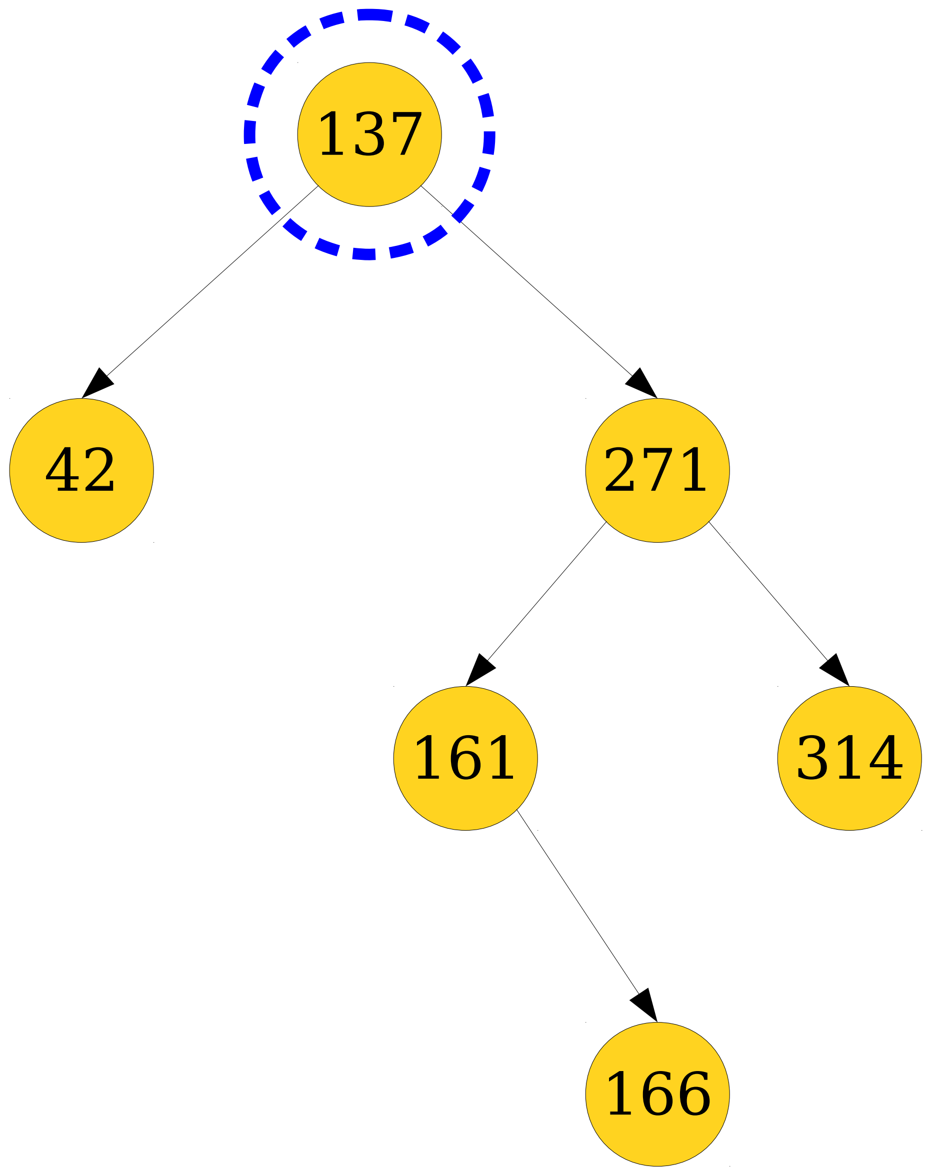

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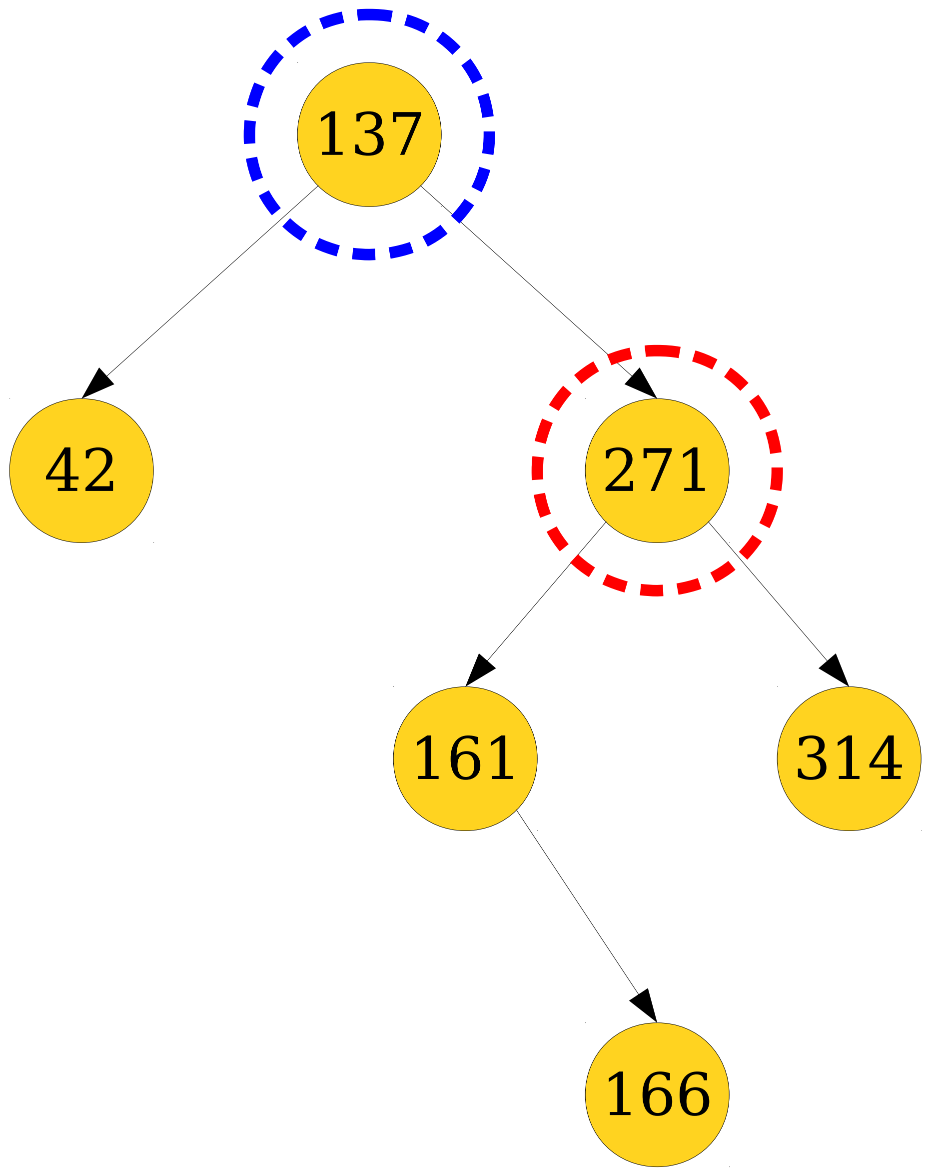

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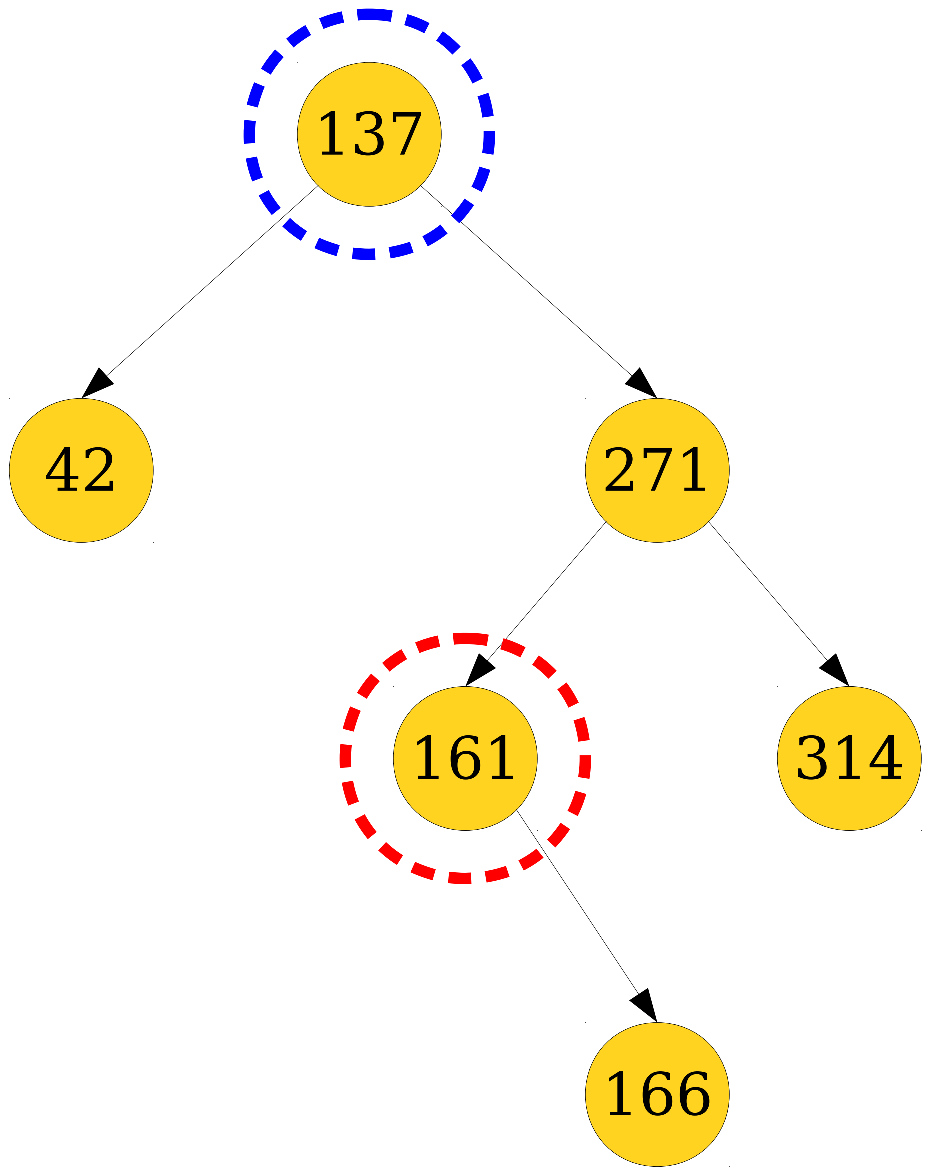

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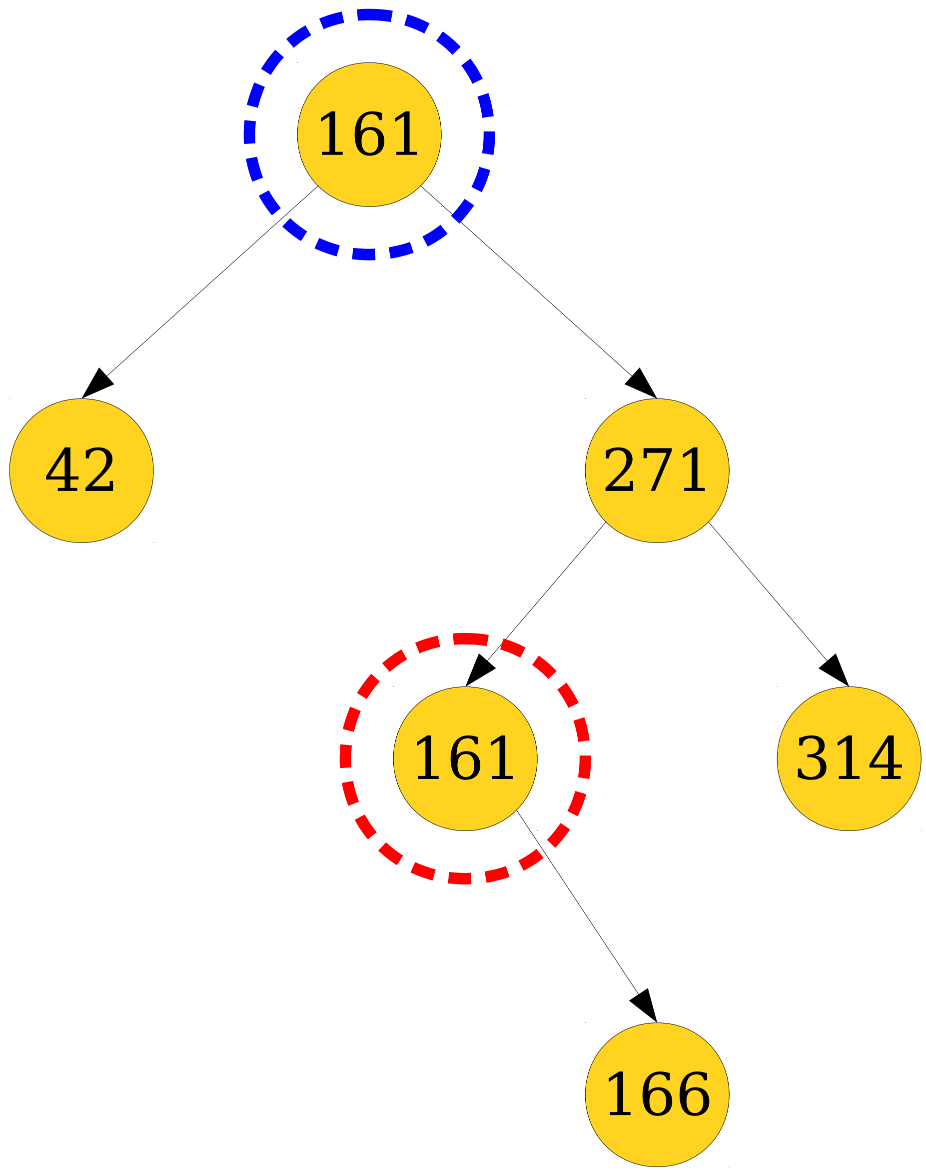

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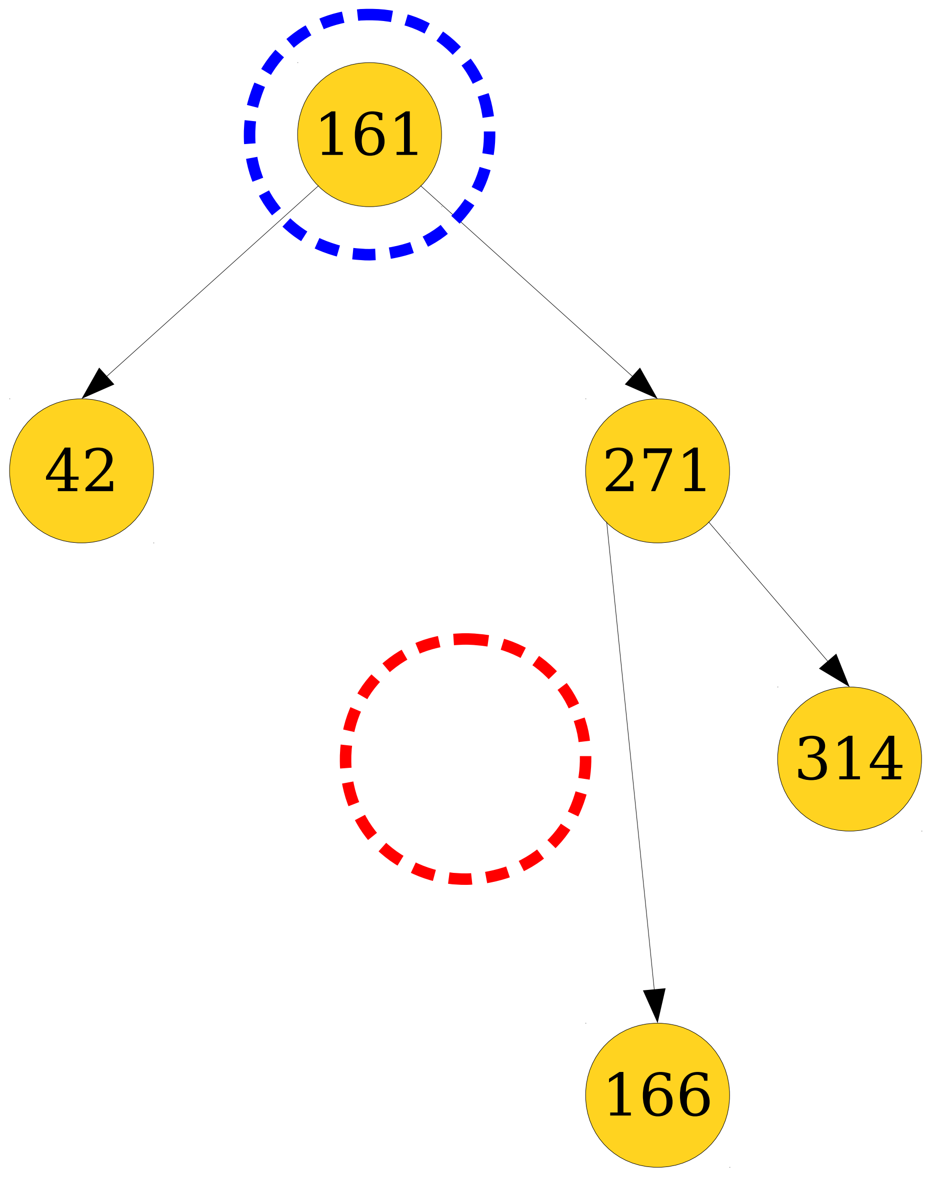

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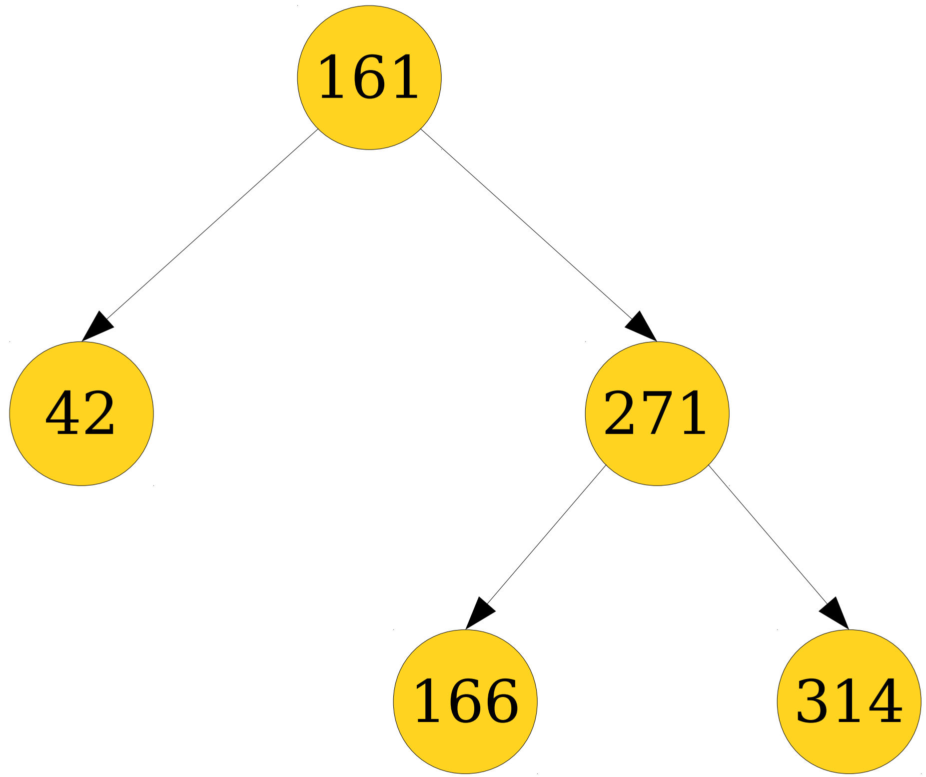

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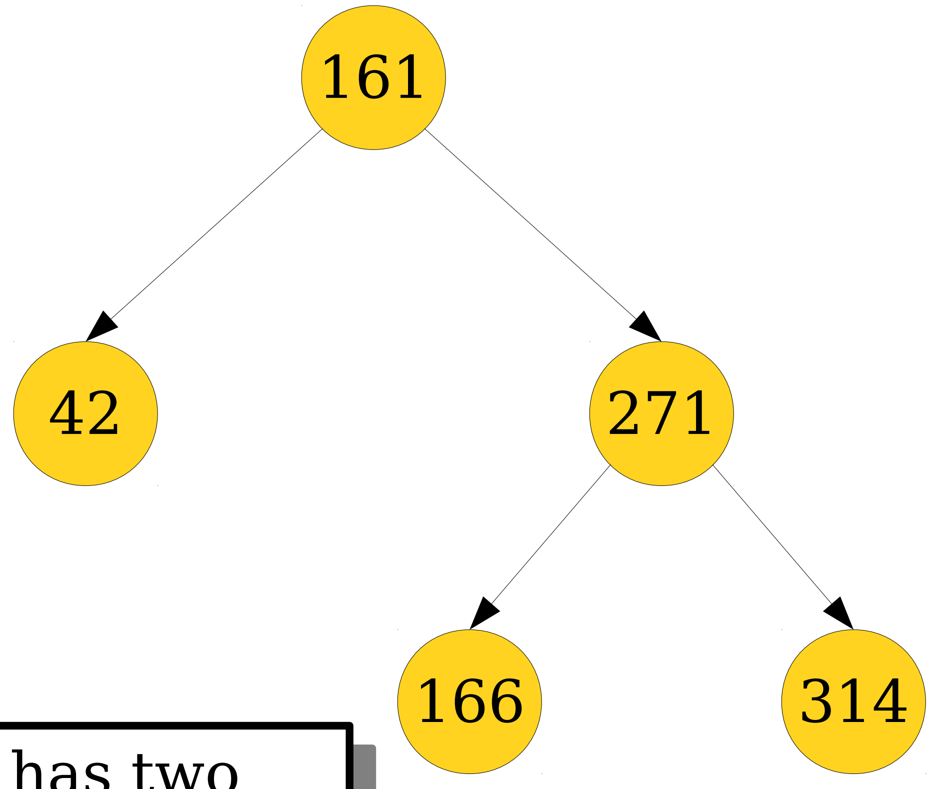


Case 3: If the node has two children, find its inorder successor (which has zero or one child), replace the node's key with its successor's key, then delete its successor.

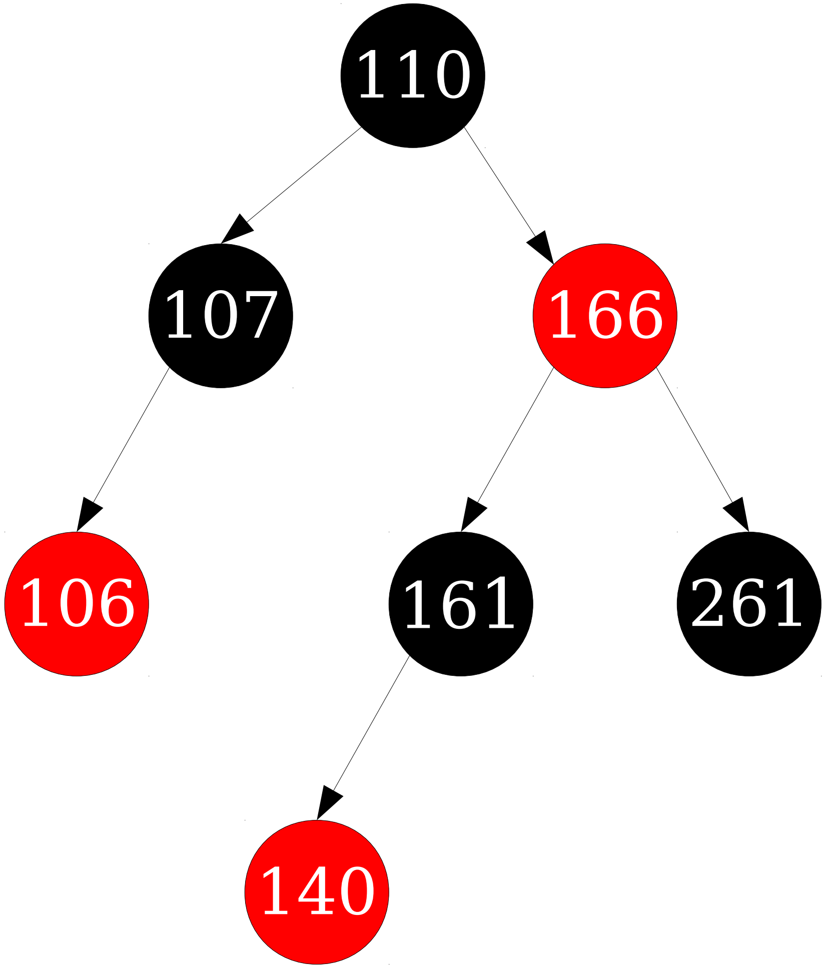
Runtime Analysis

- The time complexity of all these operations is $O(h)$, where h is the height of the tree.
  - Represents the longest path we can take.
- In the best case, $h = O(\log n)$ and all operations take time $O(\log n)$.
- In the worst case, $h = \Theta(n)$ and some operations will take time $\Theta(n)$.
- ***Challenge:*** How do you efficiently keep the height of a tree low?

A Glimpse of Red/Black Trees

Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



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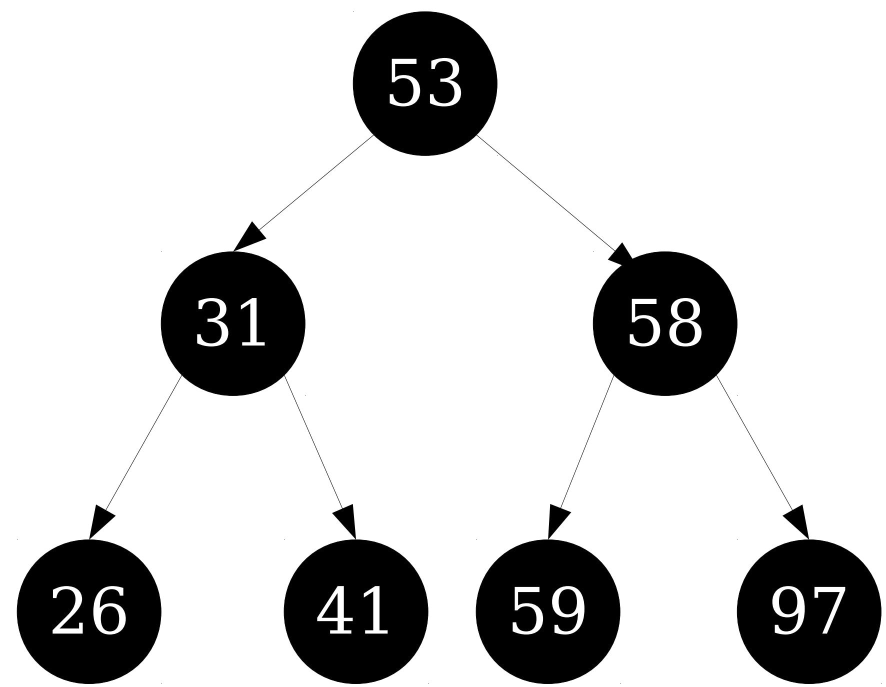

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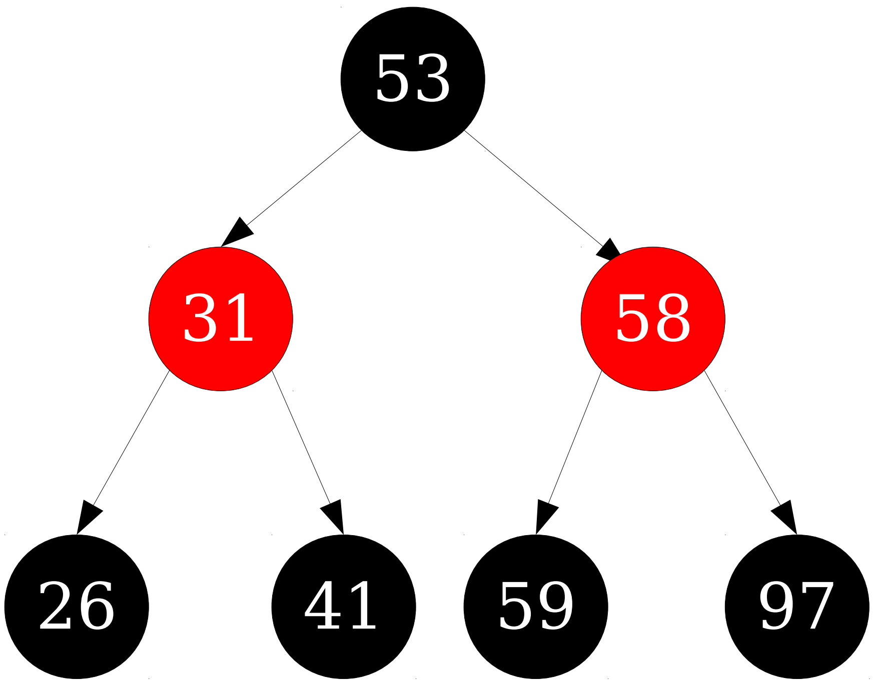

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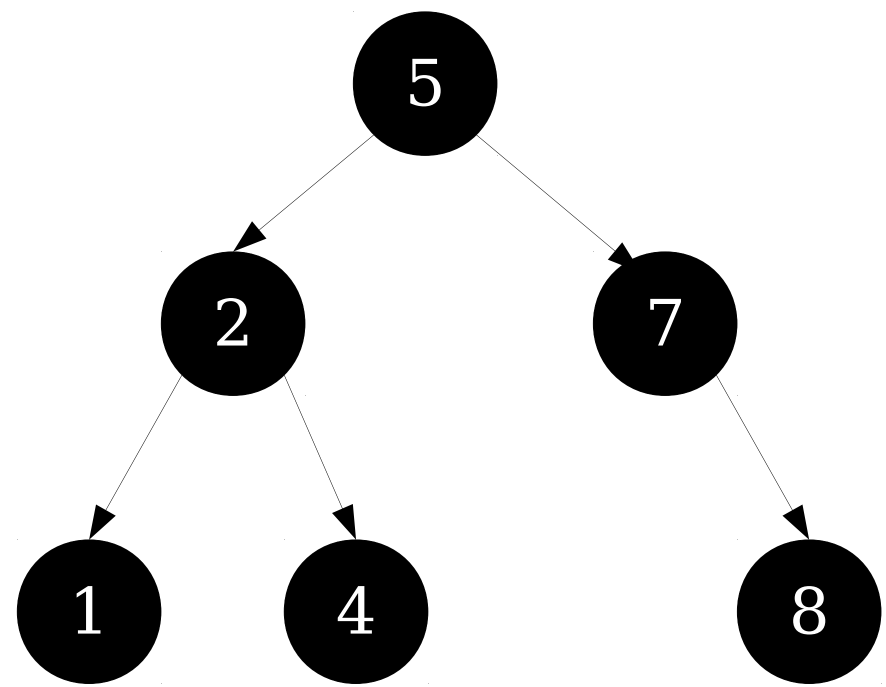

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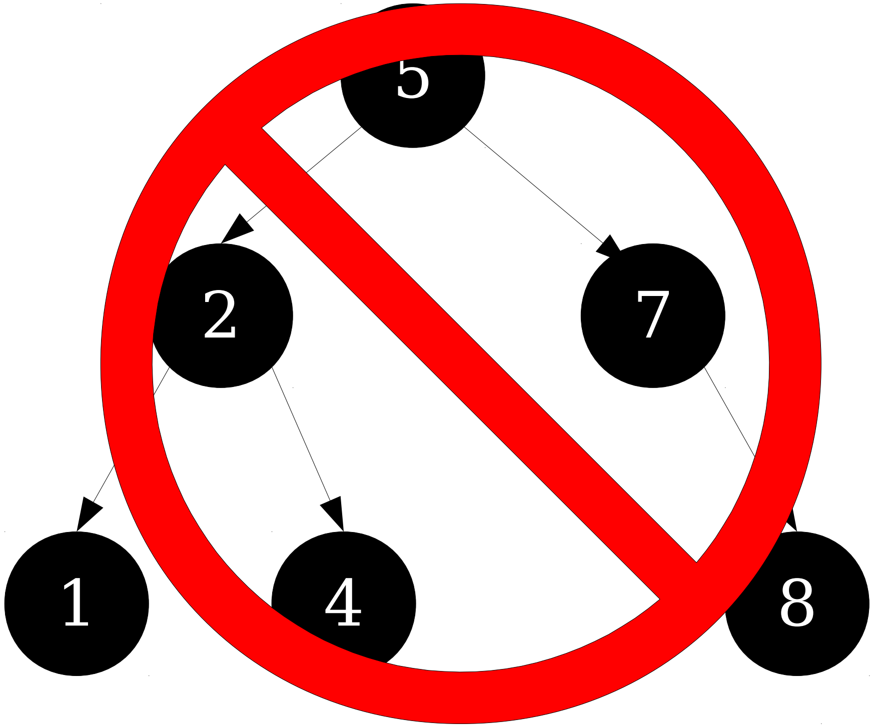

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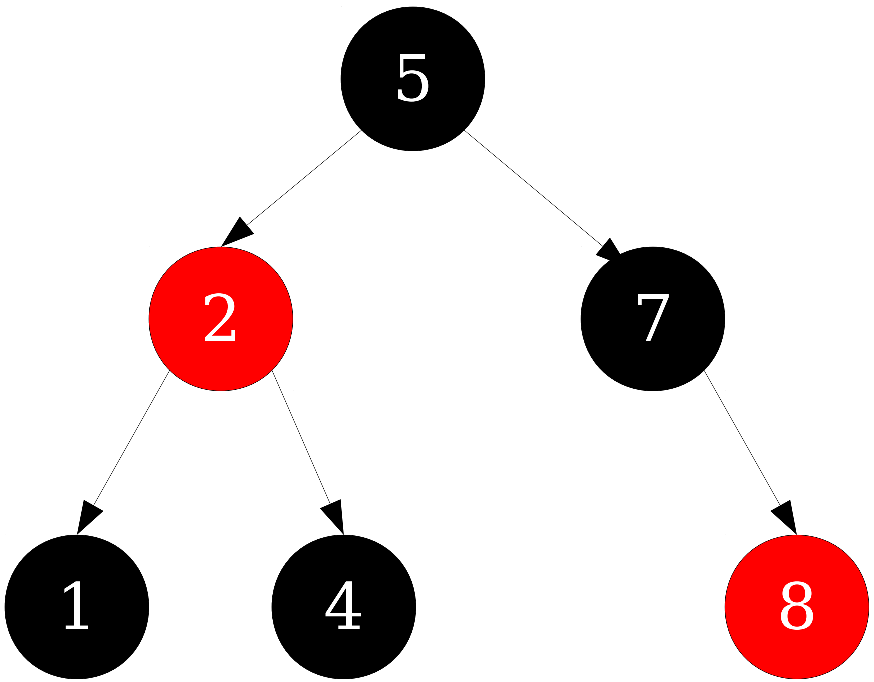
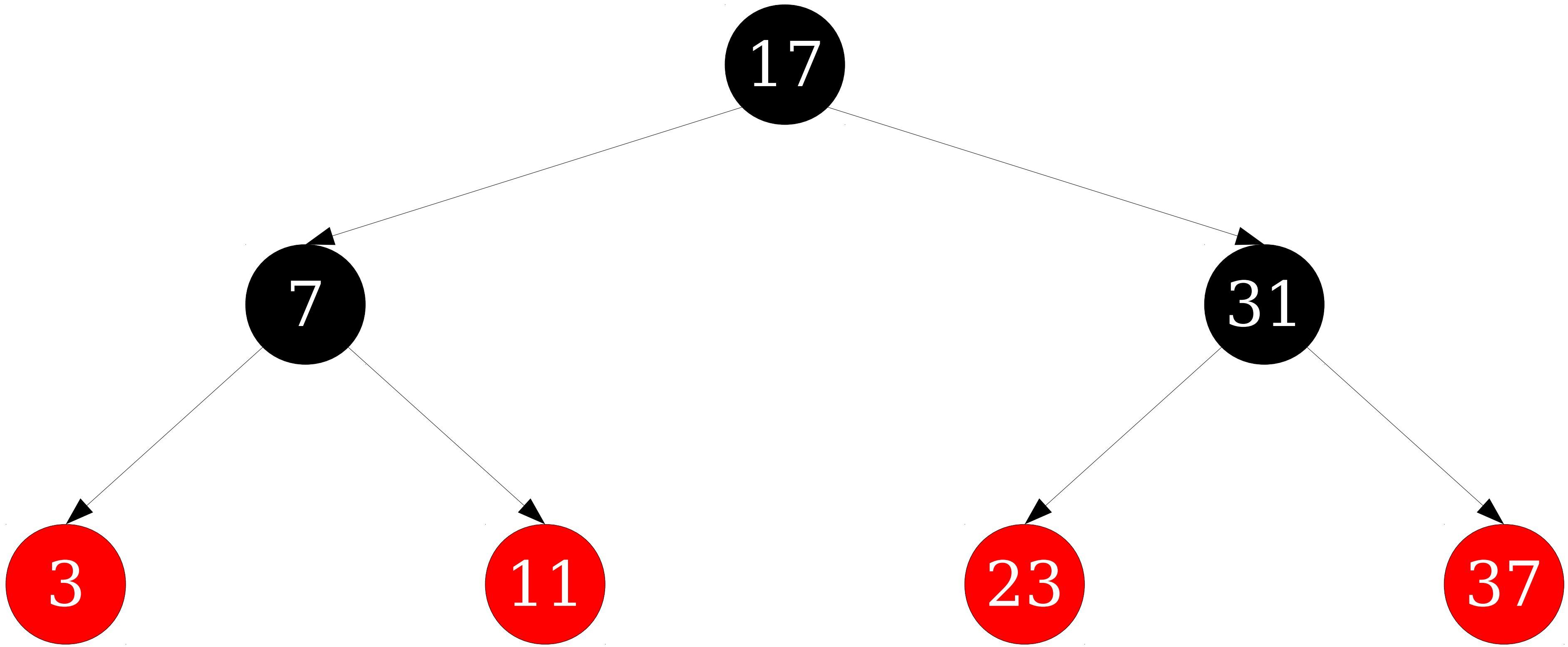


Red/Black Trees

- ***Theorem:*** Any red/black tree with n nodes has height O(log n).
  - We could prove this now, but there's a *much* simpler proof of this we'll see later on.
- Given a fixed red/black tree, lookups can be done in time O(log n).

Mutating Red/Black Trees



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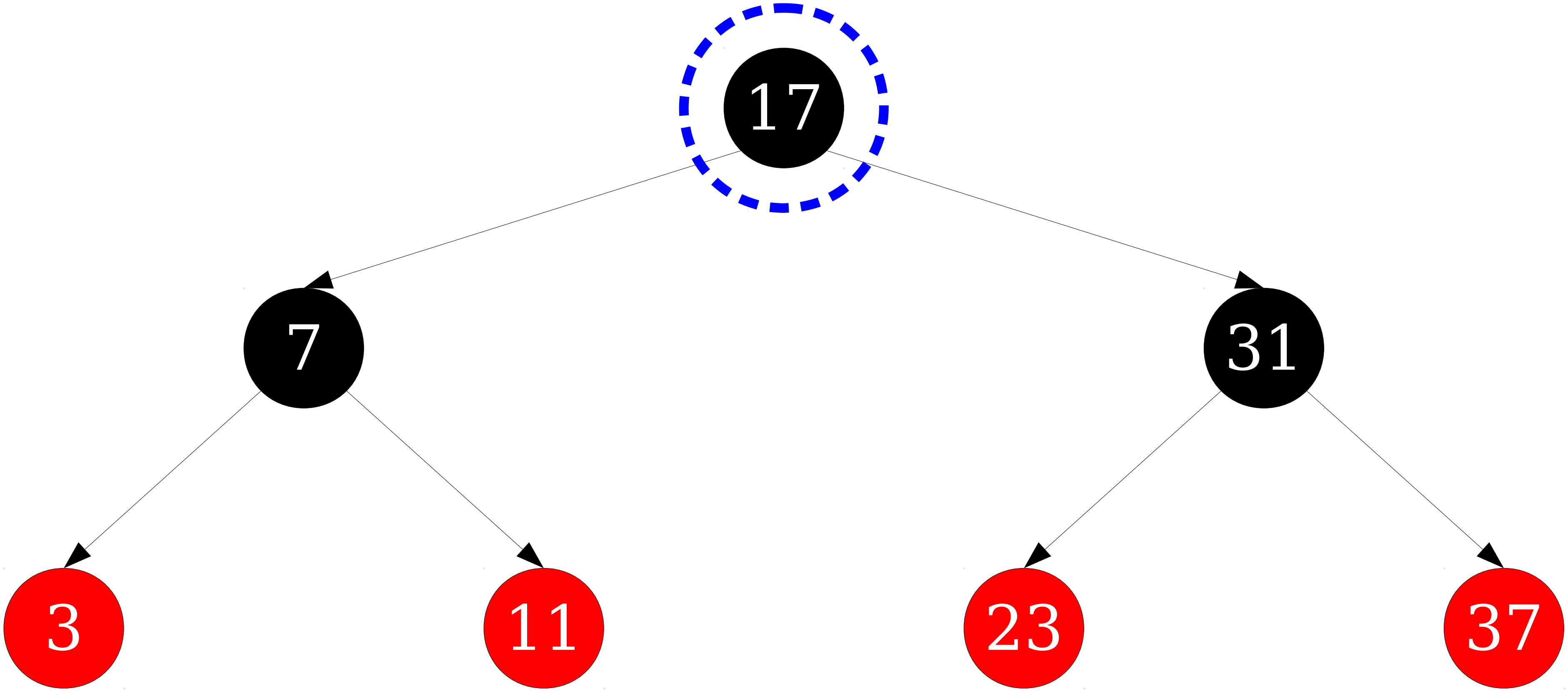

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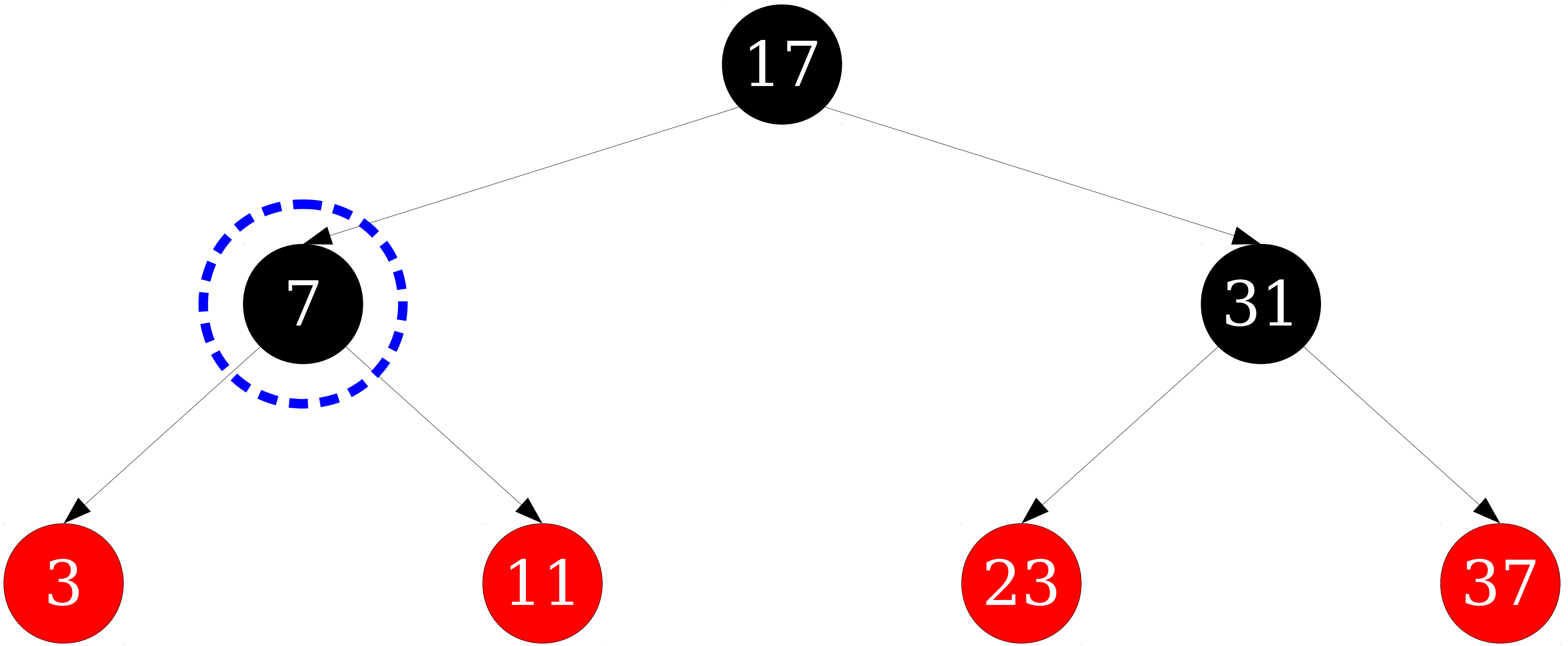

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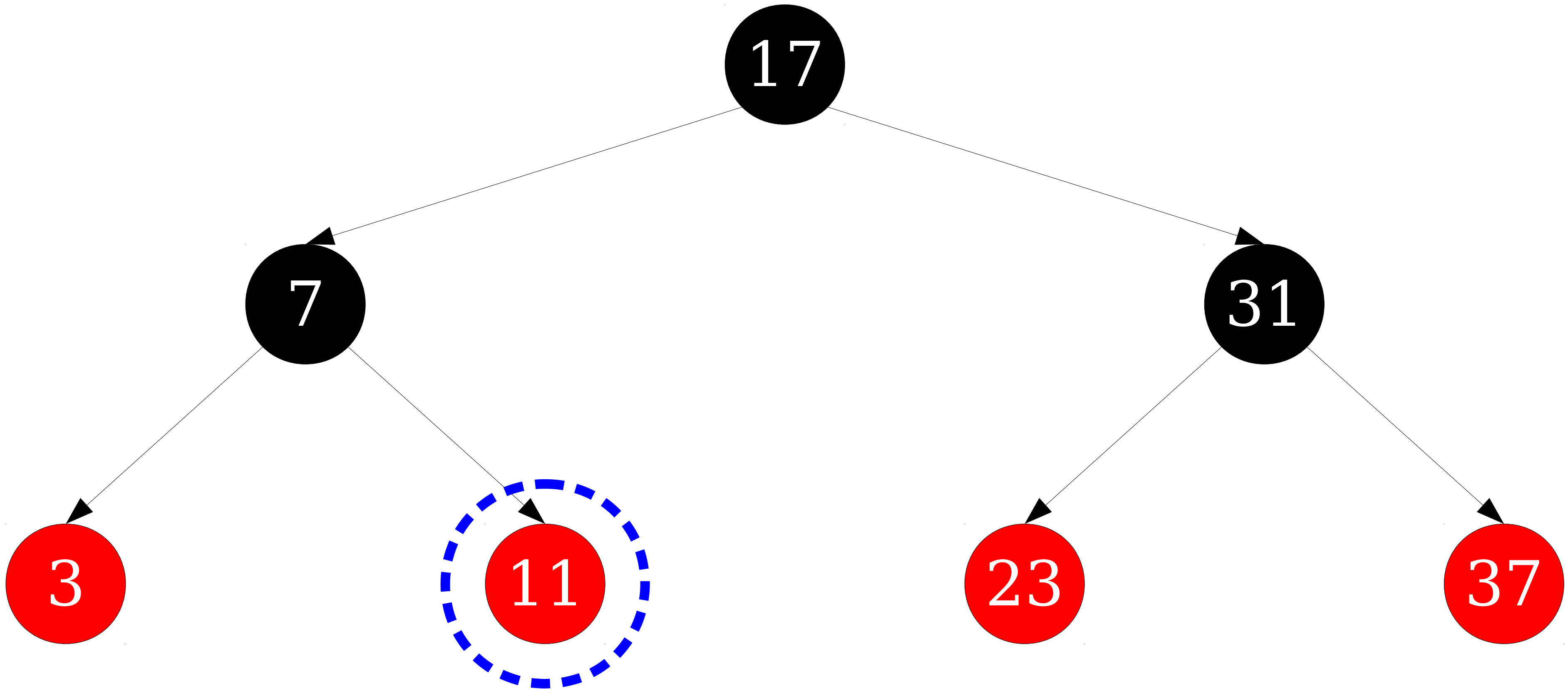

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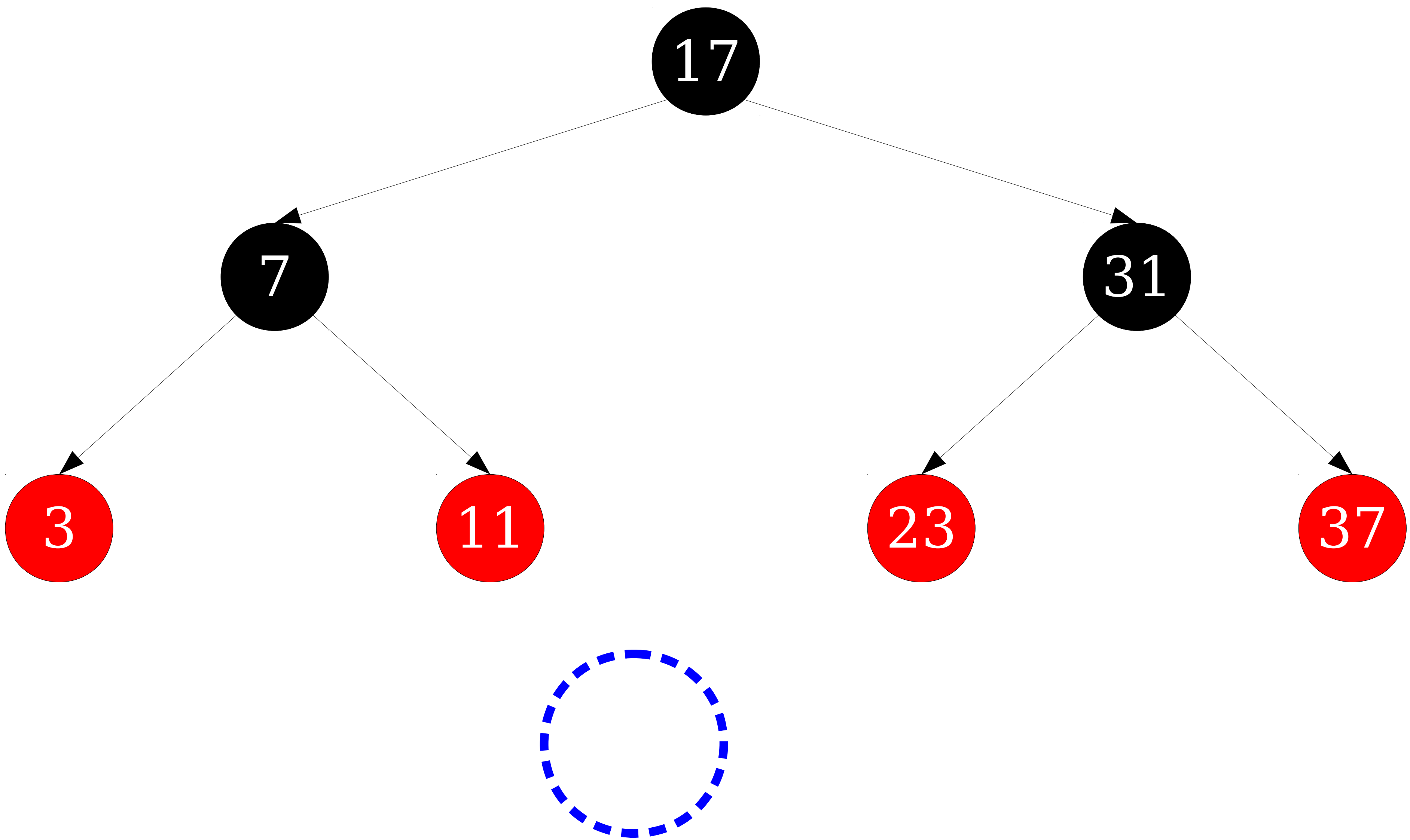

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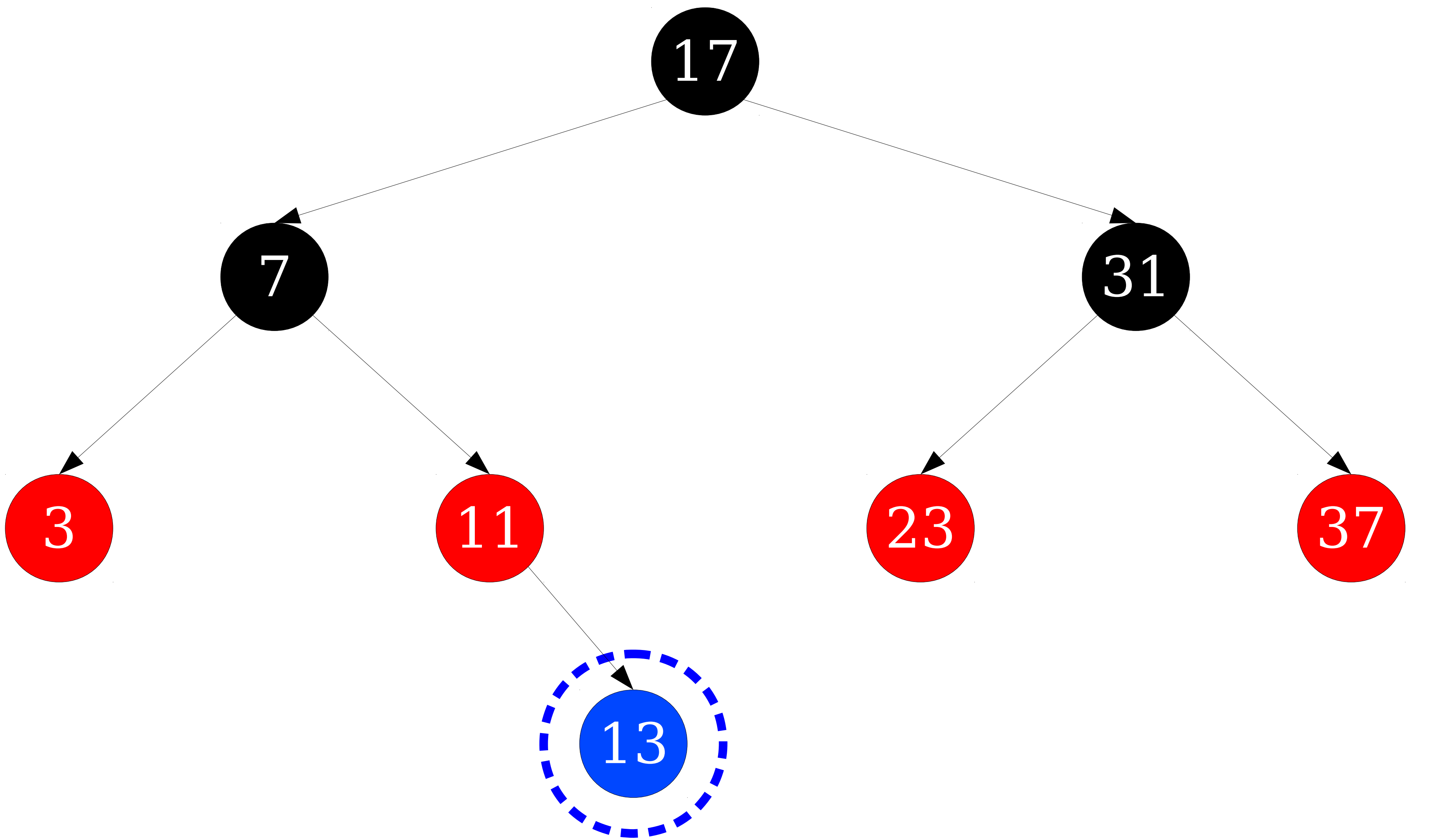

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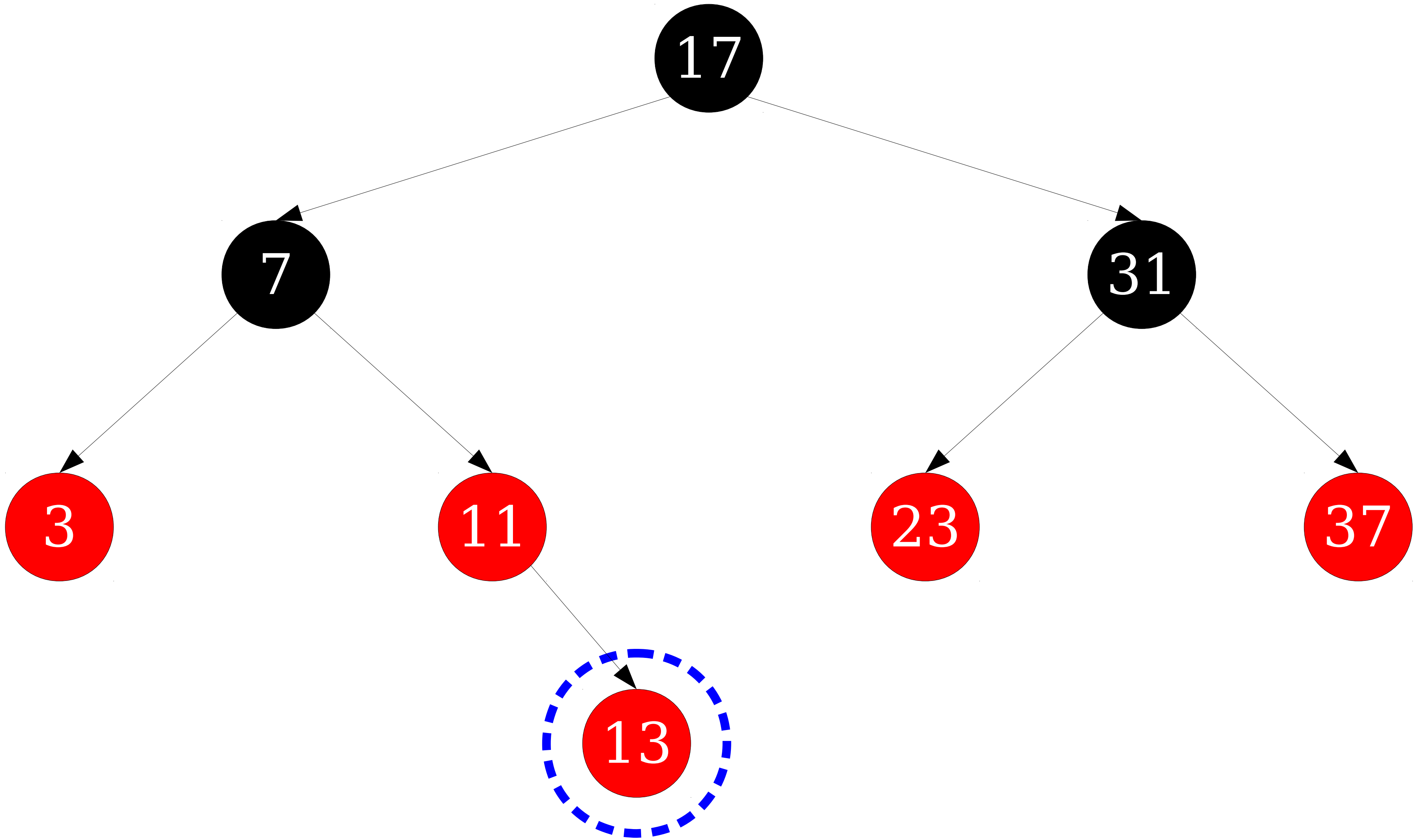

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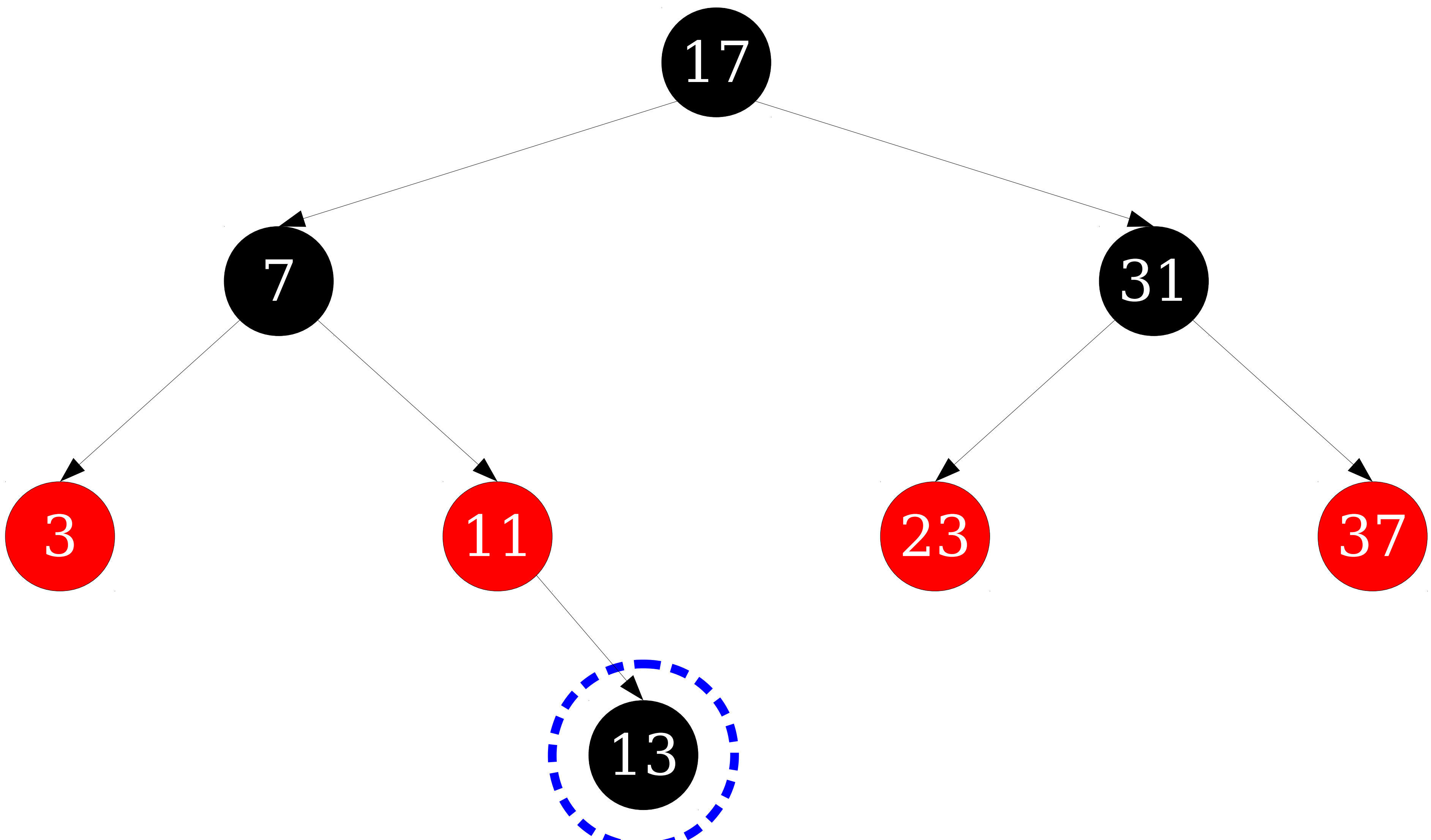

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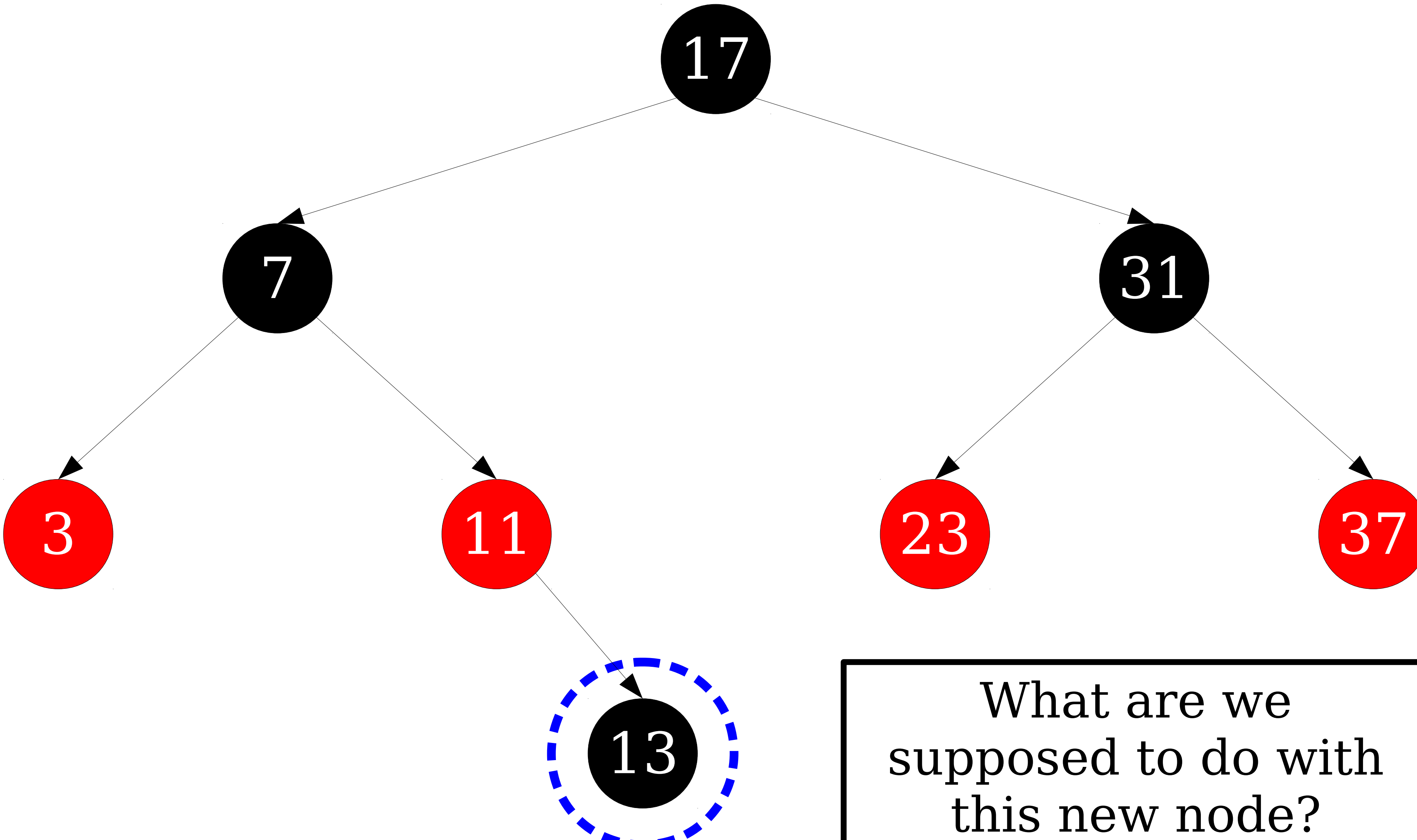

What are we supposed to do with this new node?

Mutating Red/Black Trees

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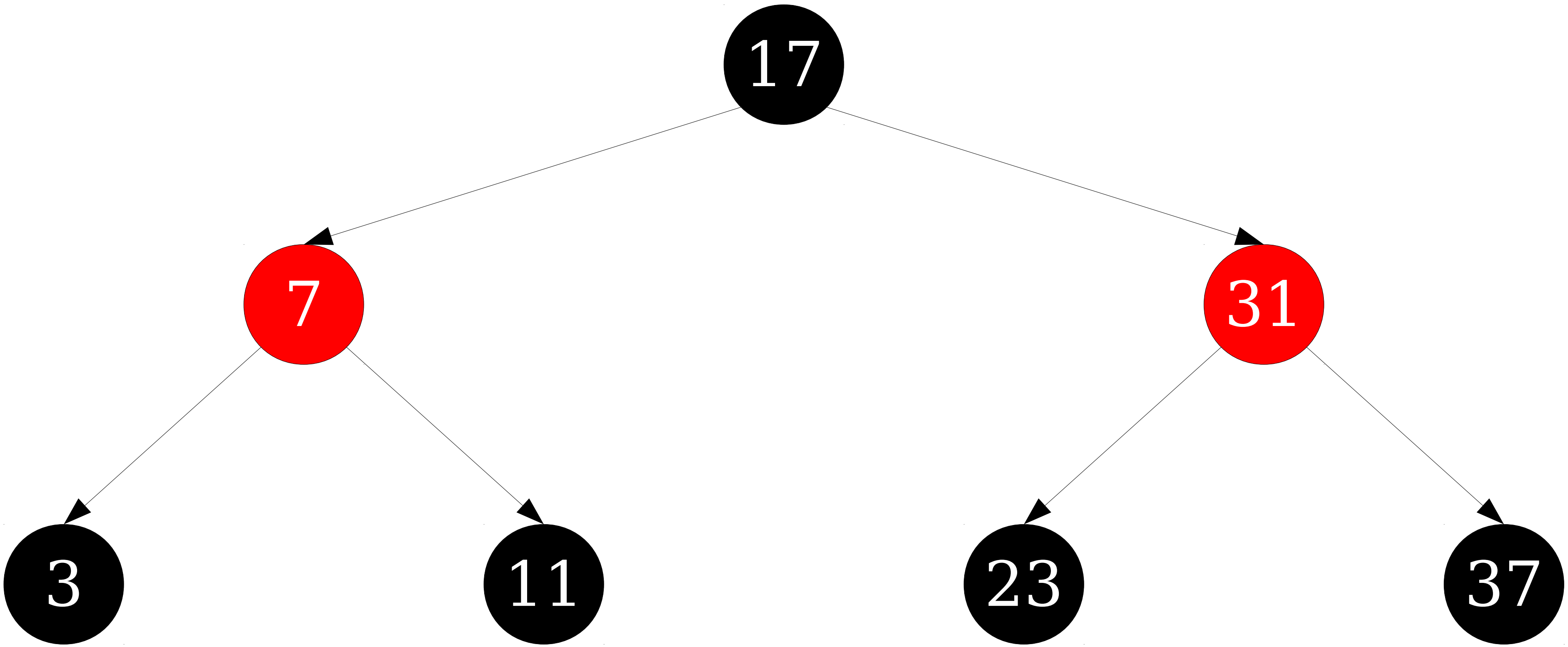

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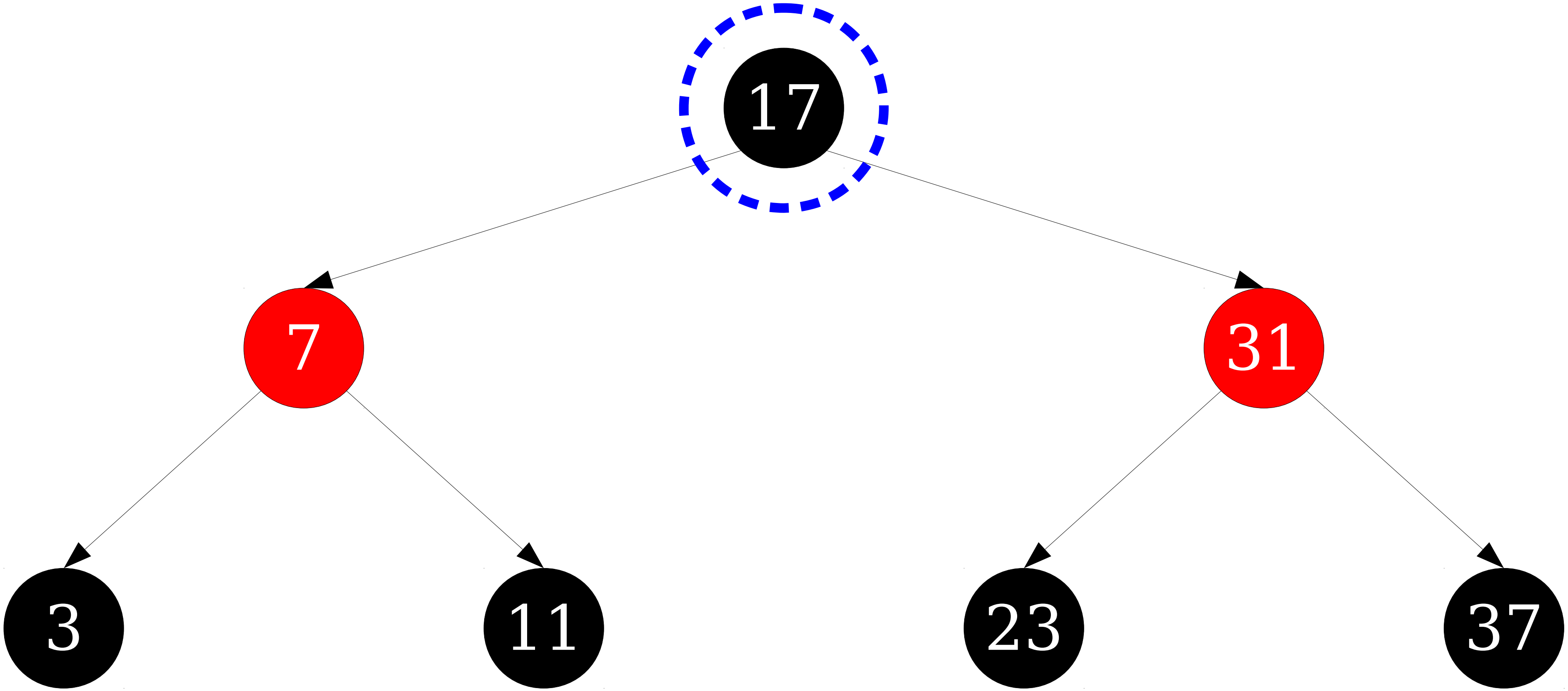

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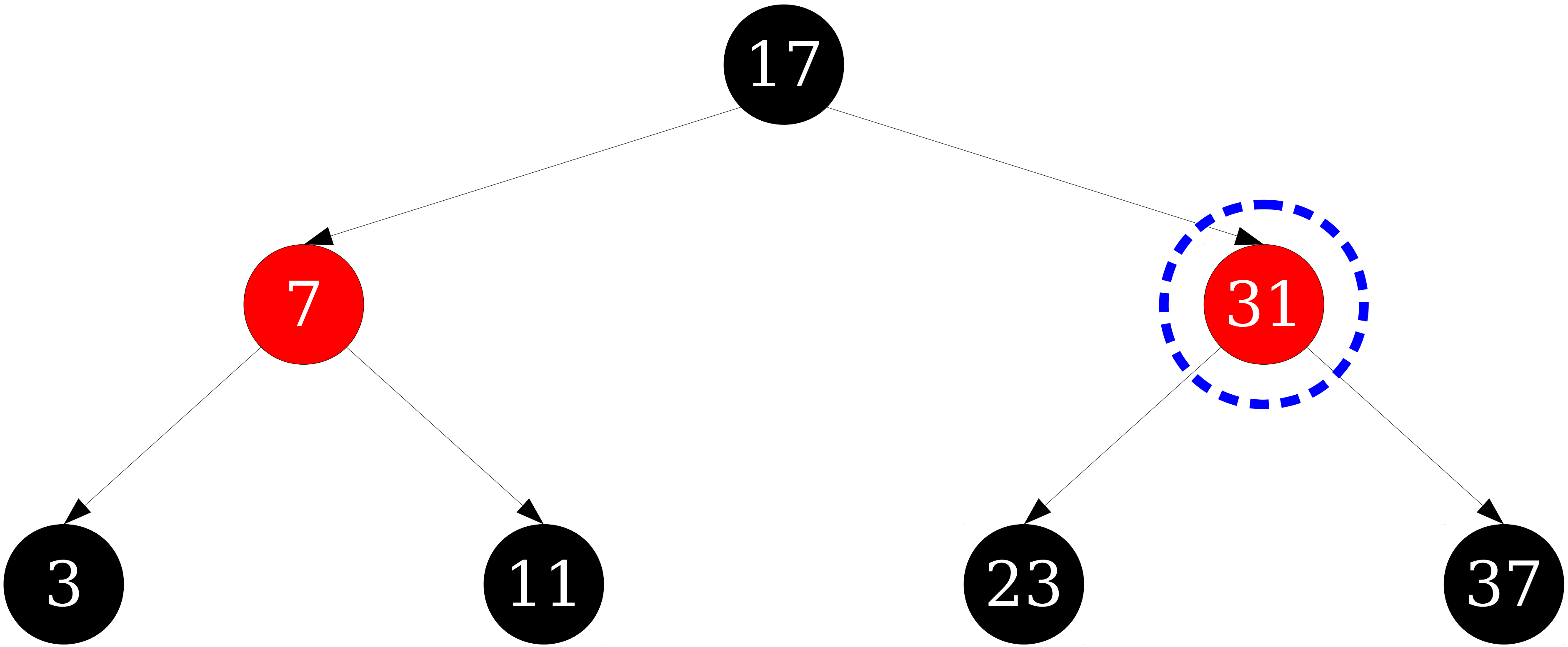

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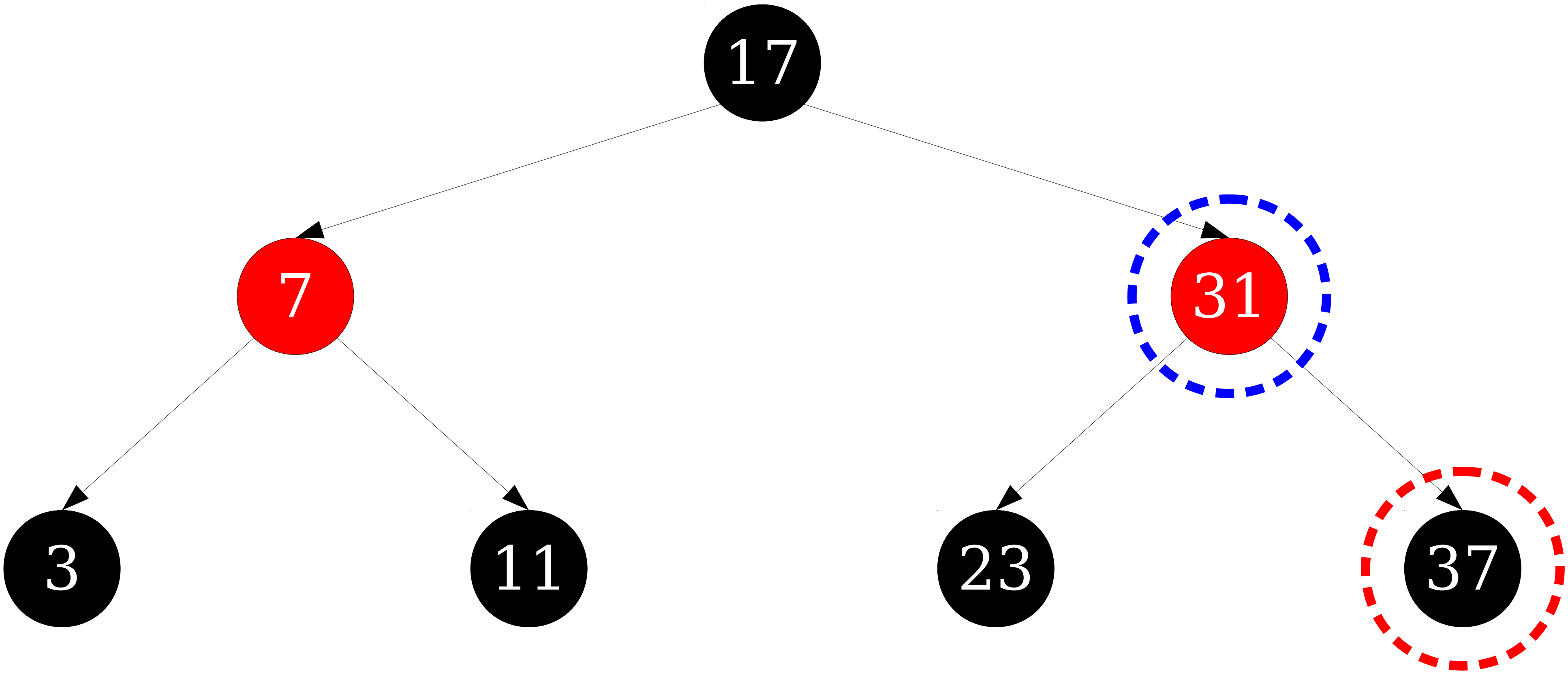
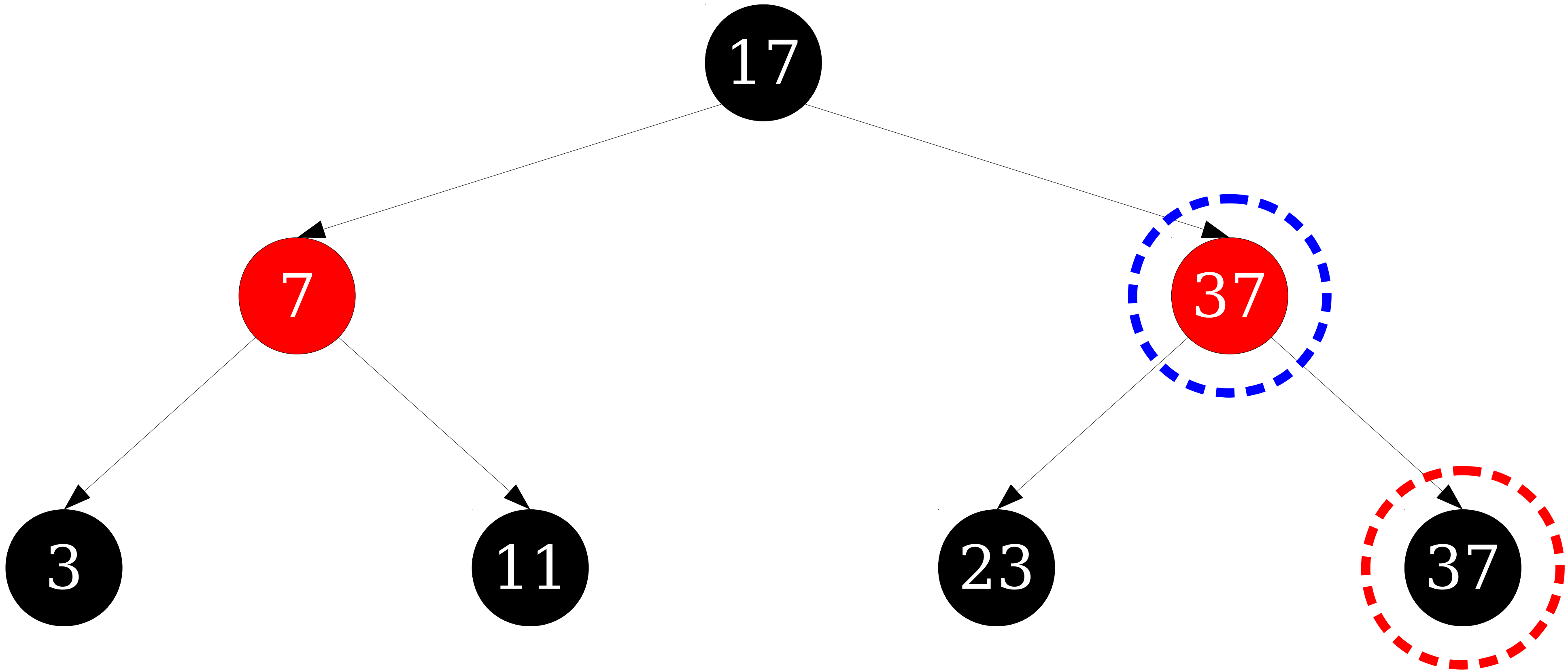


Mutating Red/Black Trees



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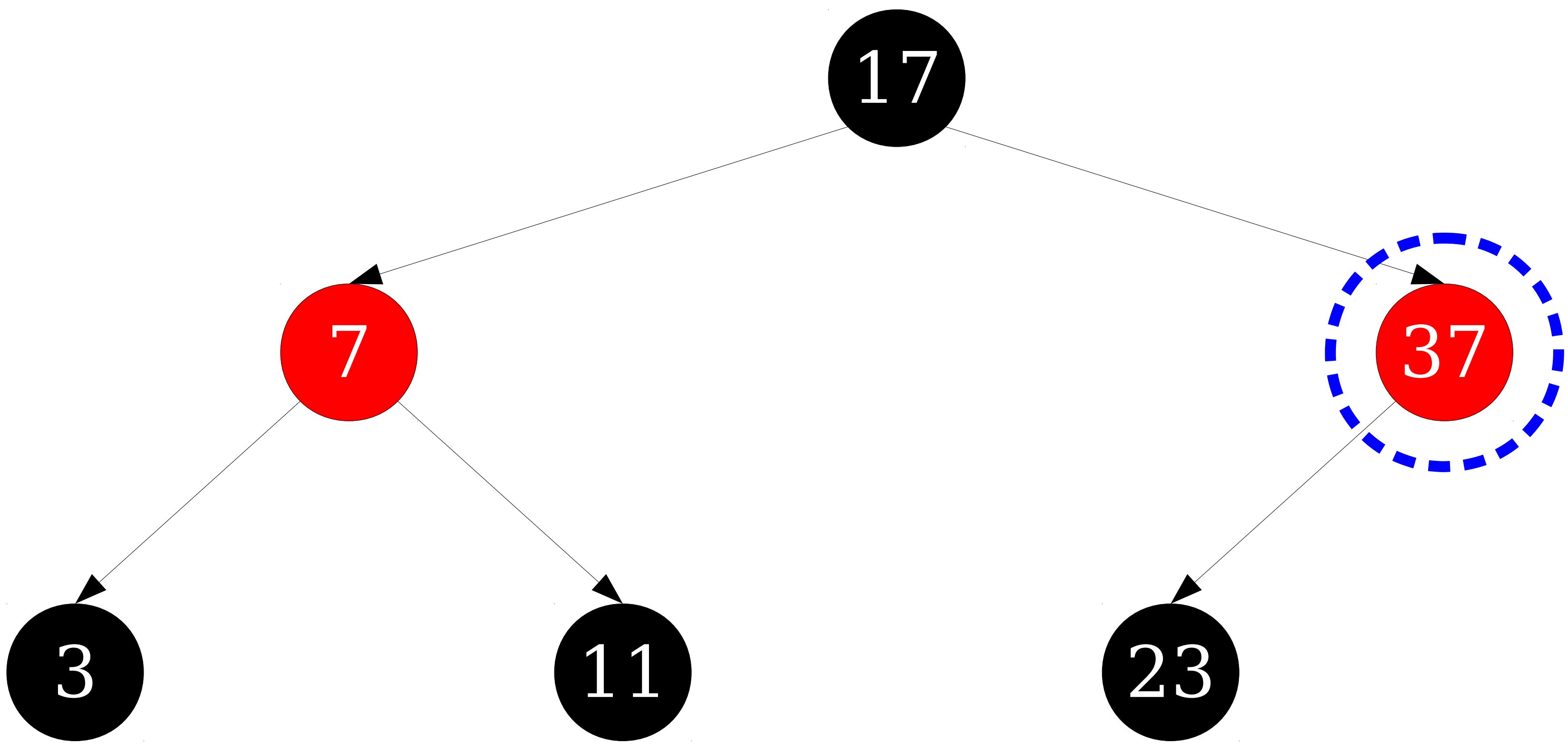

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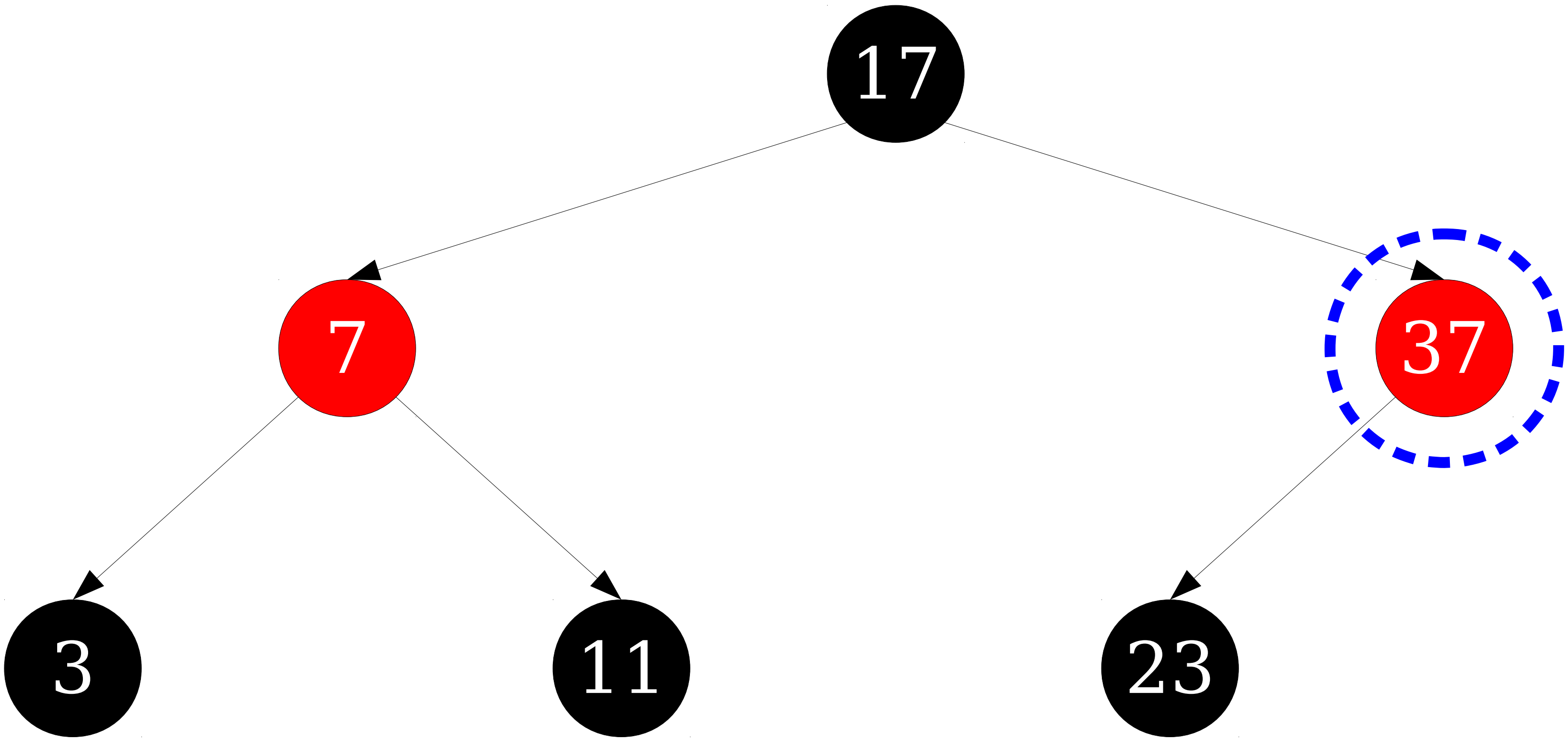


How do we fix up the black-height property?

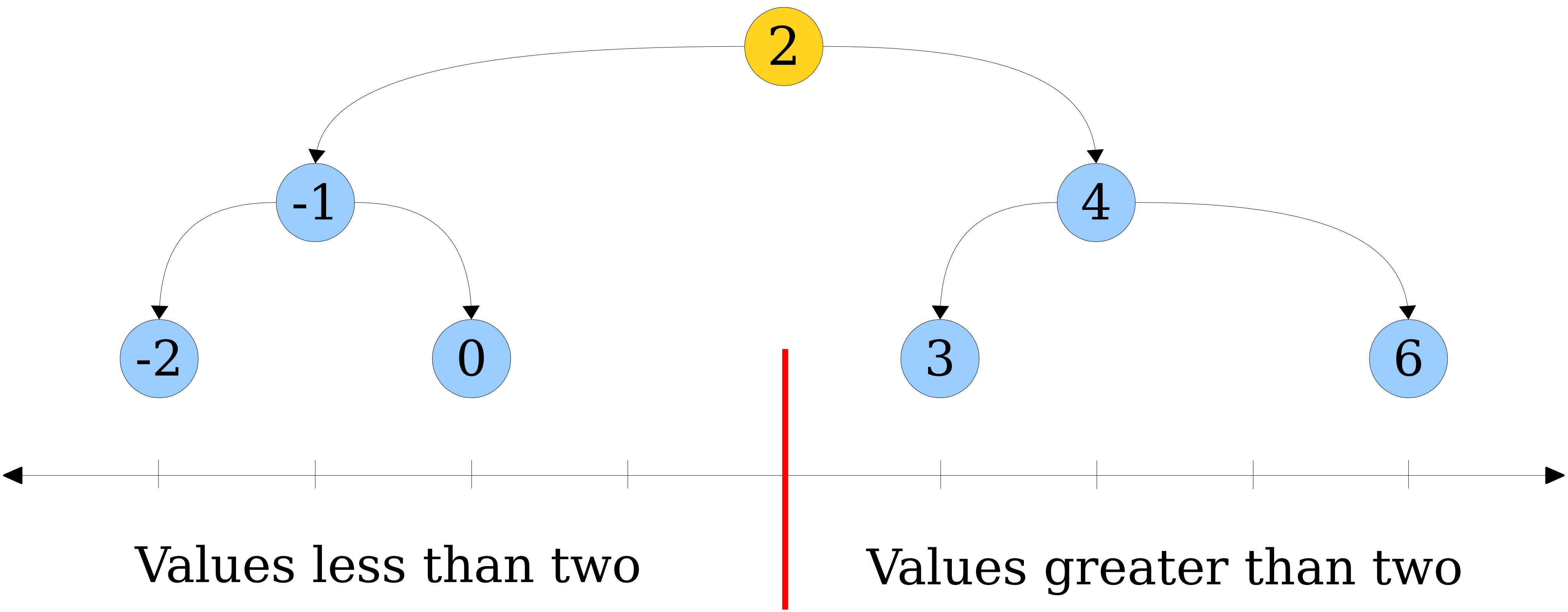
Fixing Up Red/Black Trees

- ***The Good News:*** After doing an insertion or deletion, can locally modify a red/black tree in time O(log *n*) to fix up the red/black properties.
- ***The Bad News:*** There are a *lot* of cases to consider and they're not trivial.
- Some questions:
  - How do you memorize / remember all the different types of rotations?
  - How on earth did anyone come up with red/black trees in the first place?

B-Trees

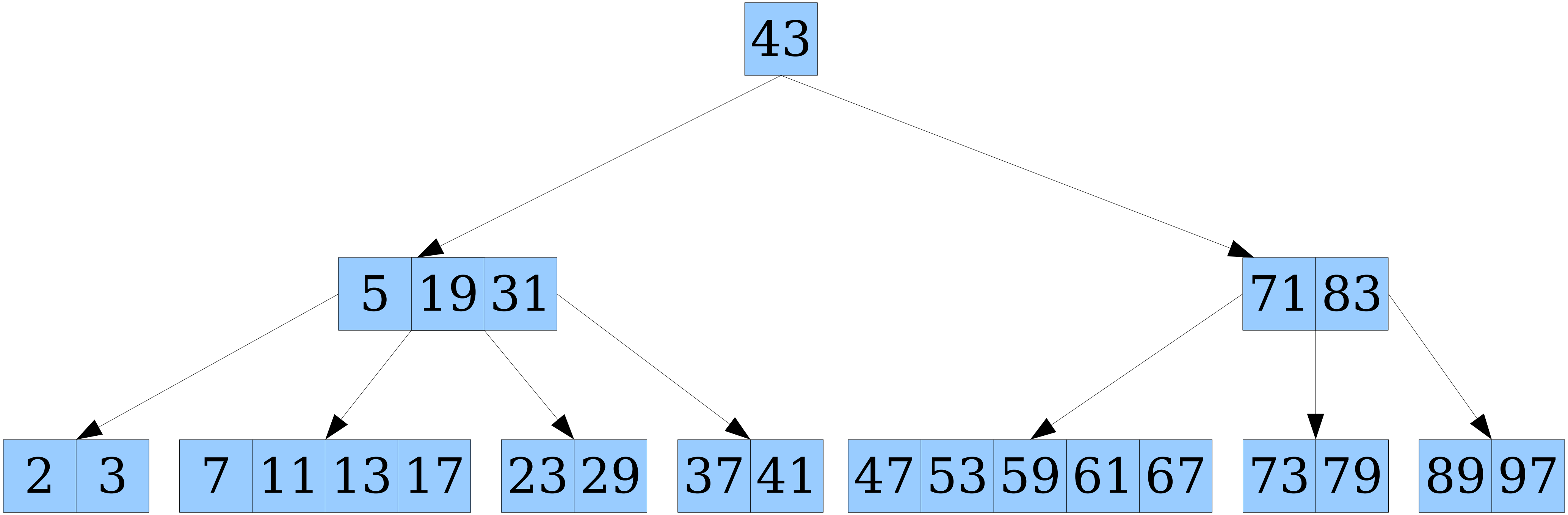
Generalizing BSTs

- In a binary search tree, each node stores a single key.
- That key splits the "key space" into two pieces, and each subtree stores the keys in those halves.



Generalizing BSTs

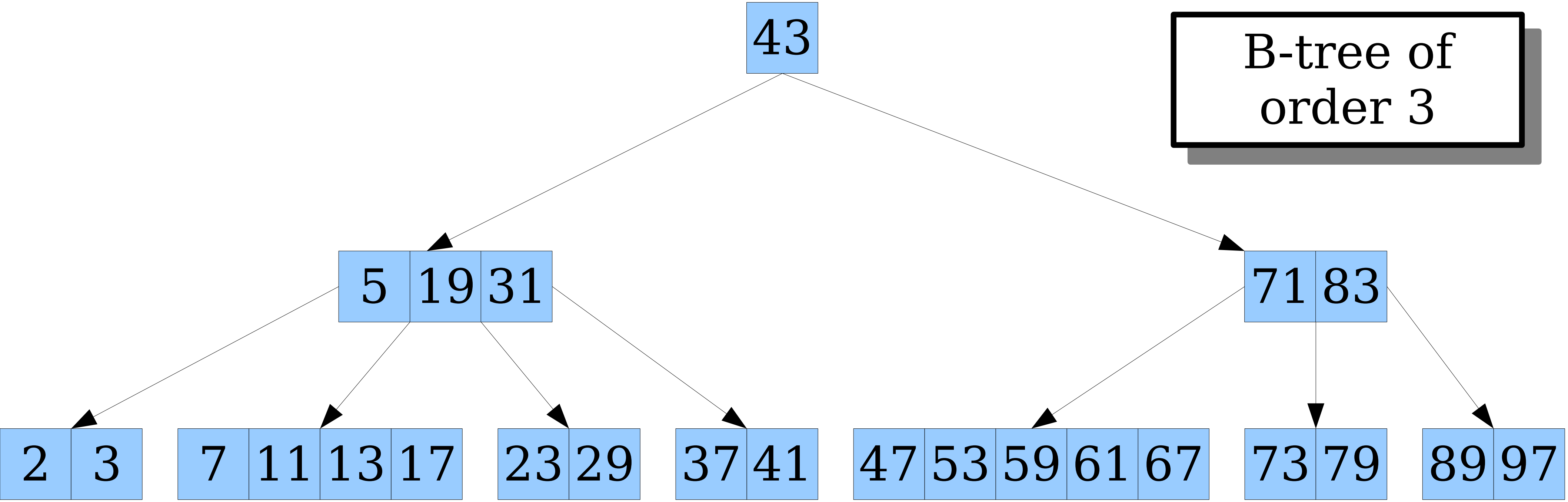
- In a ***multiway search tree***, each node stores an arbitrary number of keys in sorted order.



- In a node with k keys splits the "key space" into $k + 1$ pieces, and each subtree stores the keys in those pieces.

One Solution: **B-Trees**

- A ***B-tree of order b*** is a multiway search tree with the following properties:
  - All leaf nodes are stored at the same depth.
  - All non-root nodes have between b – 1 and 2b – 1 keys.
  - The root node has been 1 and 2b – 1 keys.
  - All root-null paths through the tree pass through the same number of nodes.



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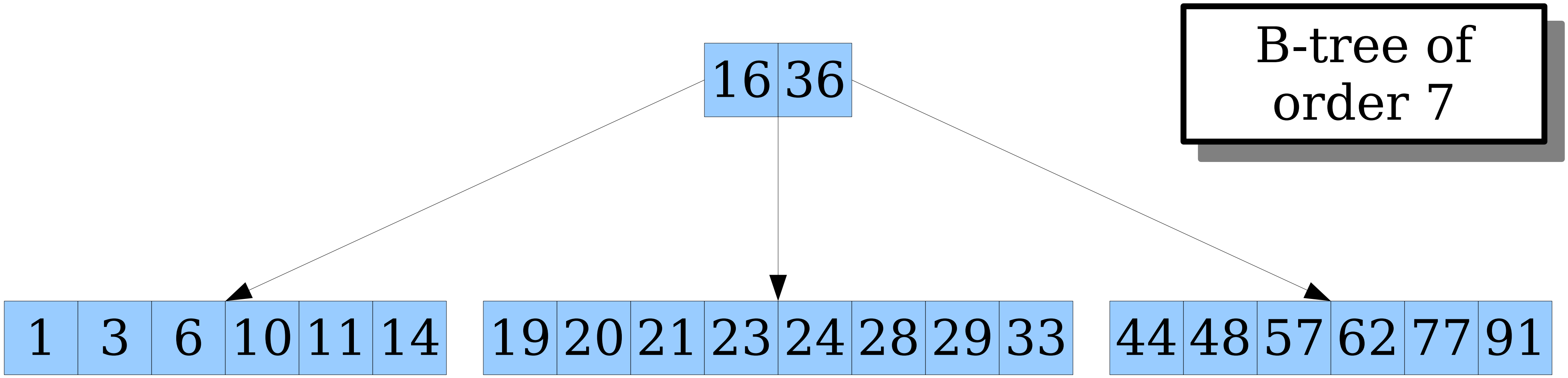

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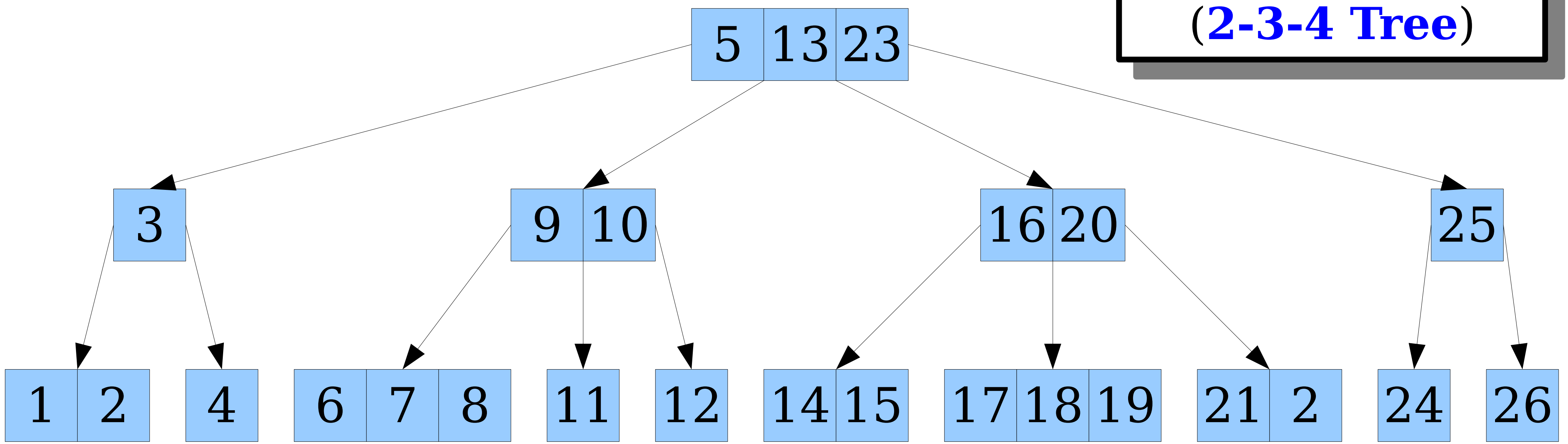
B-tree of order 2
(**2-3-4 Tree**)



The Tradeoff

- Because B-tree nodes can have multiple keys, when performing a search, insertion, or deletion, we have to spend more work inside each node.
- Insertion and deletion can be expensive – for large *b*, we might have to shuffle thousands or millions of keys over!
- Why would you use a B-tree?

Memory Tradeoffs

- There is an enormous tradeoff between *speed* and *size* in memory.
- SRAM (the stuff registers are made of) is fast but very expensive:
  - Can keep up with processor speeds in the GHz.
  - As of 2010, cost is $5/MB. *(Anyone know a good source for a more recent price?)*
  - Good luck buying 1TB of the stuff!
- Hard disks are cheap but very slow:
  - As of 2016, you can buy a 2TB hard drive for about $70.
  - As of 2016, good disk seek times are measured in ms (about two to four million times slower than a processor cycle!)

The Memory Hierarchy

- ***Idea:*** Try to get the best of all worlds by using multiple types of memory.

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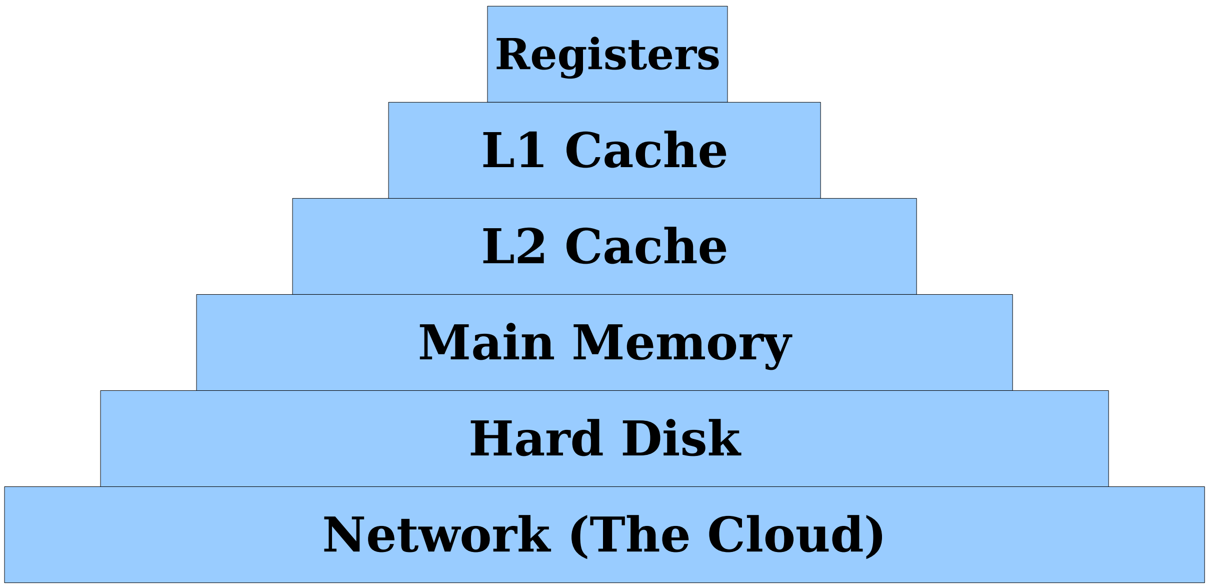


The Memory Hierarchy

- ***Idea:*** Try to get the best of all worlds by using multiple types of memory.

Registers	256B - 8KB
L1 Cache	16KB – 64KB
L2 Cache	1MB - 4MB
Main Memory	4GB – 256GB
Hard Disk	1TB+
Network (The Cloud)	*Lots*

The Memory Hierarchy

- ***Idea:*** Try to get the best of all worlds by using multiple types of memory.

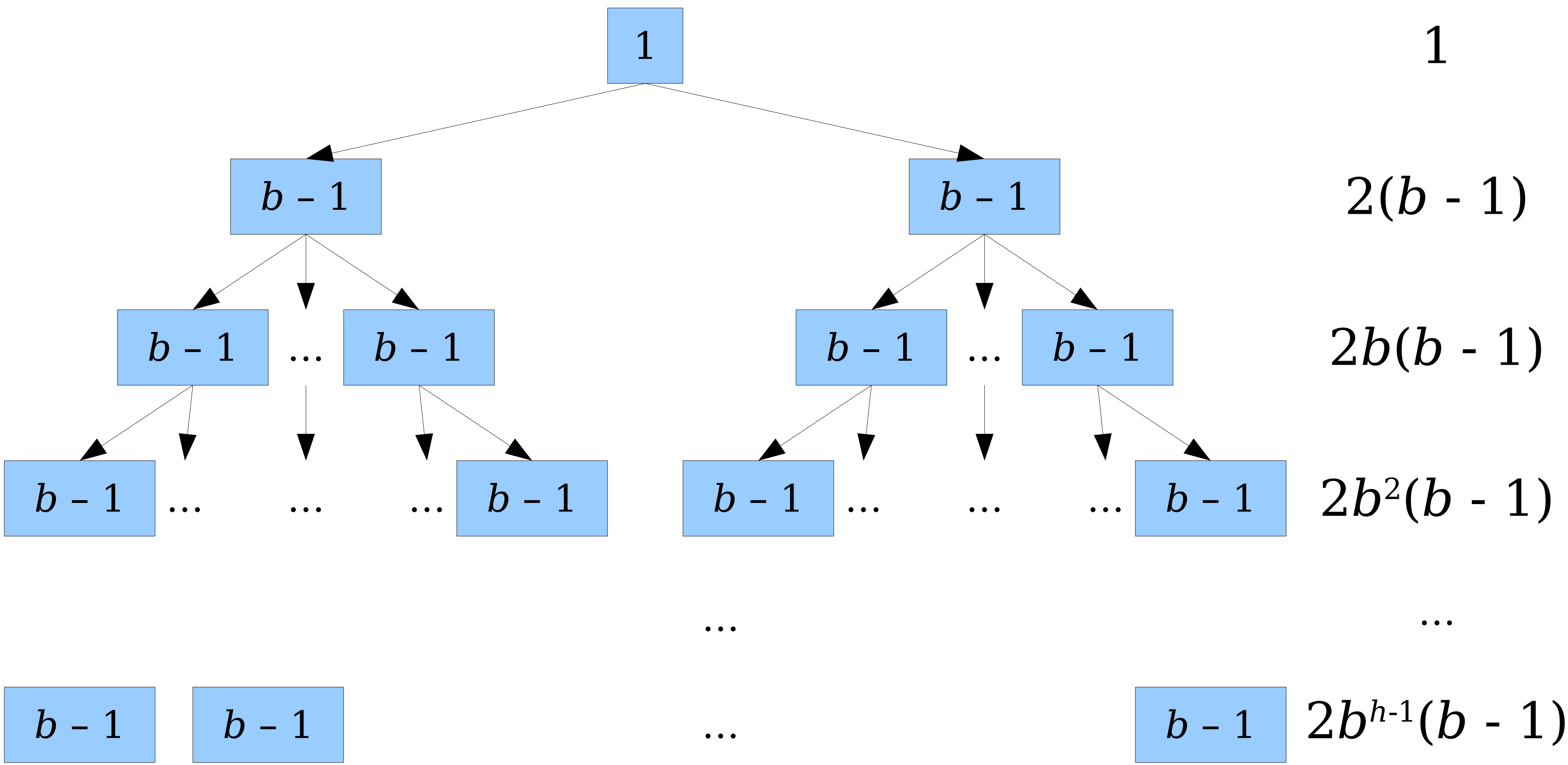
Registers	256B - 8KB	0.25 – 1ns
L1 Cache	16KB – 64KB	1ns – 5ns
L2 Cache	1MB - 4MB	5ns – 25ns
Main Memory	4GB – 256GB	25ns – 100ns
Hard Disk	1TB+	3 – 10ms
Network (The Cloud)	*Lots*	10 – 2000ms

Why B-Trees?

- Because B-trees have a huge branching factor, they're great for on-disk storage.
  - Disk block reads/writes are glacially slow.
  - The high branching factor minimizes the number of blocks to read during a lookup.
  - Extra work scanning inside a block offset by these savings.
- Used extensively in databases, file systems, etc.
  - Typically, use a **B+-tree** rather than a B-tree, but idea is similar.
- Recently, have been gaining traction for main-memory data structures.
  - Memory cache effects offset extra searching costs.

The Height of a B-Tree

- What is the maximum possible height of a B-tree of order *b*?



The Height of a B-Tree

- ***Theorem:*** The maximum height of a B-tree of order b containing n nodes is $\log_b ((n + 1) / 2)$.
- ***Proof:*** Number of nodes n in a B-tree of height h is guaranteed to be at least

$$1 + 2(b - 1) + 2b(b - 1) + 2b^2(b - 1) + \dots + 2b^{h-1}(b - 1)$$

$$= 1 + 2(b - 1)(1 + b + b^2 + \dots + b^{h-1})$$

$$= 1 + 2(b - 1)((b^h - 1) / (b - 1))$$

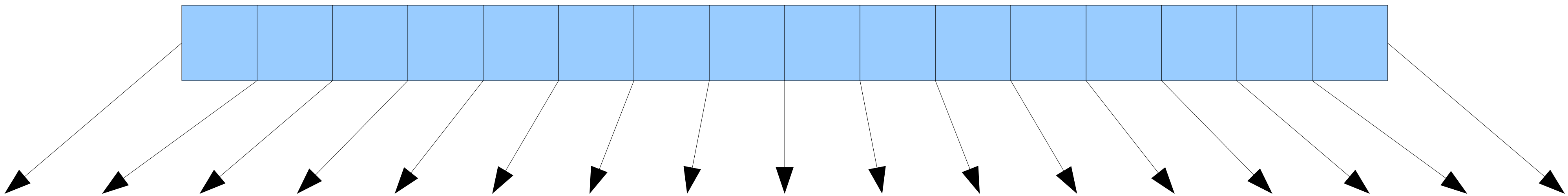
$$= 1 + 2(b^h - 1) = 2b^h - 1$$

Solving $n = 2b^h - 1$ yields $h = \log_b ((n + 1) / 2)$. ■

- ***Corollary:*** B-trees of order b have height $O(\log_b n)$.

Searching in a B-Tree

- Doing a search in a B-tree involves
  - searching the root node for the key, and
  - if it's not found, recursively exploring the correct child.



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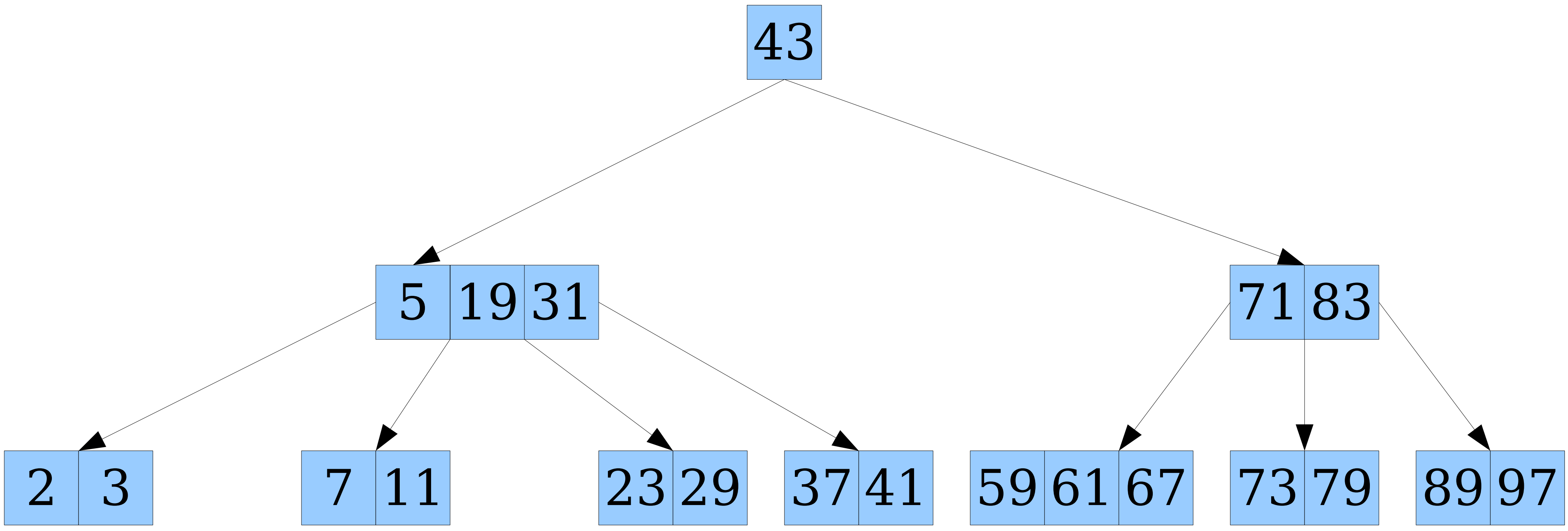
  - if it's not found, recursively exploring the correct child.
- Using binary search within a given node, can find the key or the correct child in time O(log *number-of-keys*).
- Repeat this process O(*tree-height*) times.
- Time complexity is

$$\begin{aligned} & O(\log \textit{number-of-keys} \cdot \textit{tree-height}) \\ &= O(\log b \cdot \log_b n) \\ &= O(\log b \cdot (\log n / \log b)) \\ &= \mathbf{O(\log n)} \end{aligned}$$

- Requires reading $O(\log_b n)$ blocks; this more directly accounts for the total runtime.

B-Trees are Simple

- Because nodes in a B-tree can store multiple keys, most insertions or deletions are straightforward.
- Here's a B-tree with *b* = 3 (nodes have between 2 and 5 keys):



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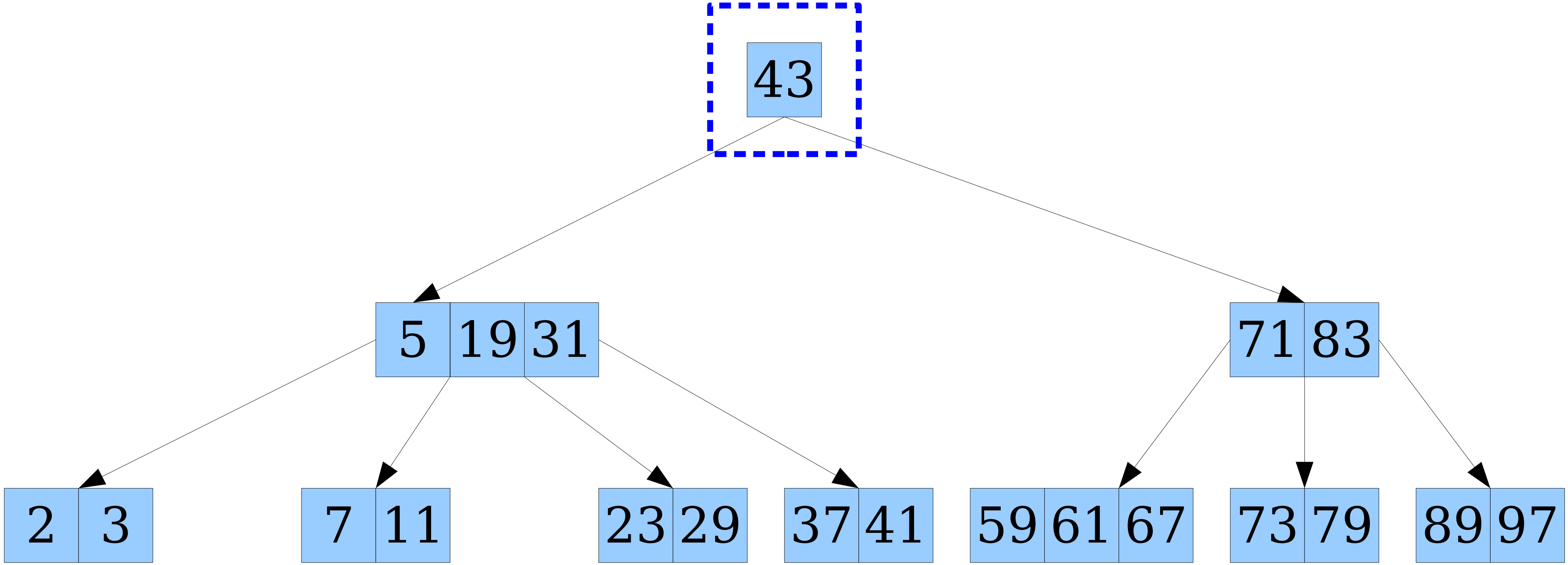

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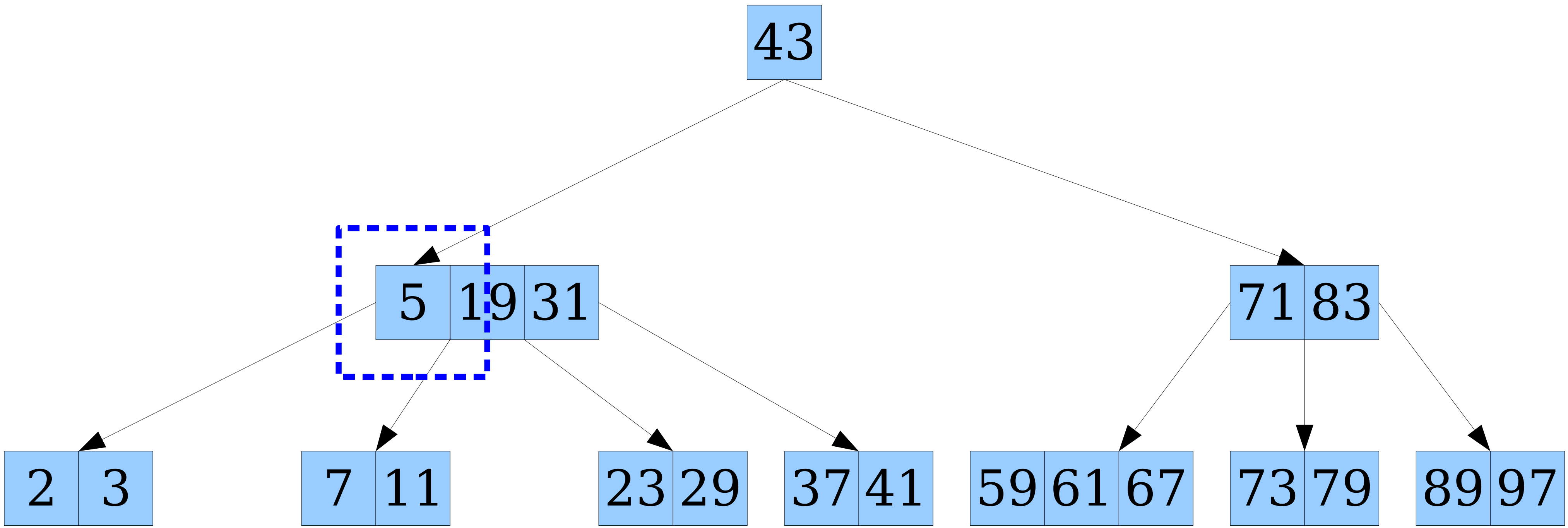

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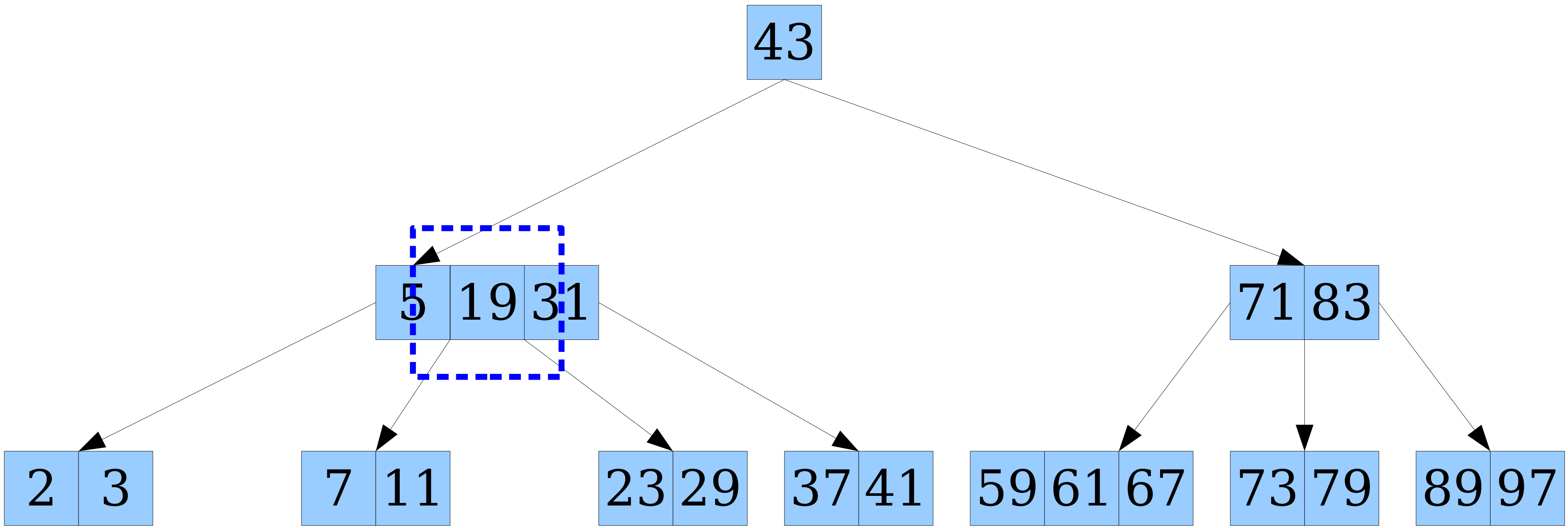

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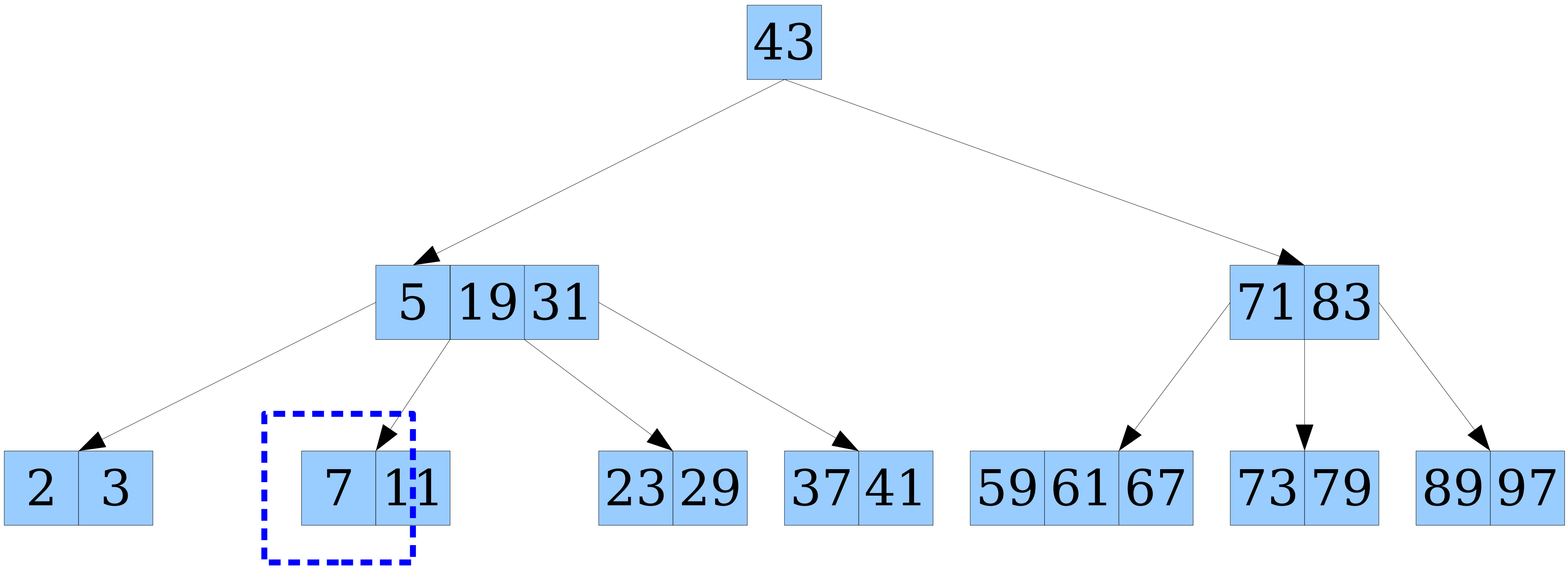

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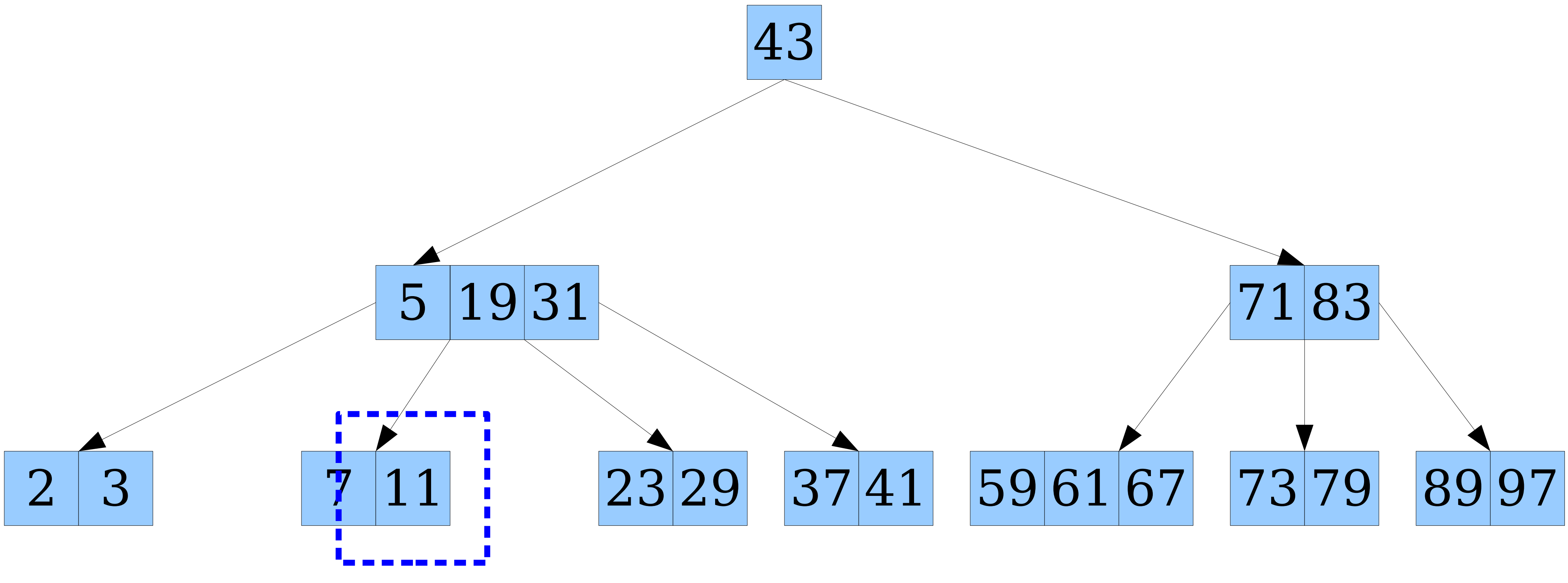

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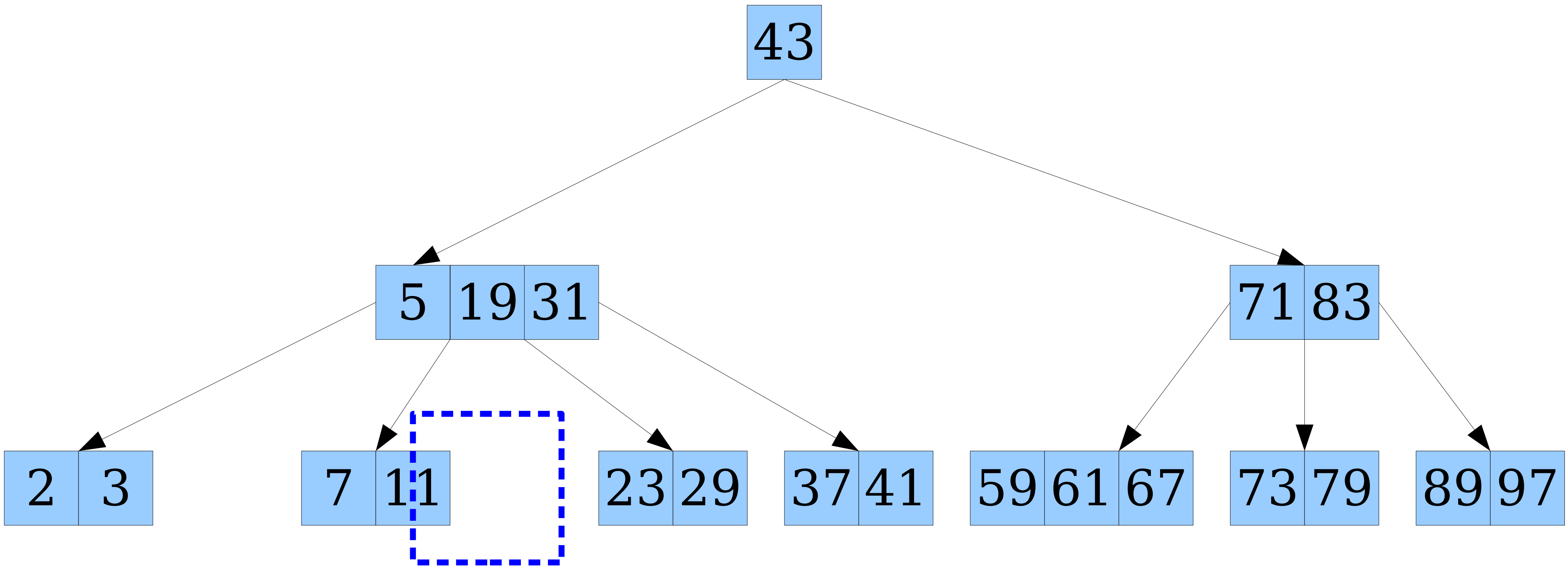

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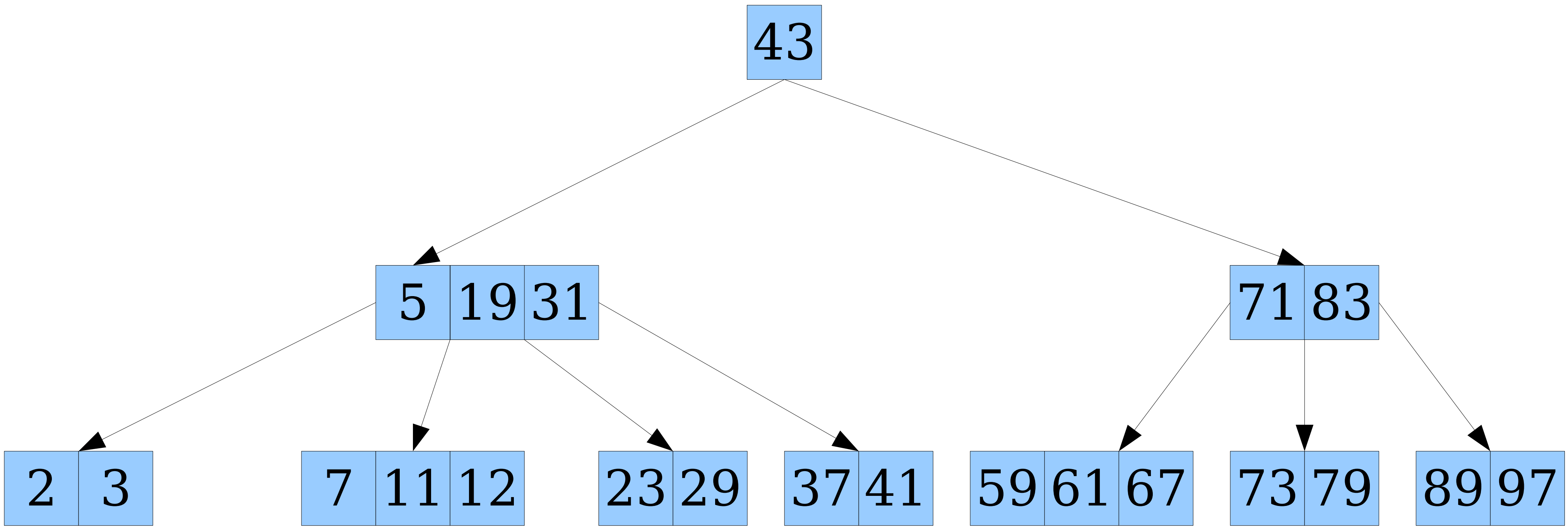

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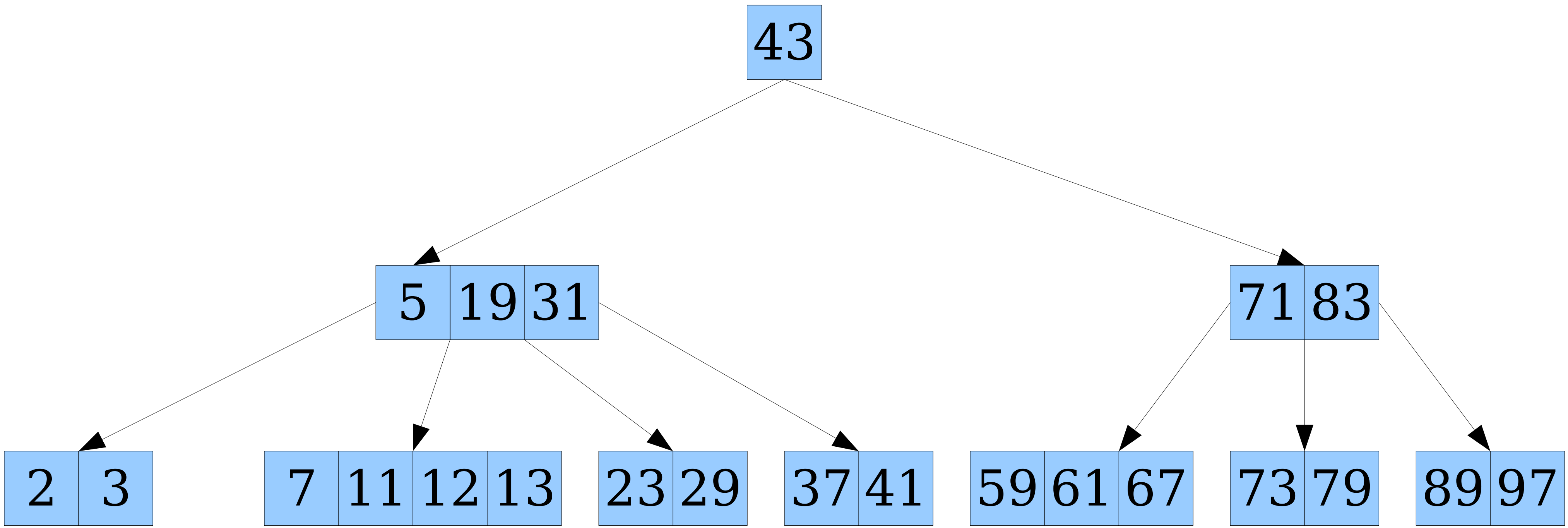

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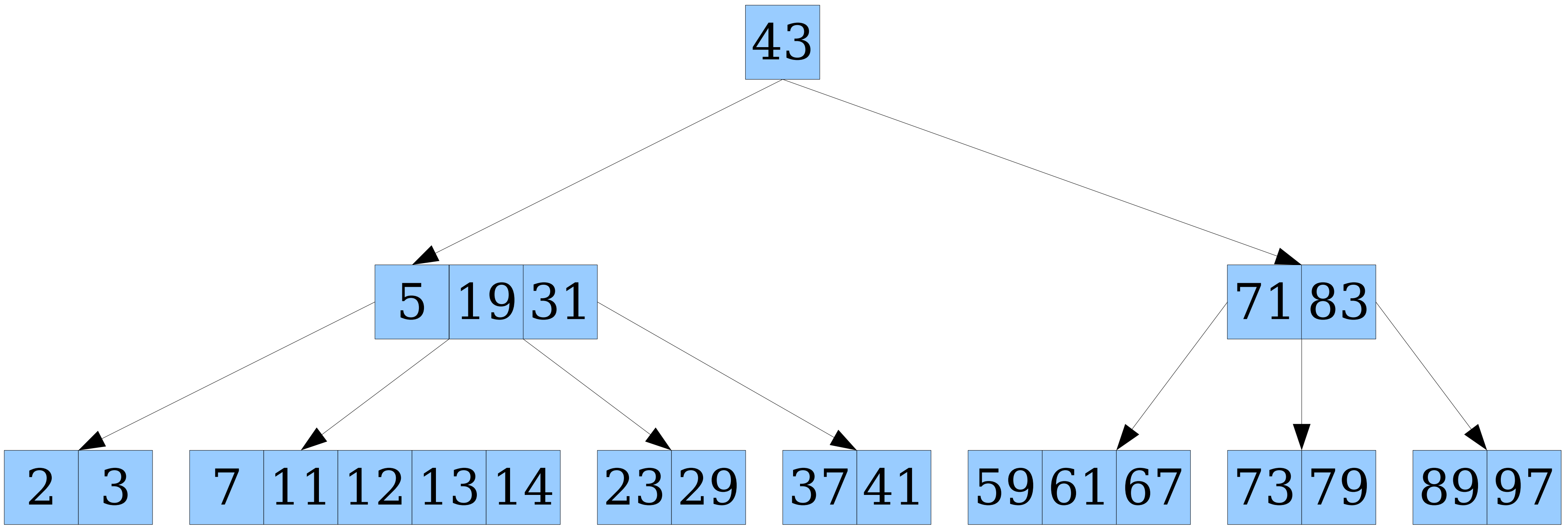

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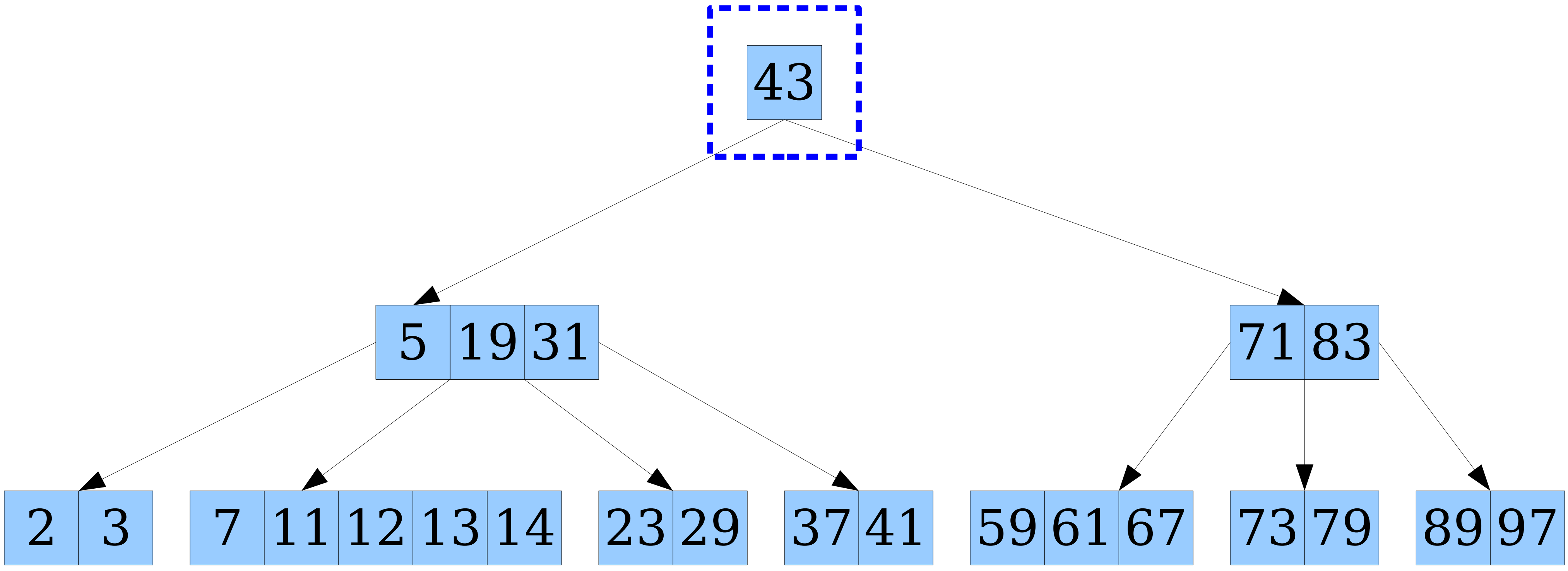

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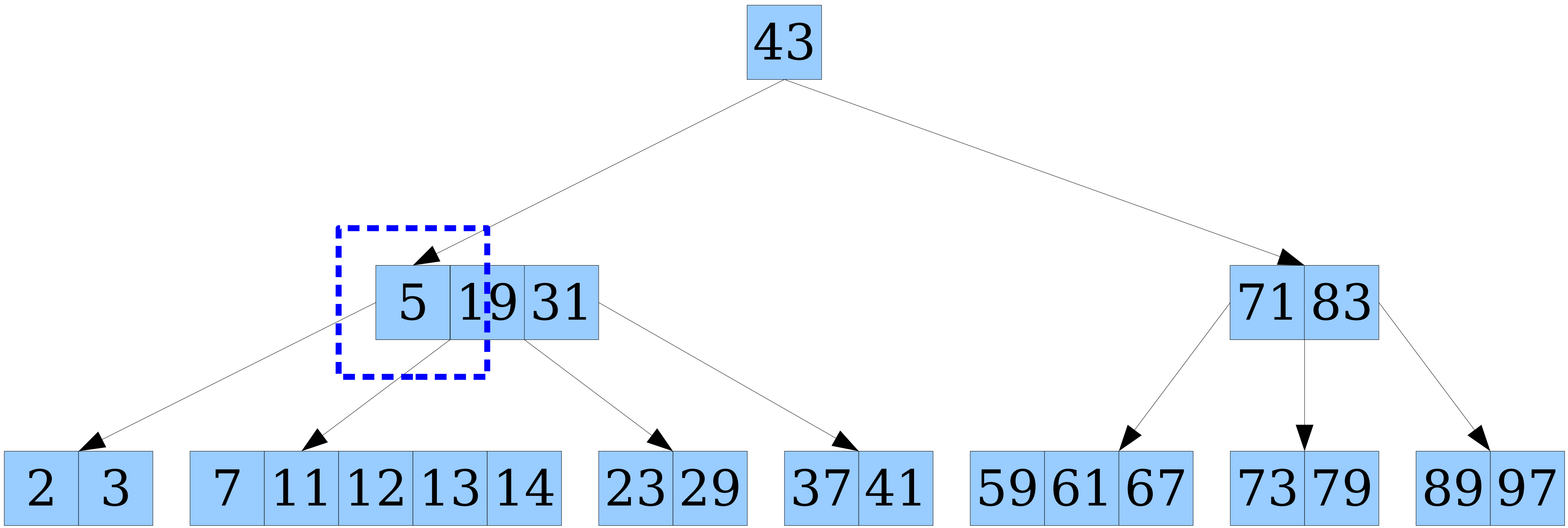

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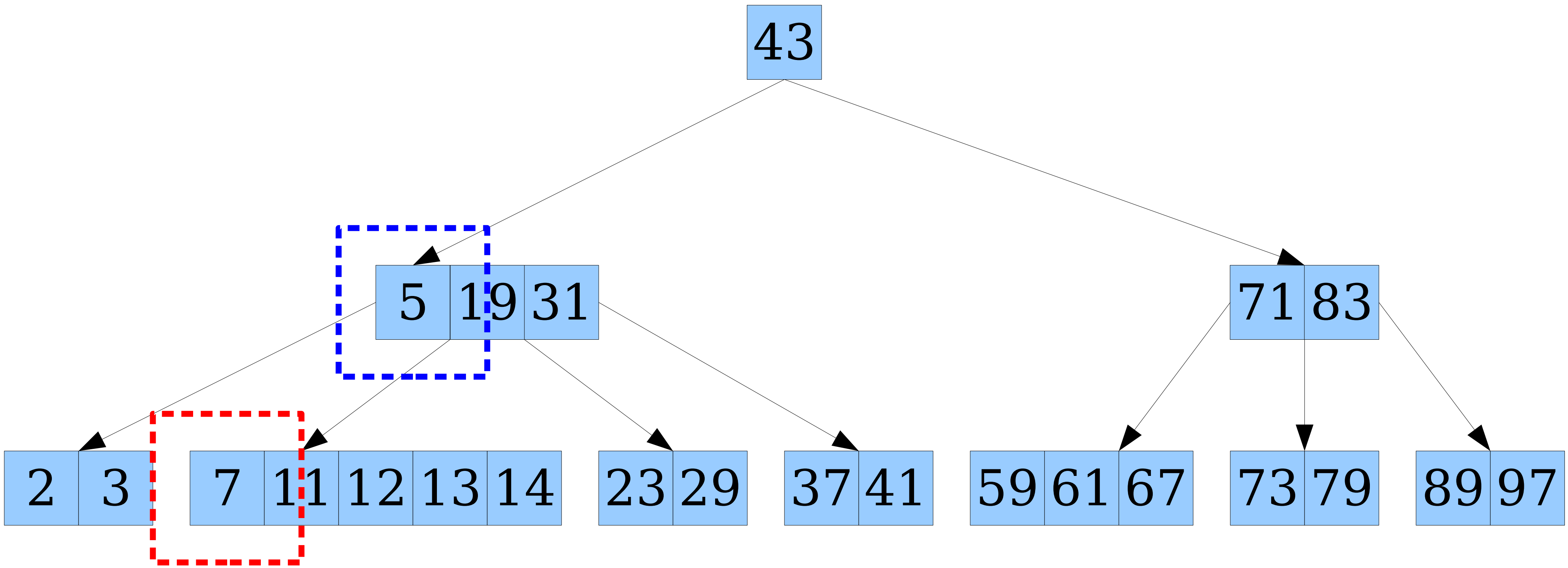

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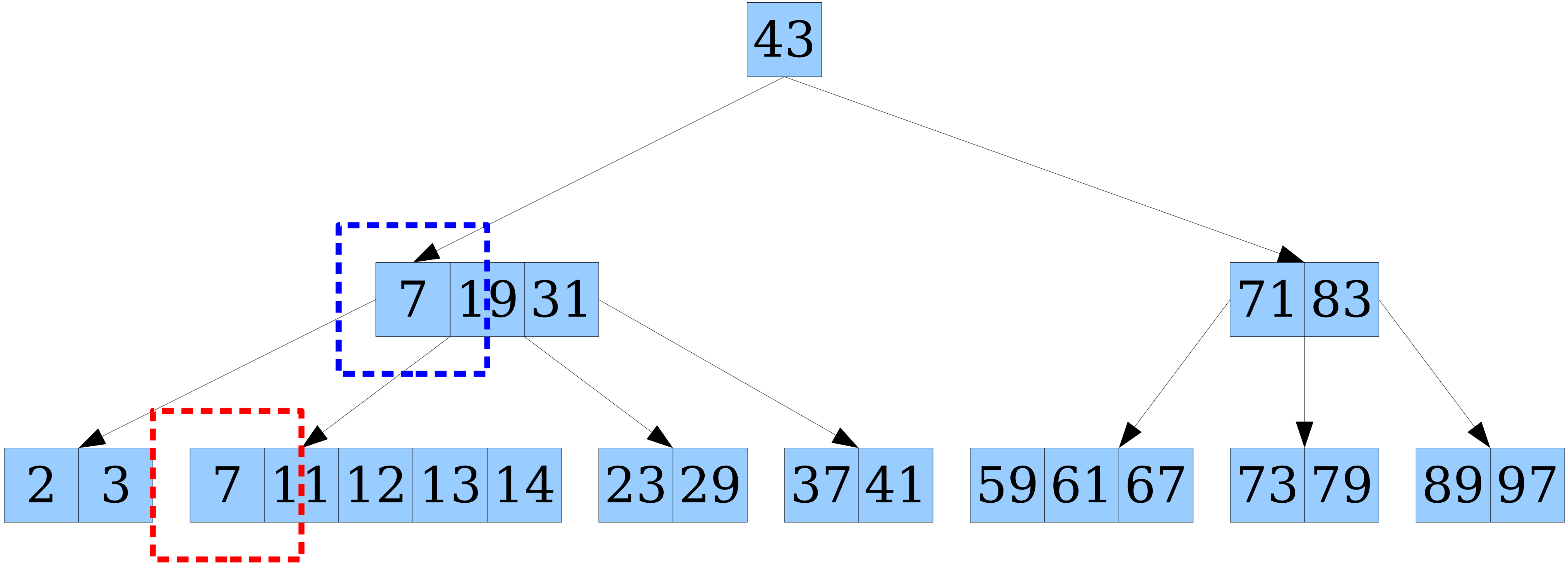

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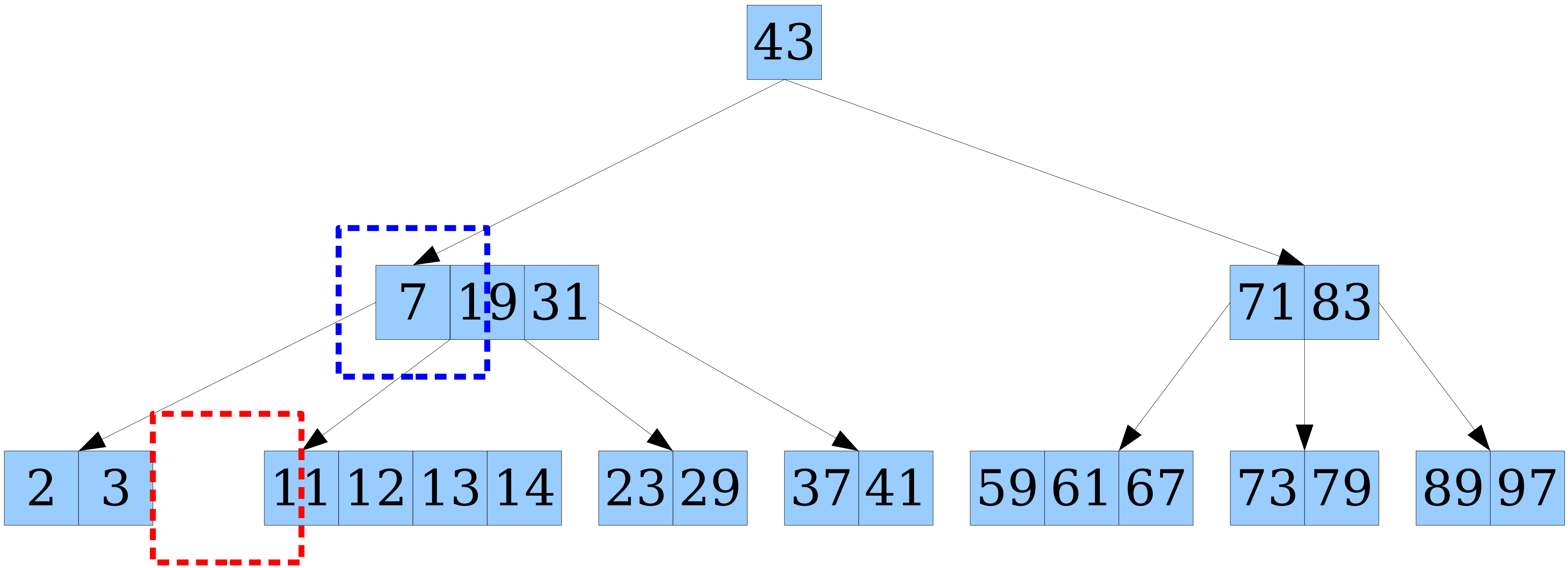

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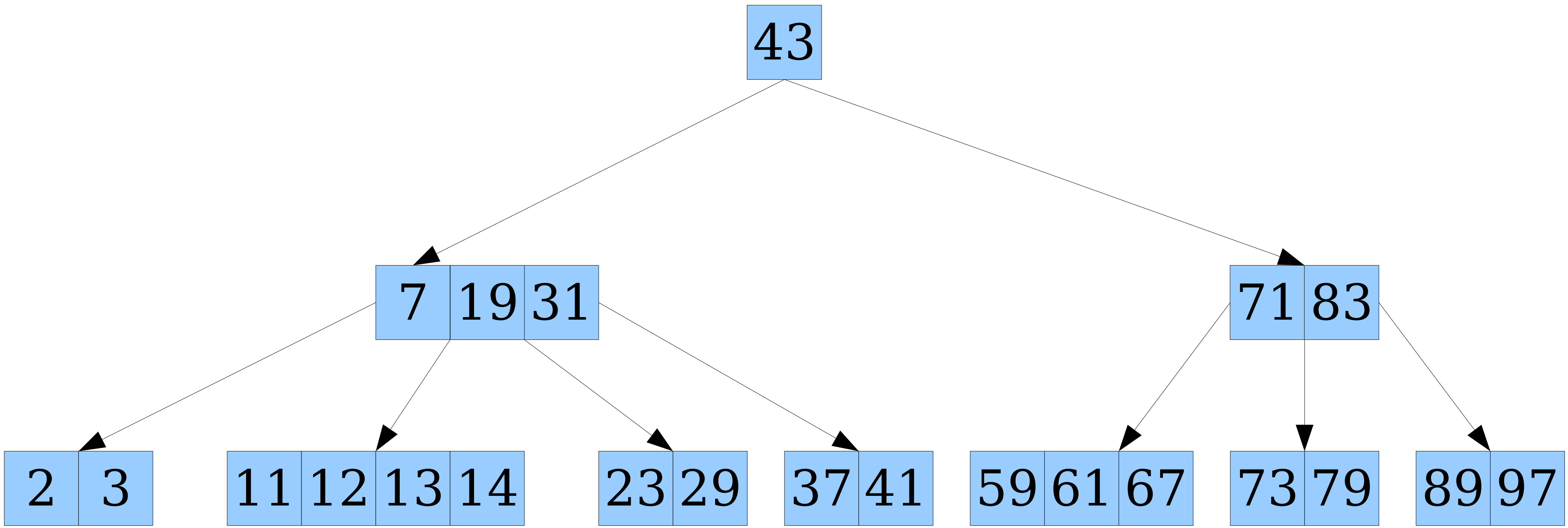

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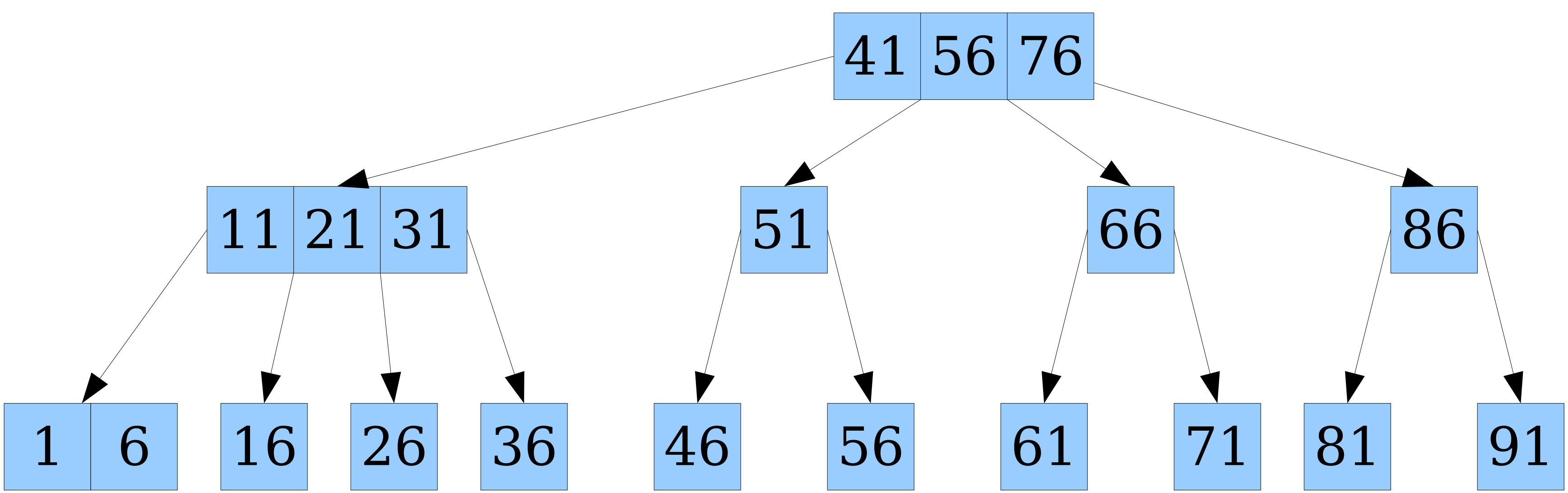
- Here's a 2-3-4 tree (each node has 1 to 3 keys).



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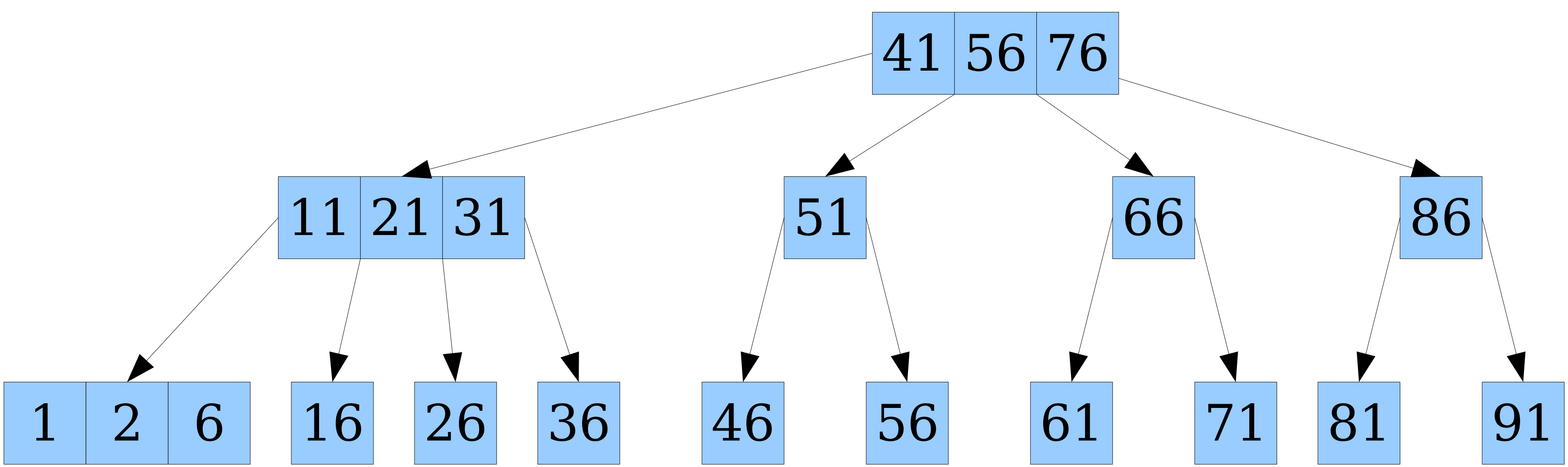

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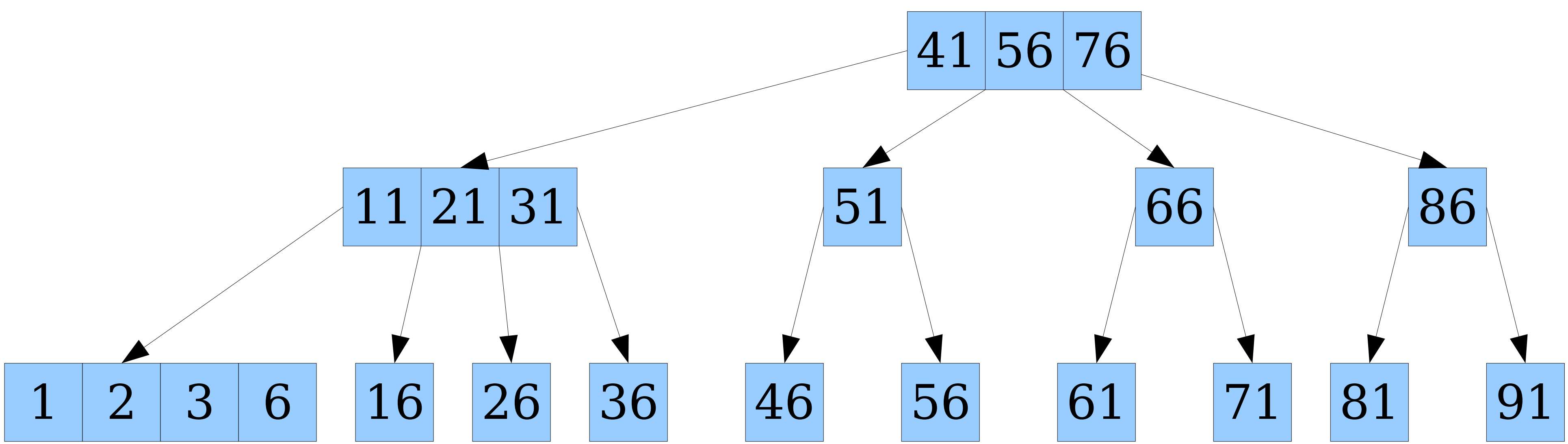

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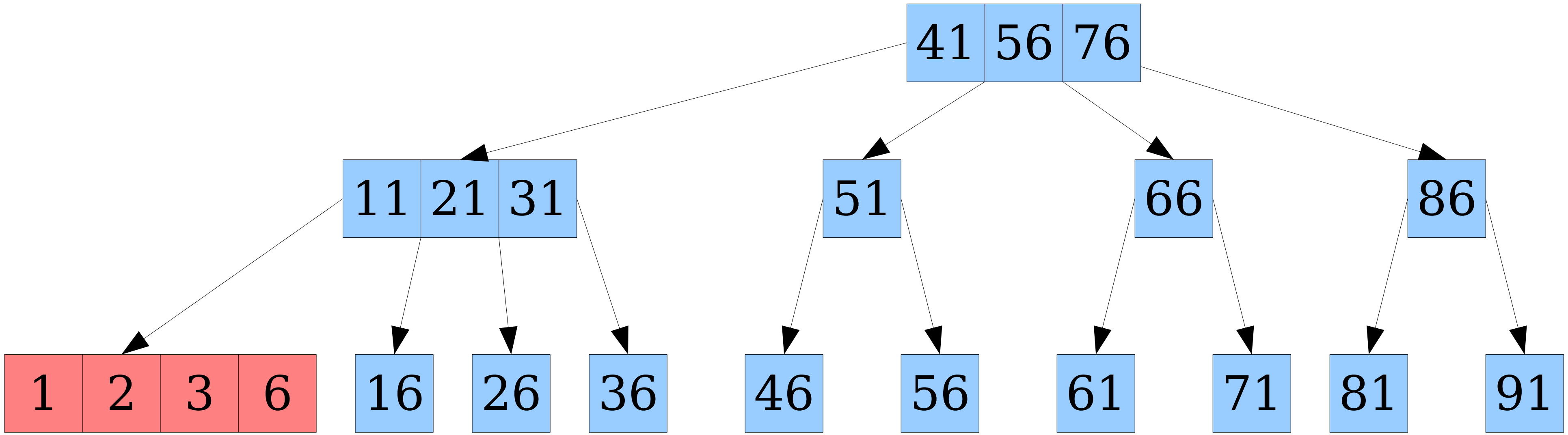

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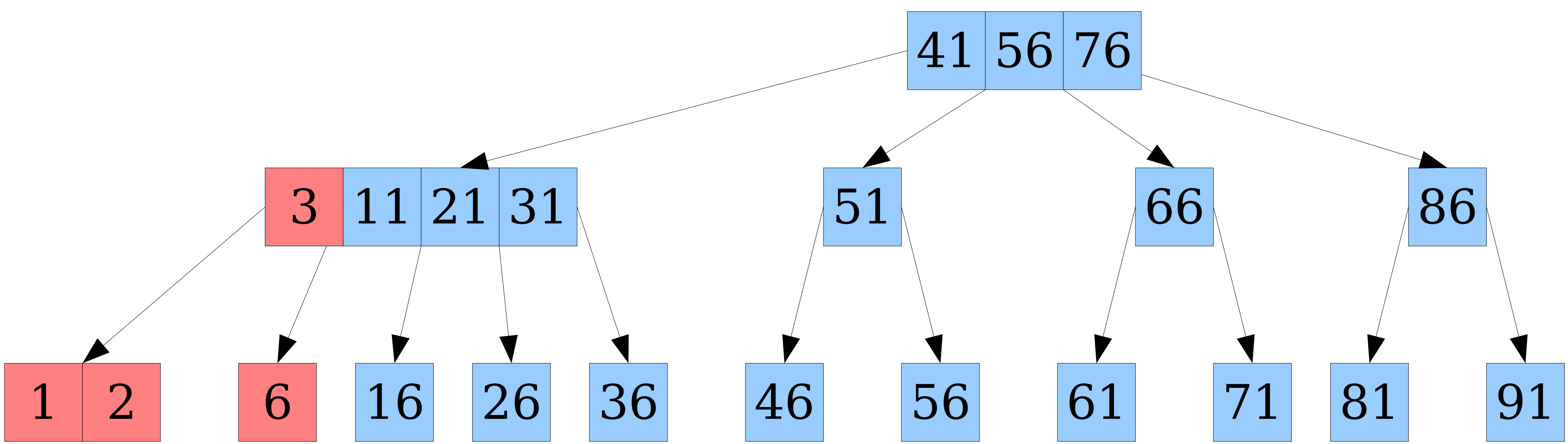

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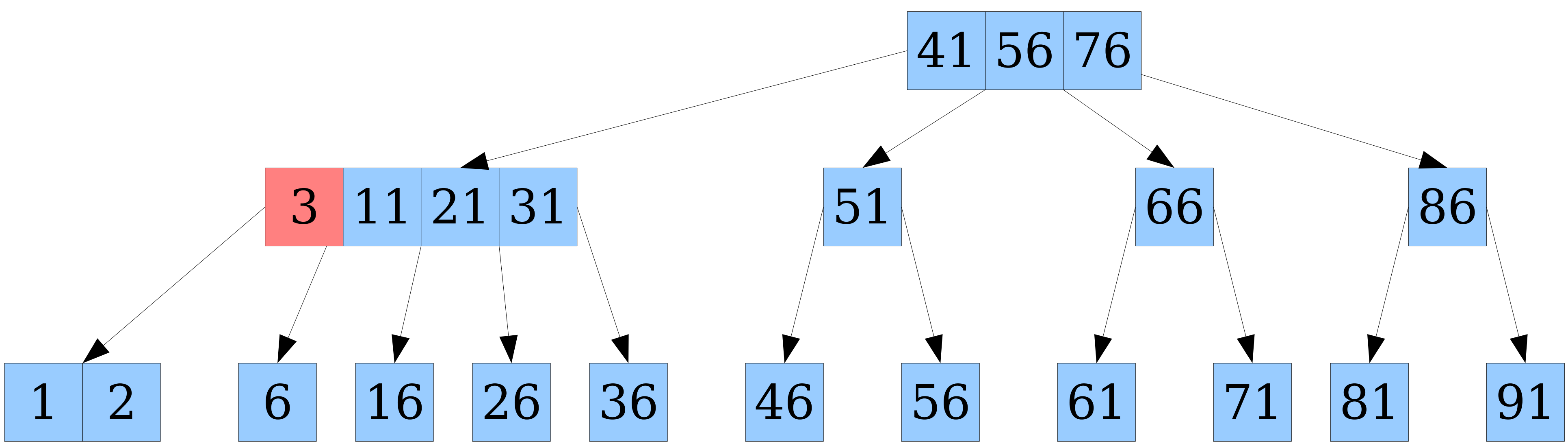

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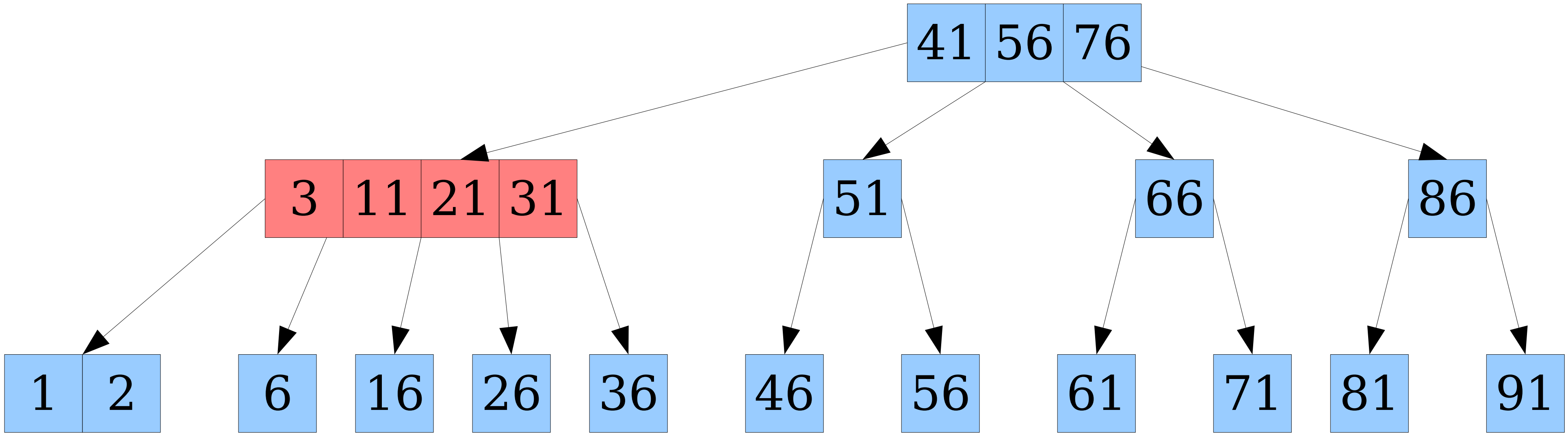

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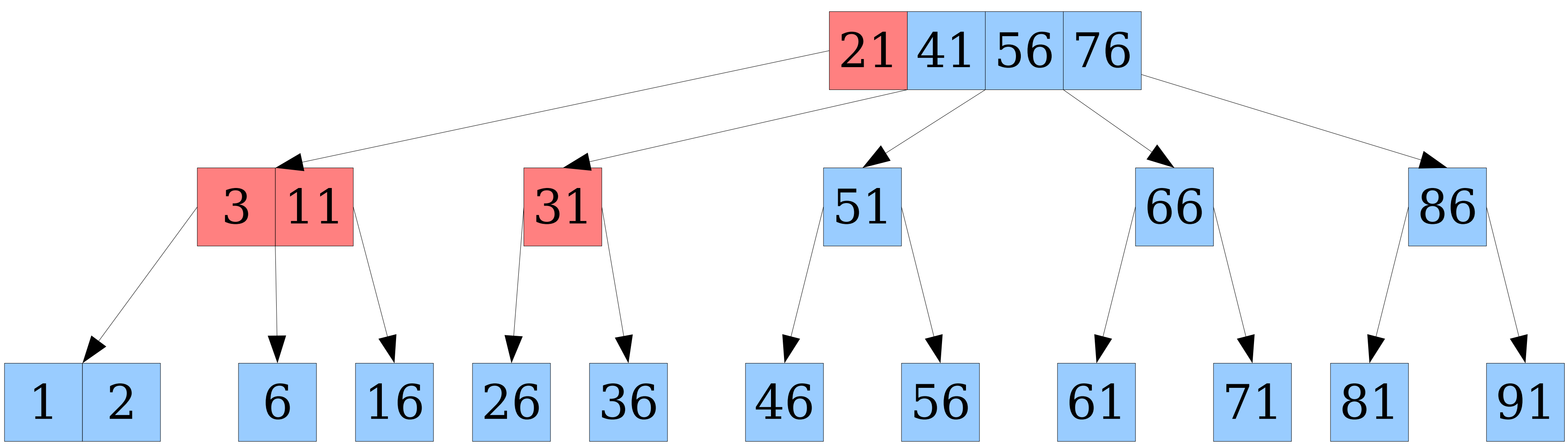

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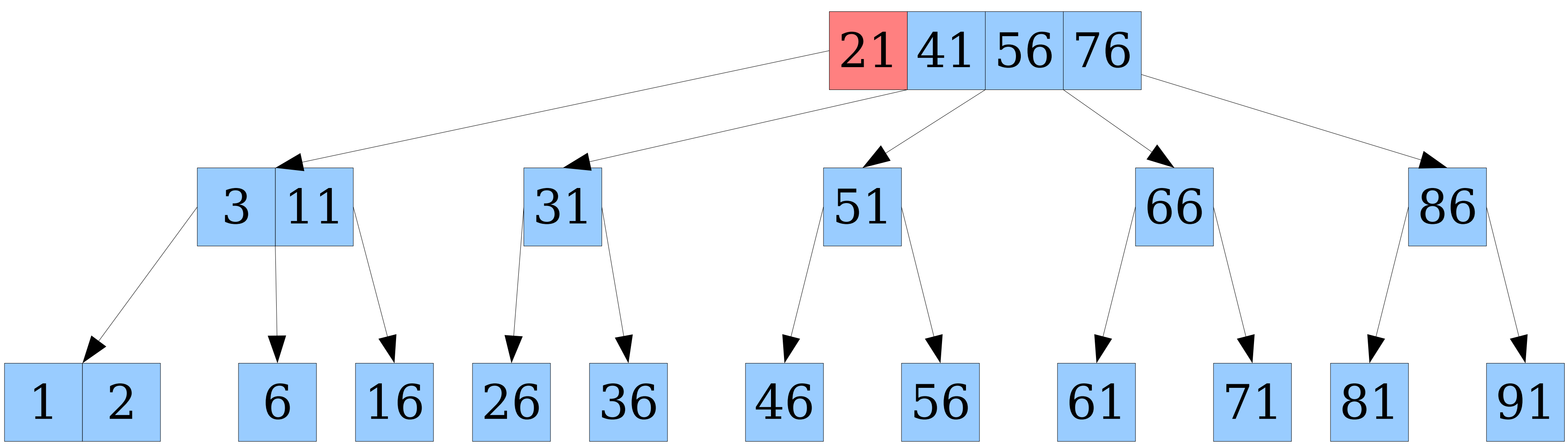

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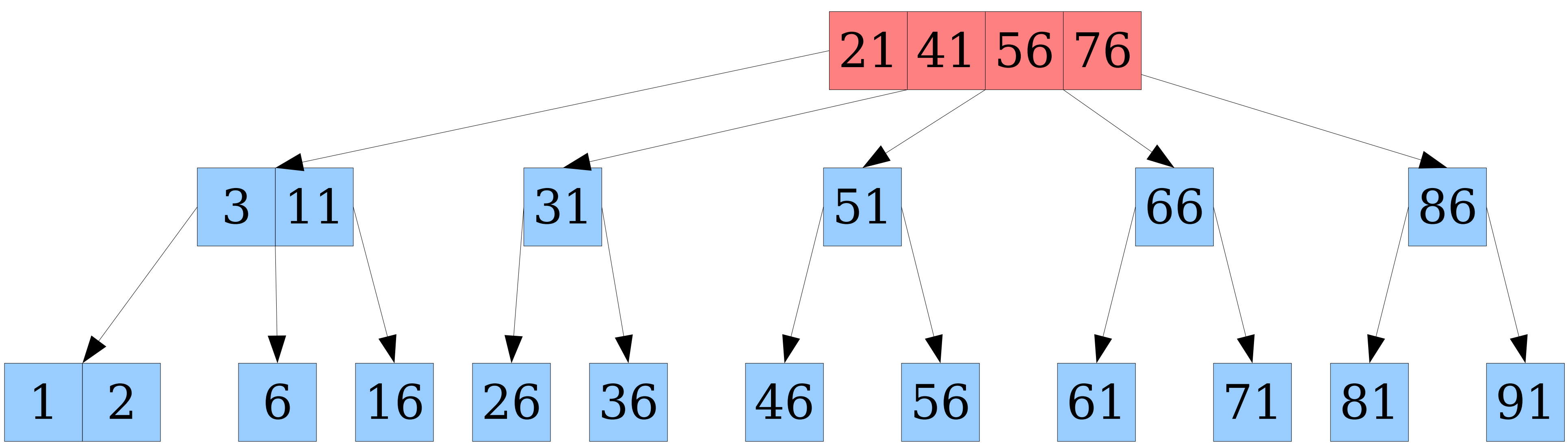

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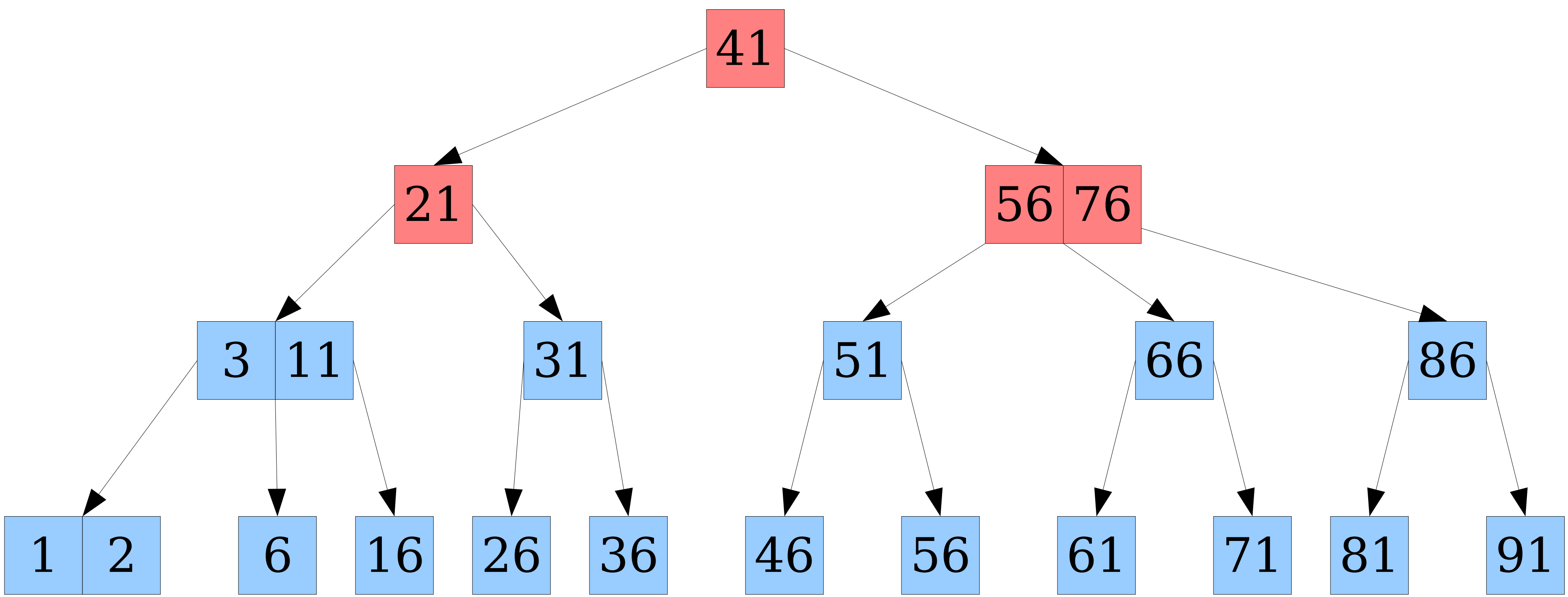

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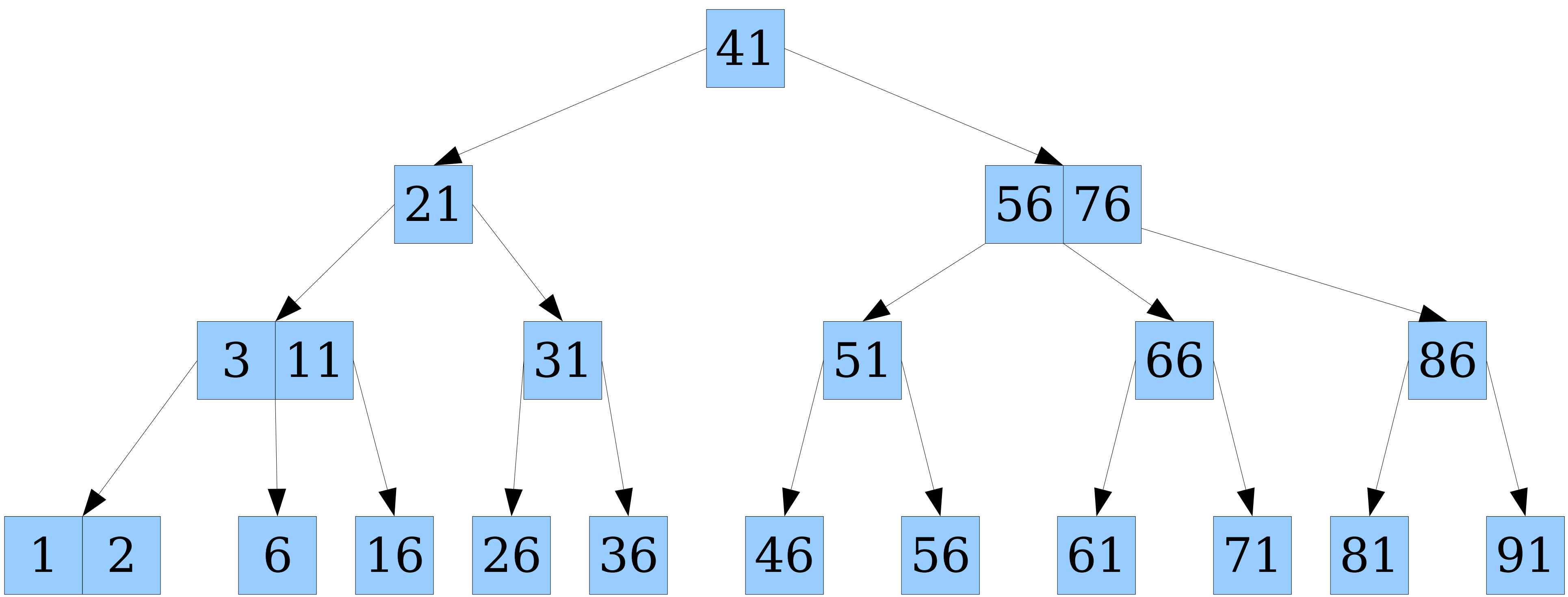

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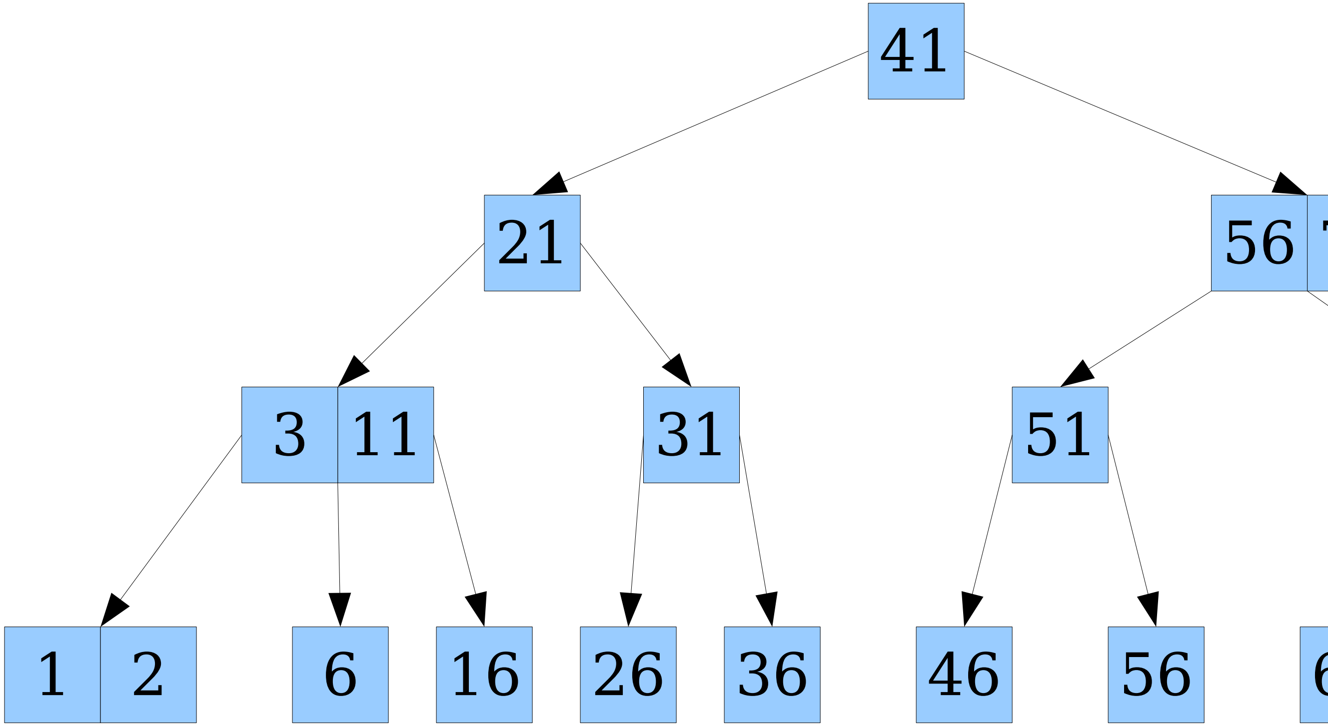
Note: B-trees grow upward, not downward.

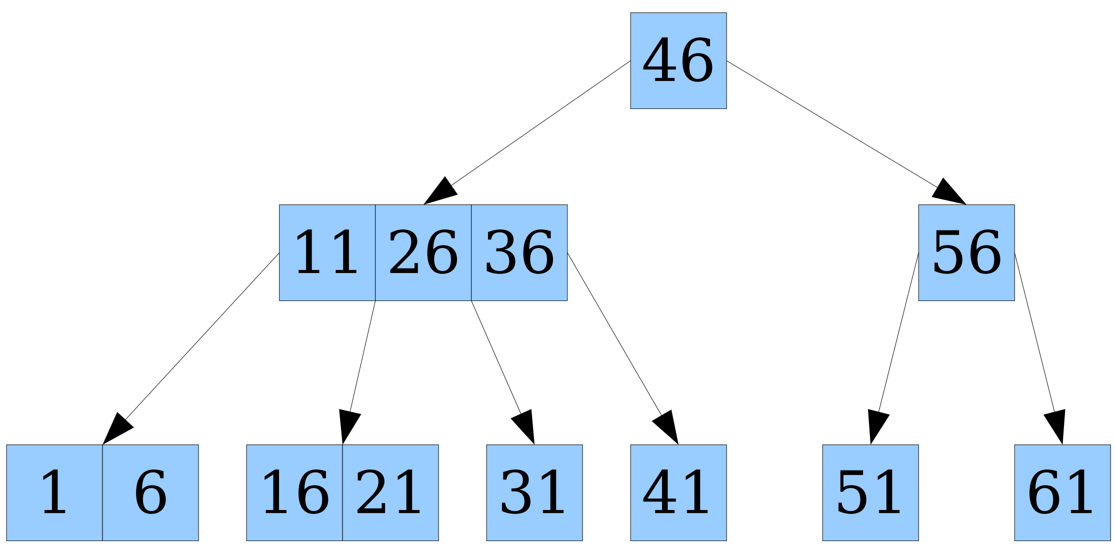


Inserting into a B-Tree

- To insert a key into a B-tree:
  - Search for the key, insert at the last-visited leaf node.
  - If the leaf is too big (contains $2b$ keys):
    - Split the node into two nodes of size b each.
    - Remove the largest key of the first block and make it the parent of both blocks.
    - Recursively add that node to the parent, possibly triggering more upward splitting.
- Time complexity:
  - $O(b)$ work per level and $O(\log_b n)$ levels.
  - Total work: $\mathbf{O(b \log_b n)}$
  - In terms of blocks read: $\mathbf{O(\log_b n)}$

The Trickier Cases

- How do you delete from a leaf that has only b – 1 keys?
- ***Idea:*** Steal keys from an adjacent nodes, or merge the nodes if both are empty.
- Again, a 2-3-4 tree:



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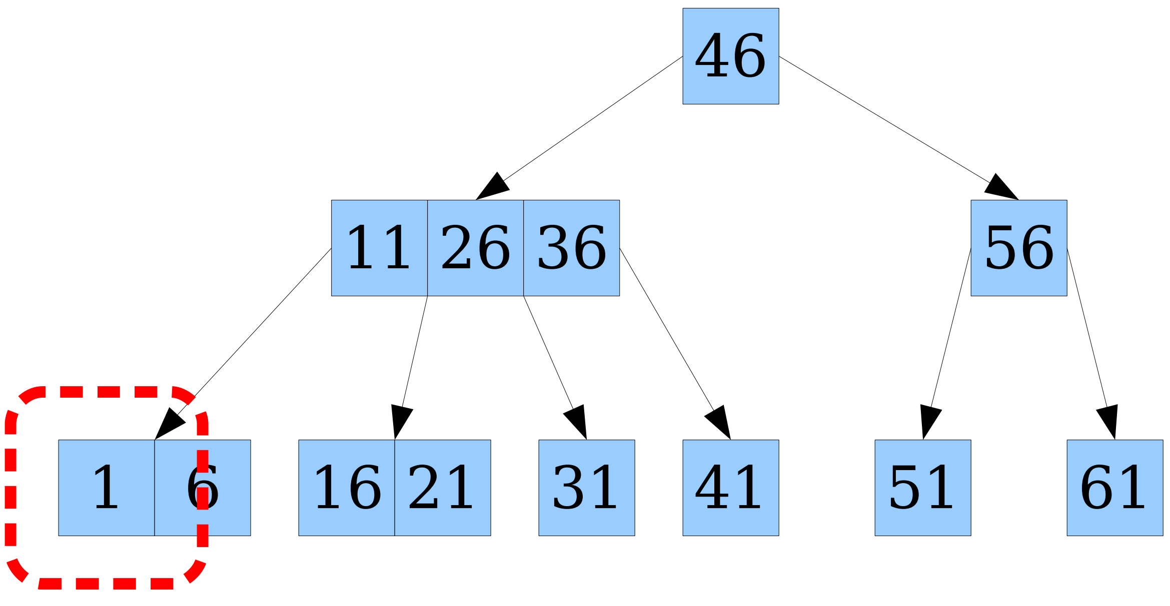

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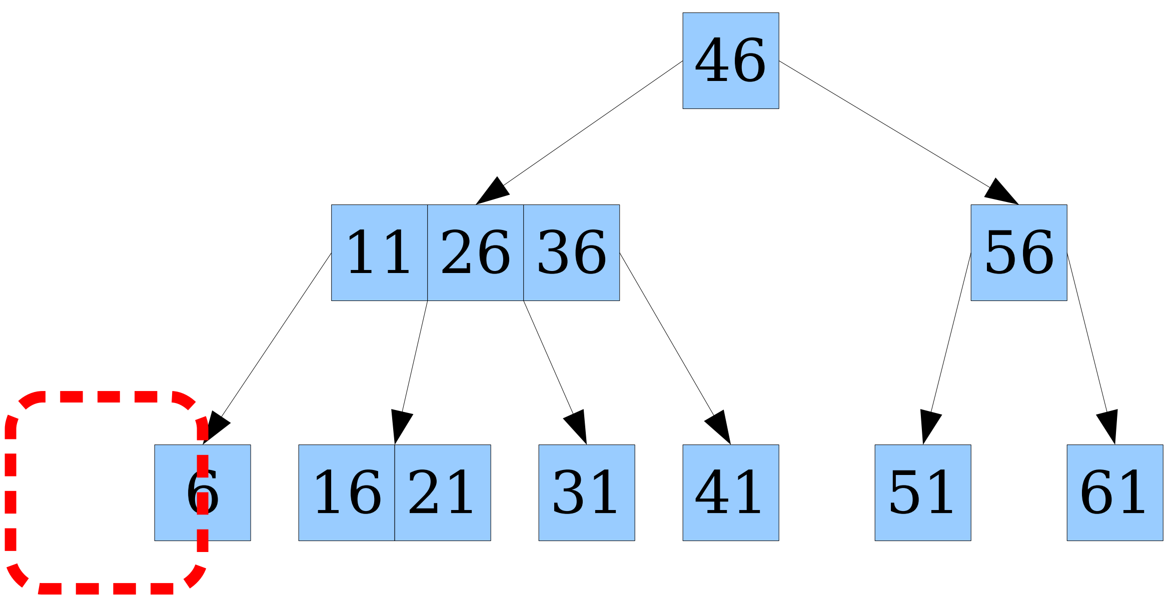

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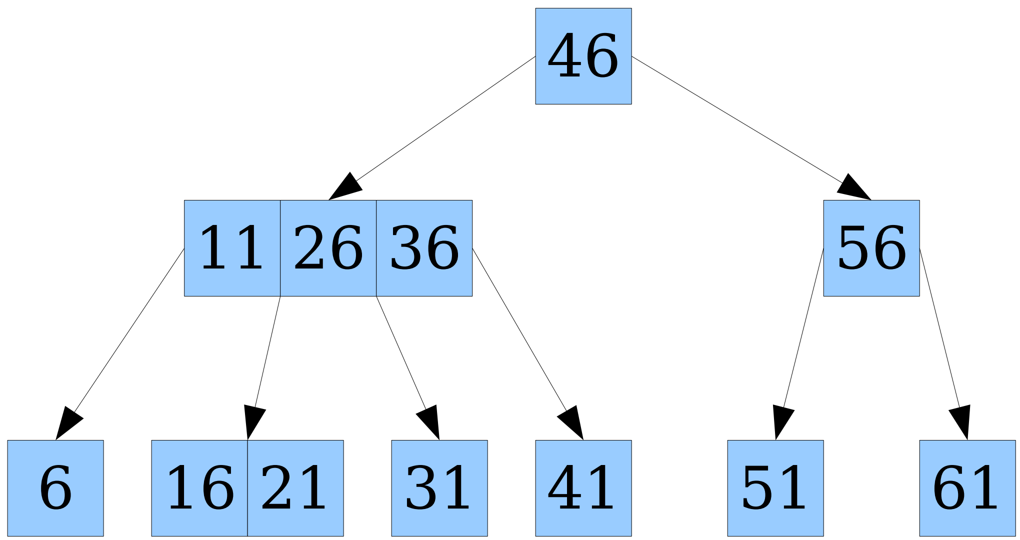

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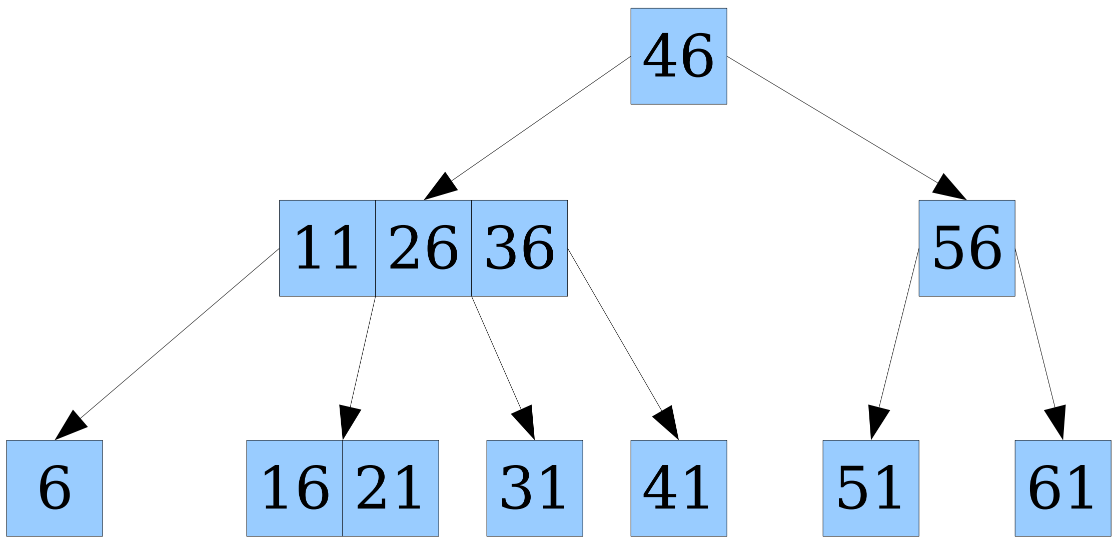

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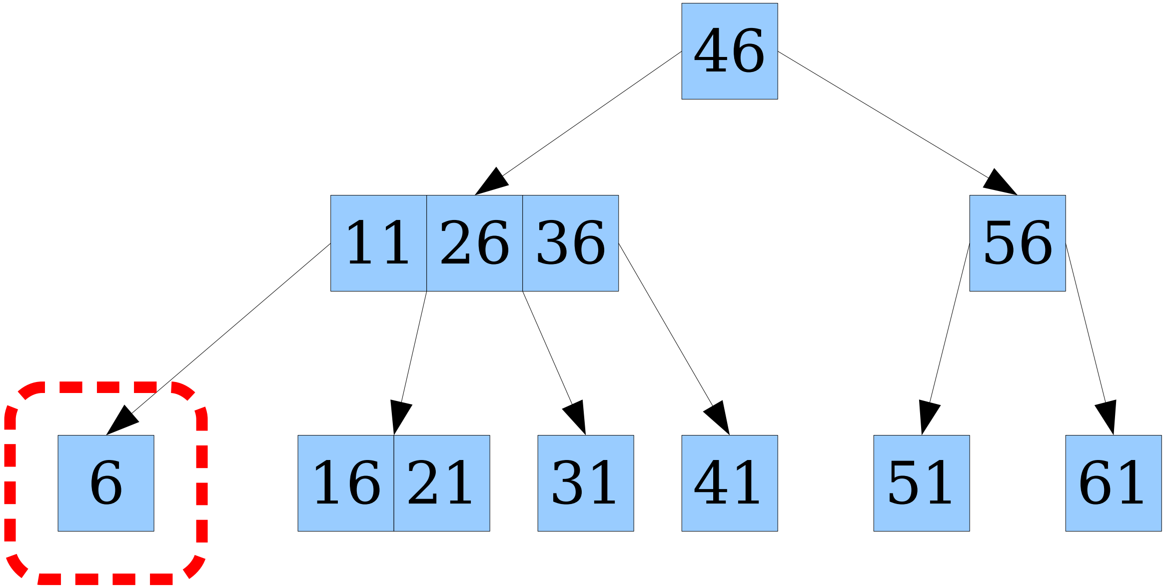

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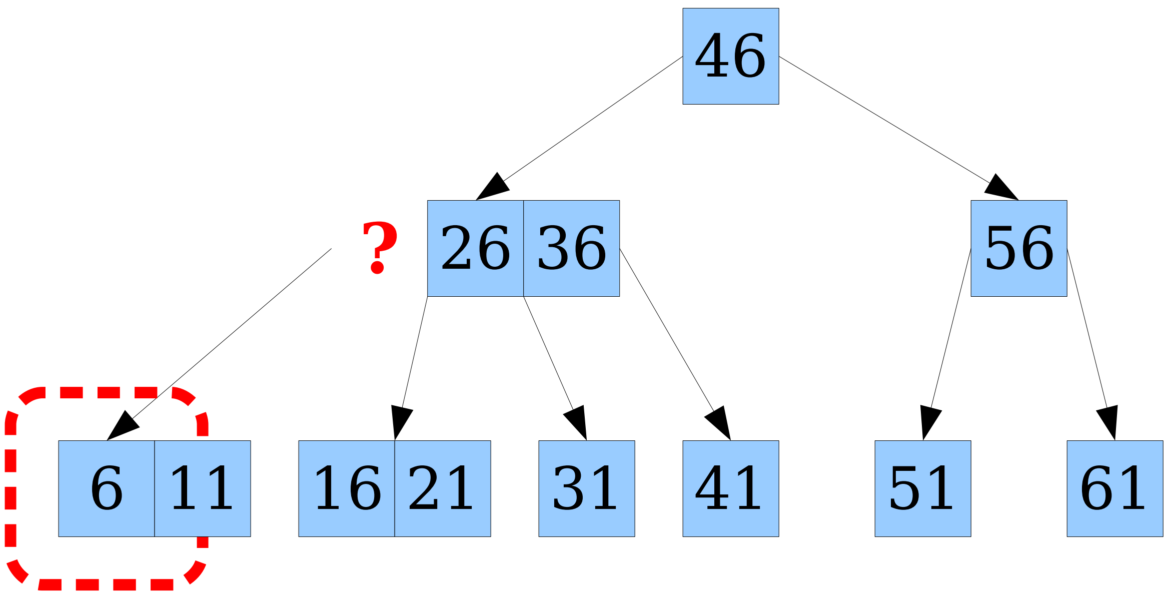

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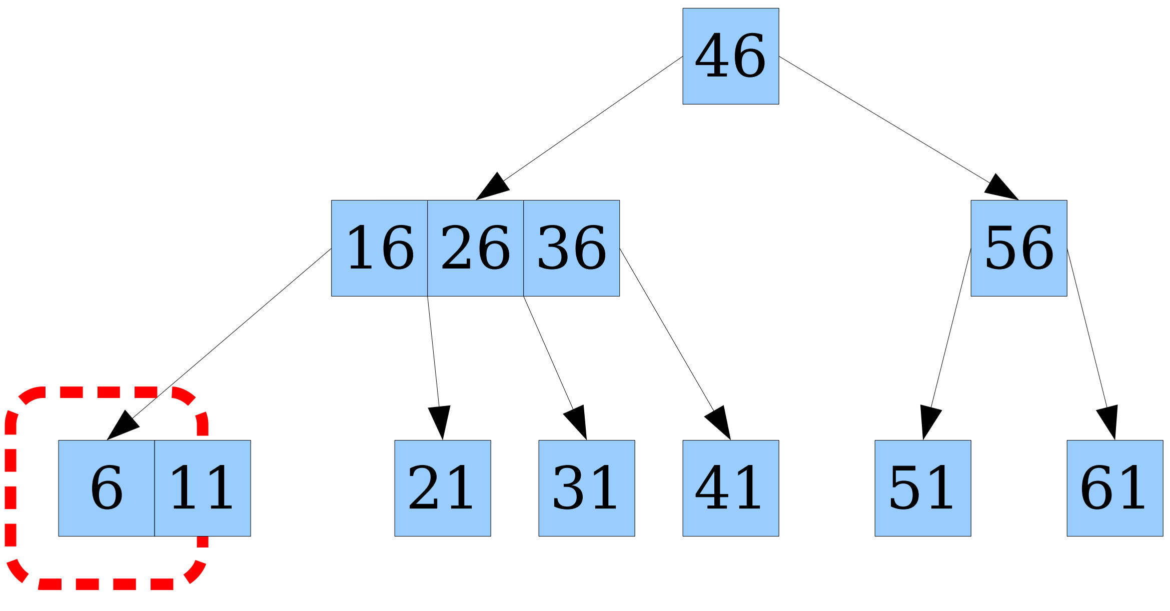

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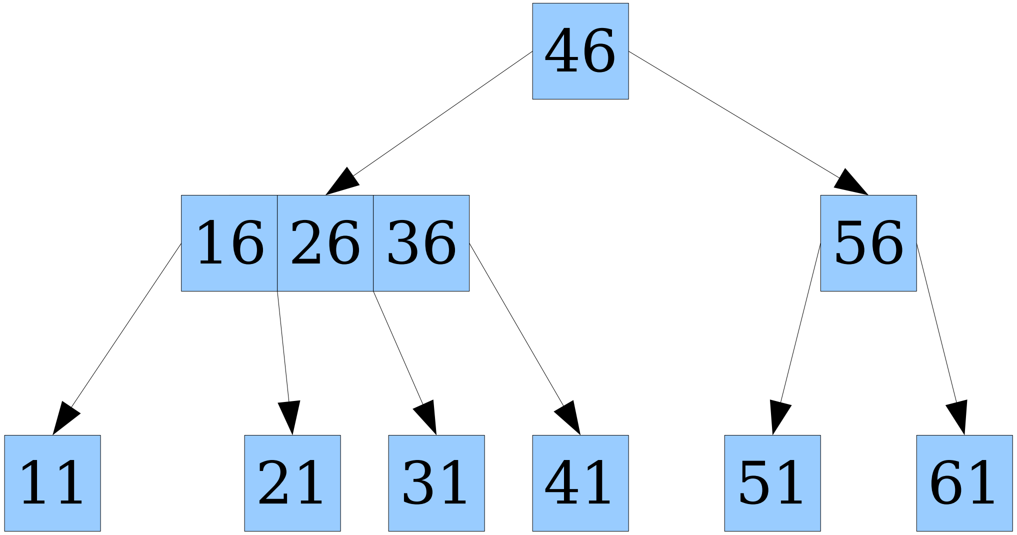

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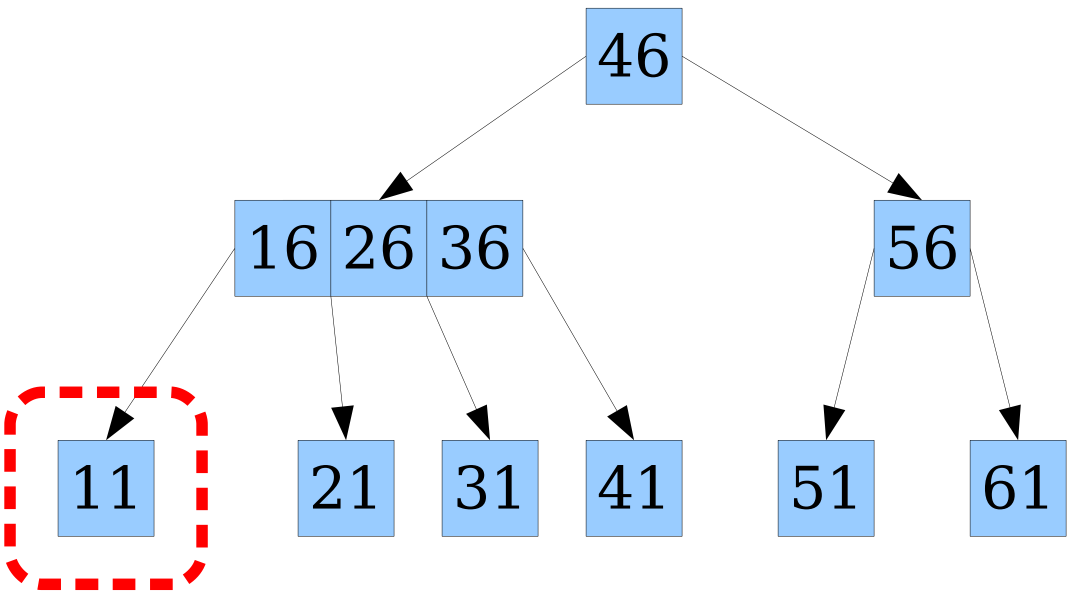

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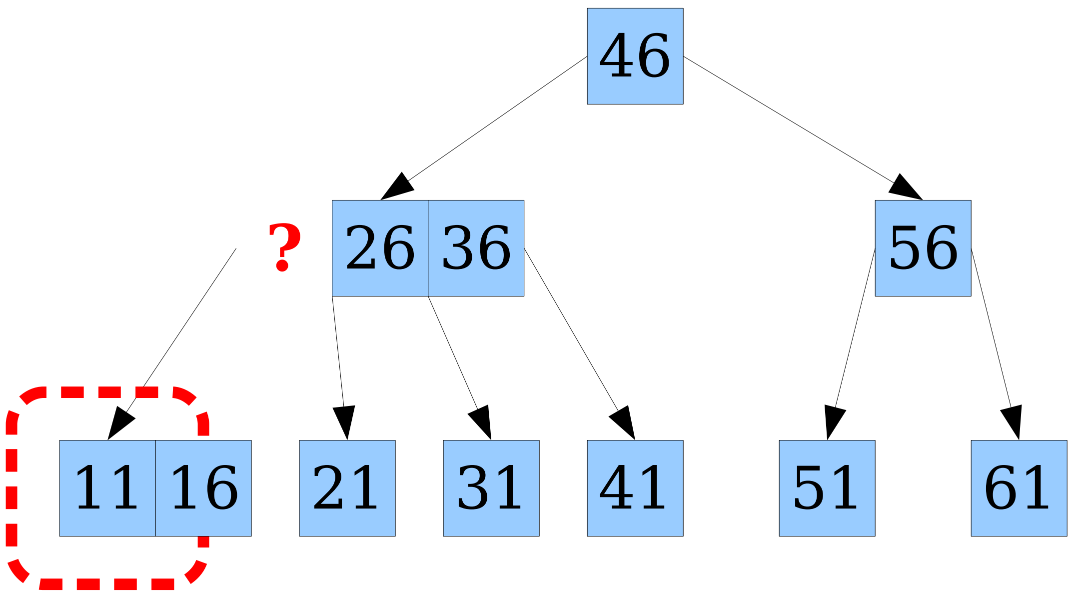

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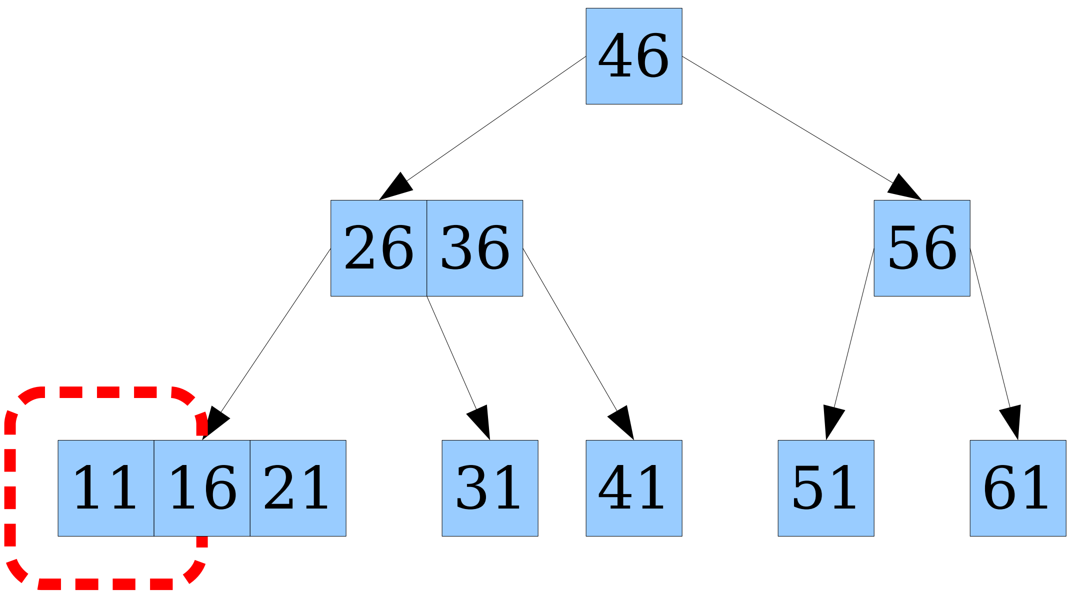

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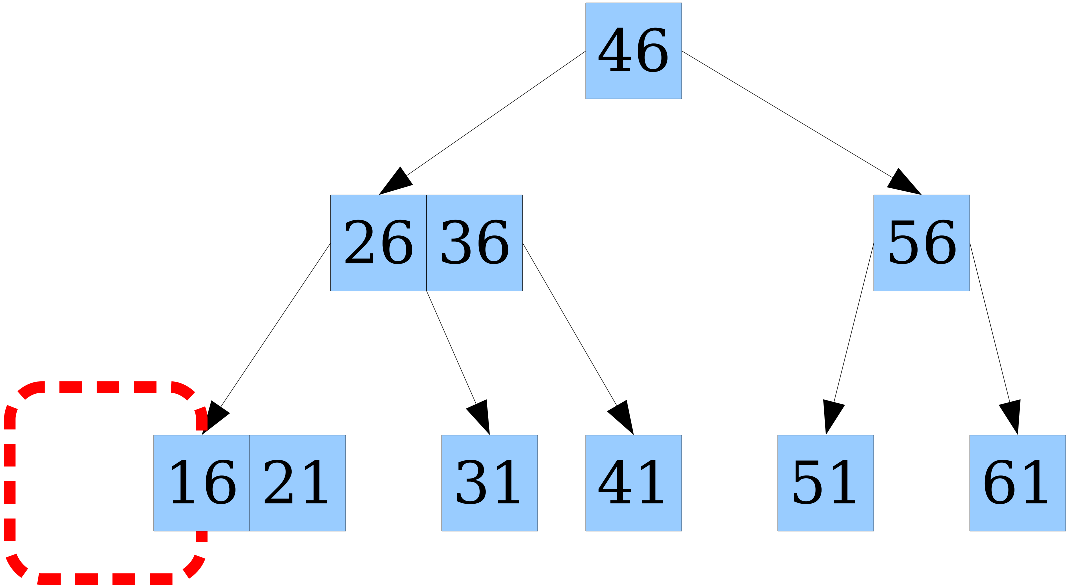

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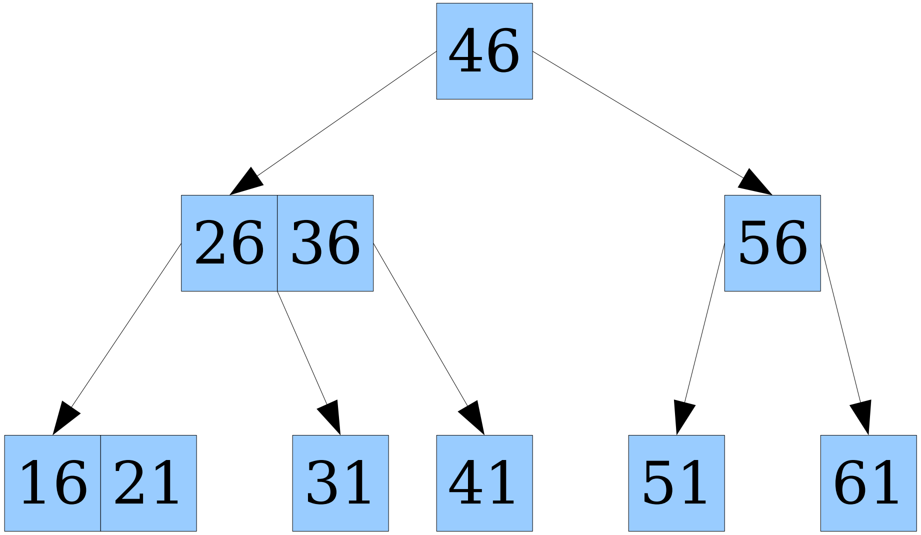

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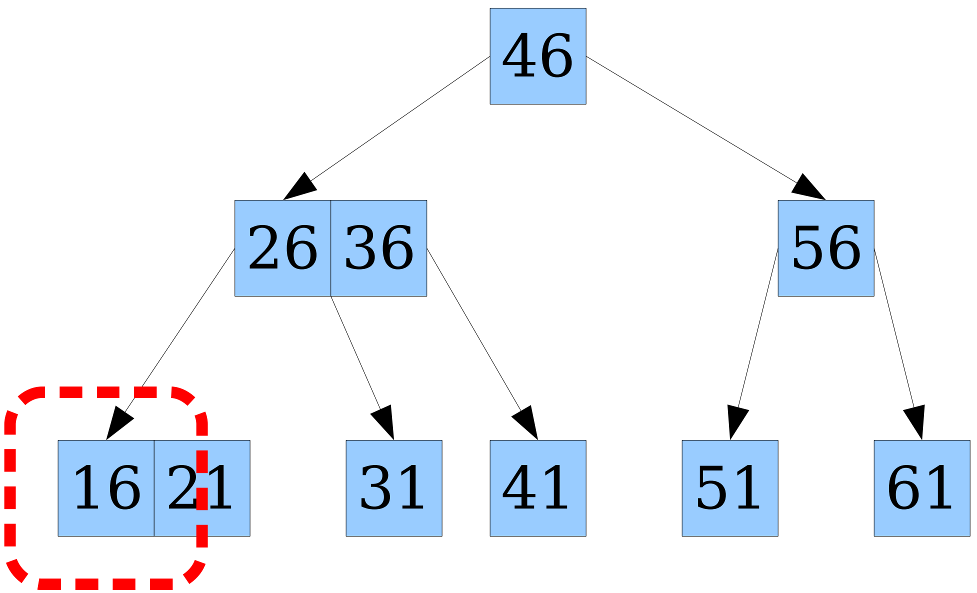

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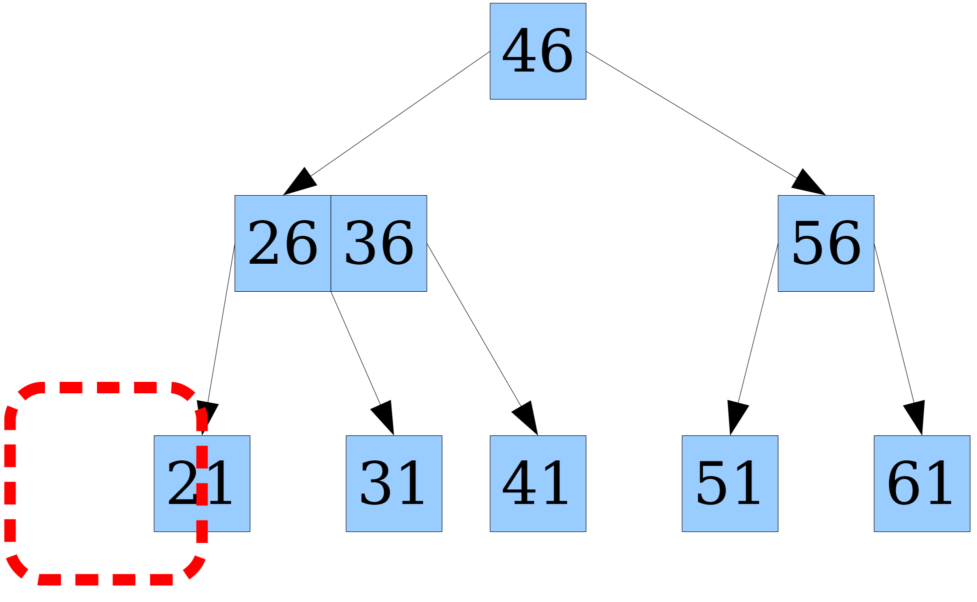

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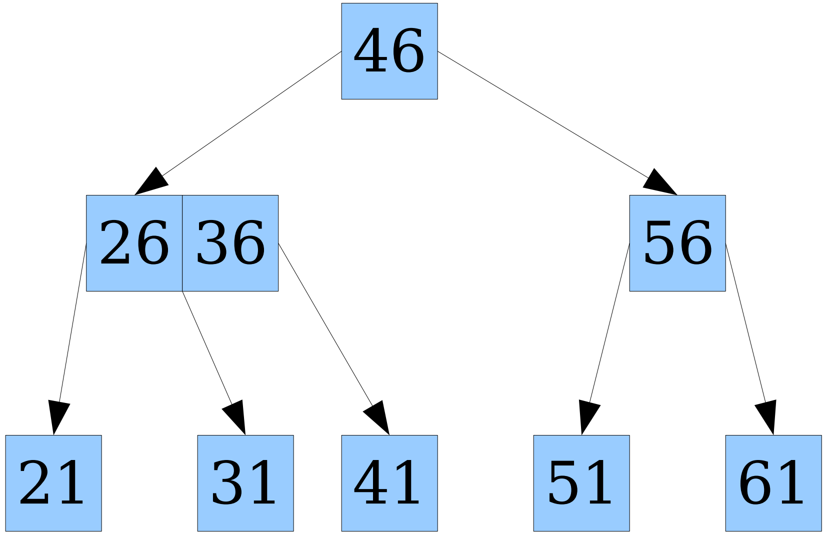

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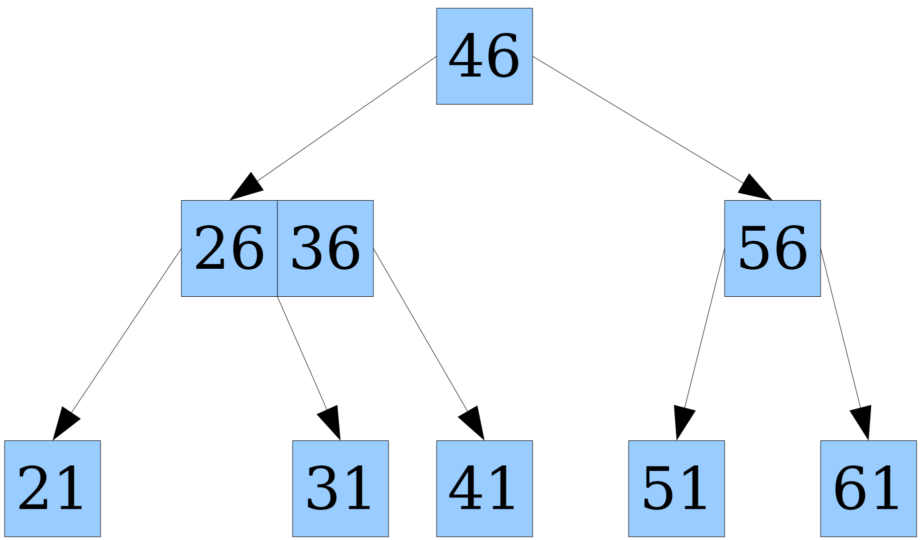

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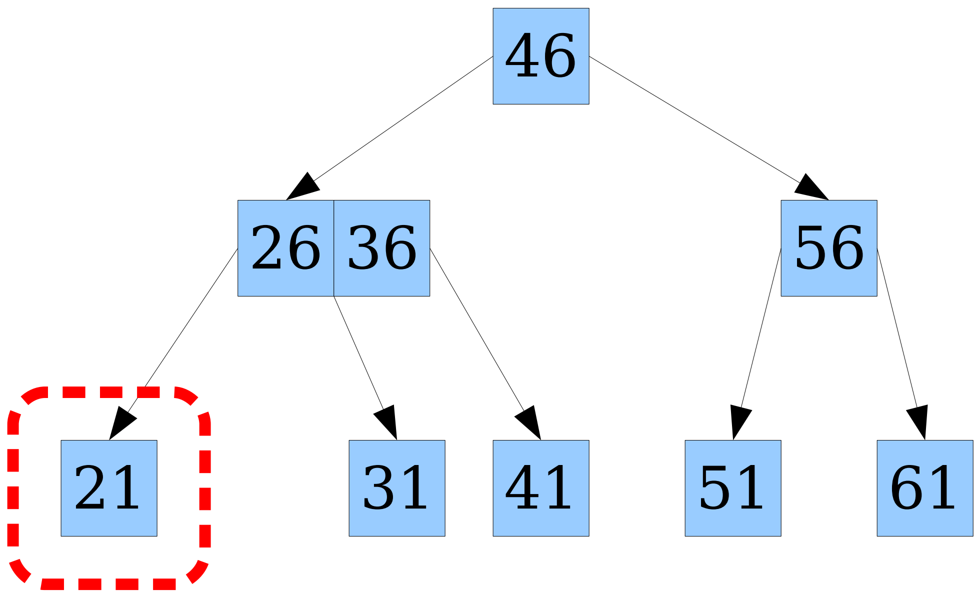

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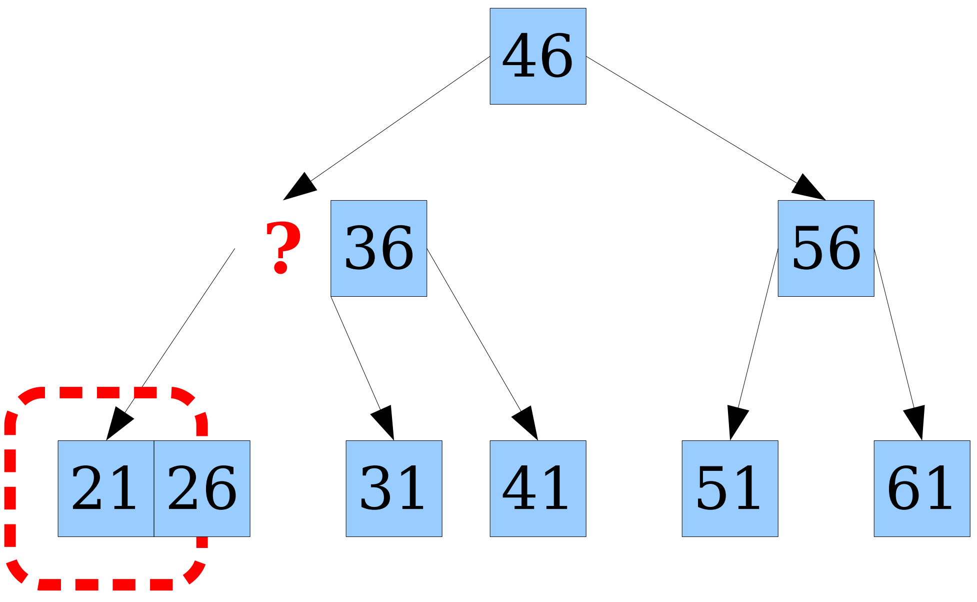

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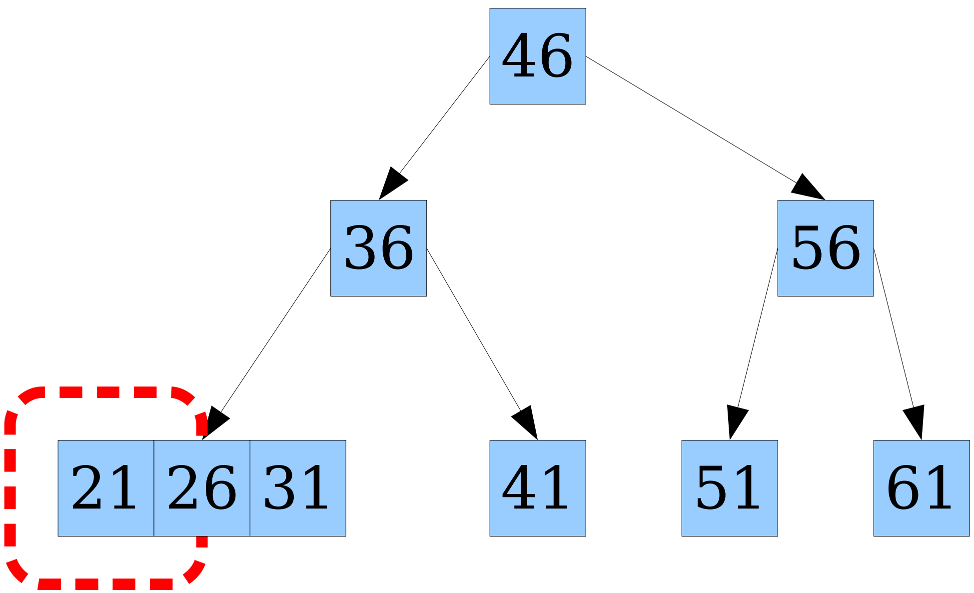

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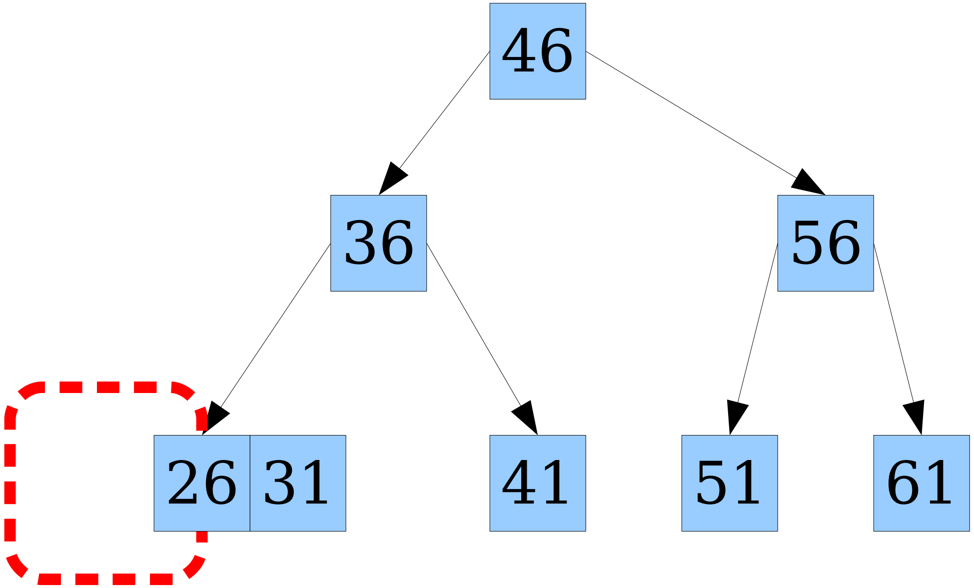

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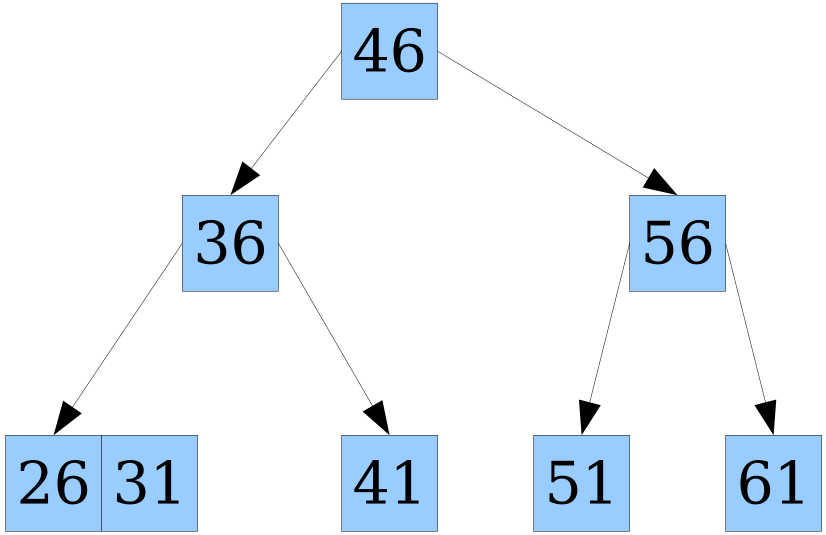

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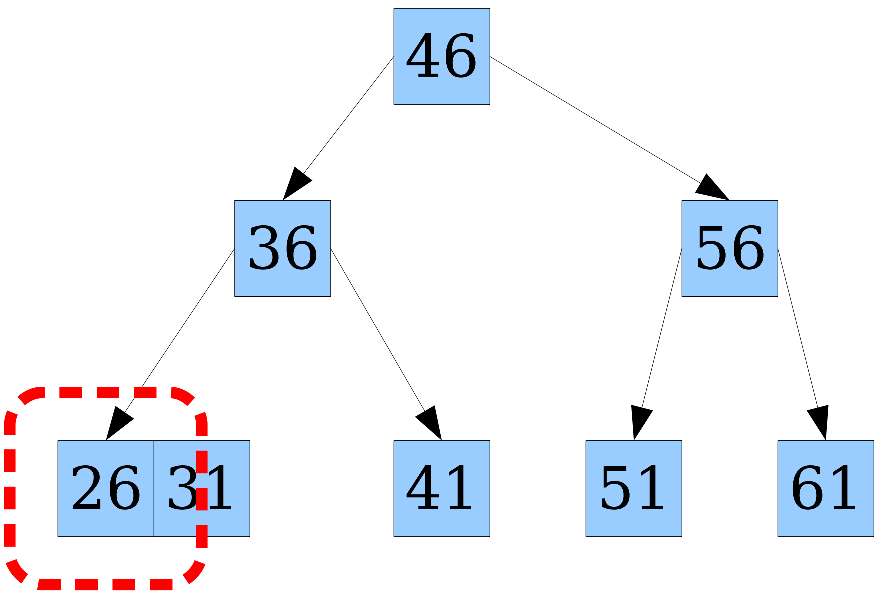

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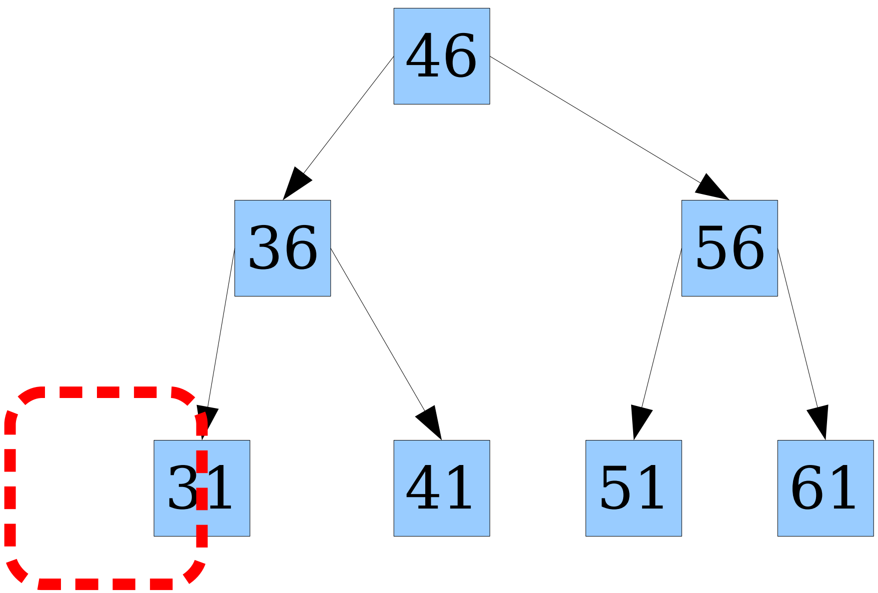

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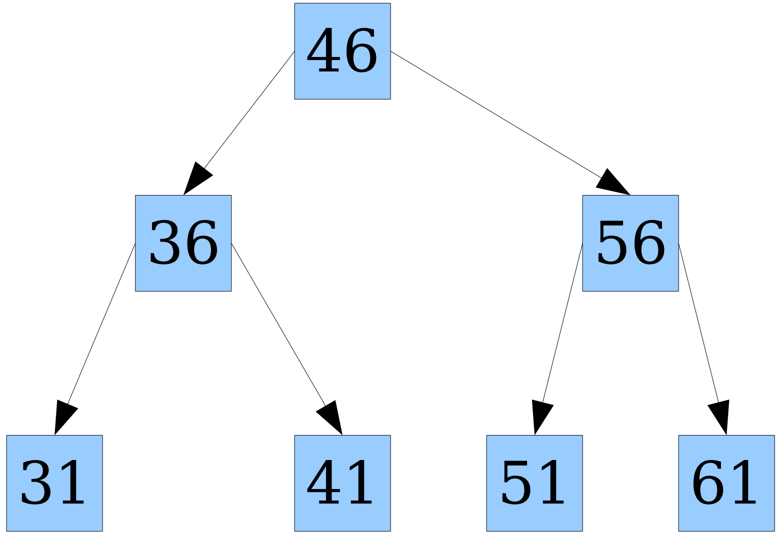

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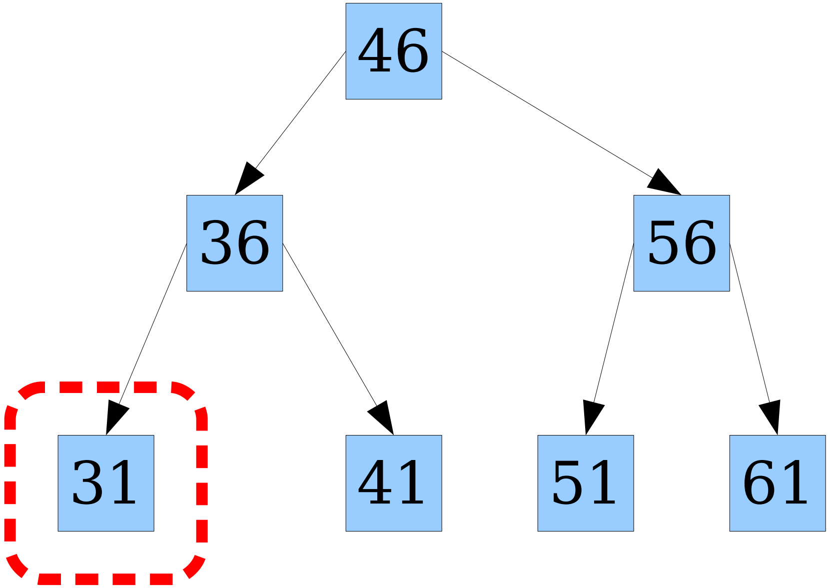

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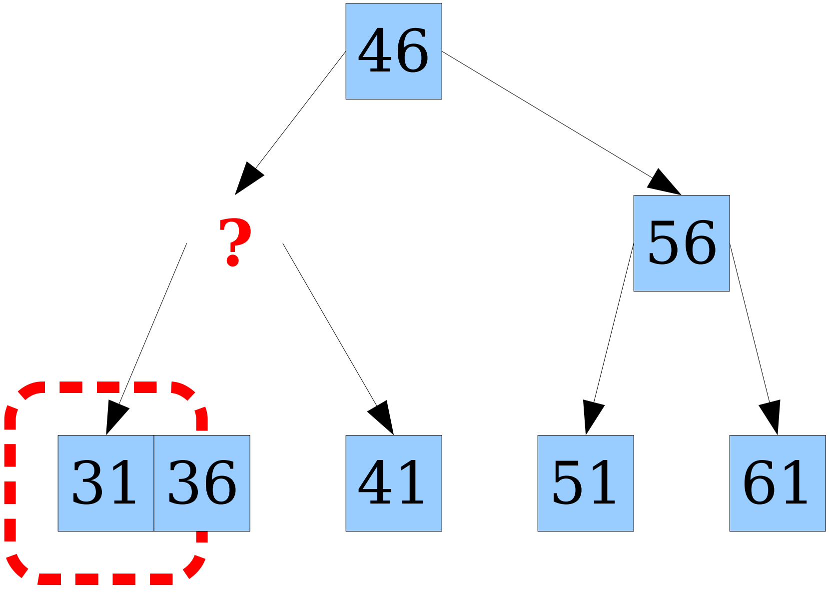

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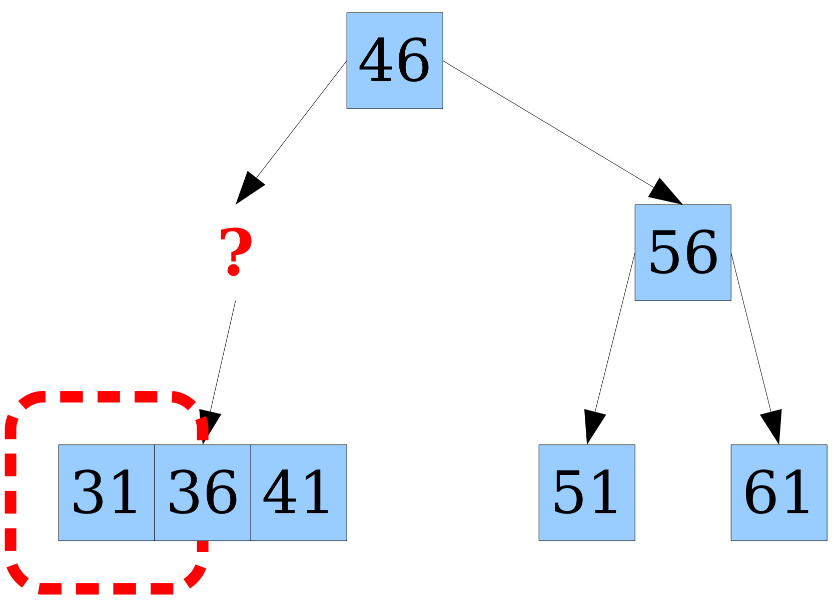

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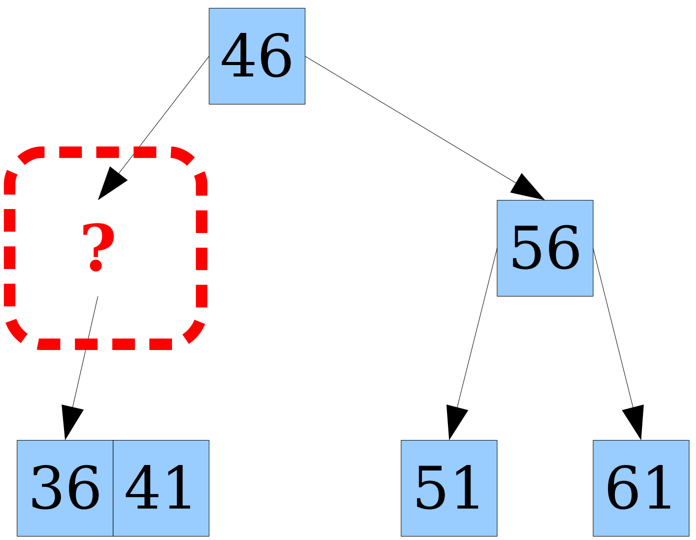

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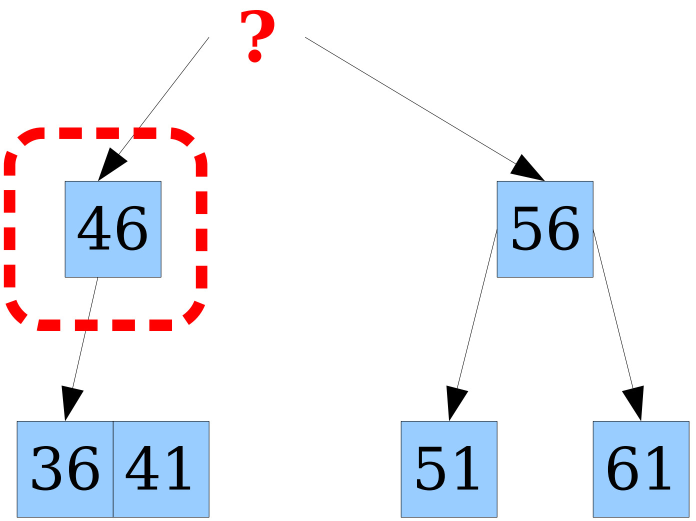

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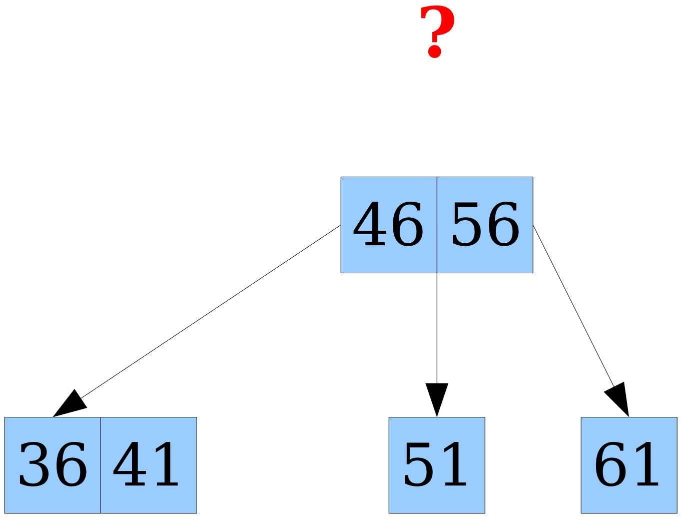

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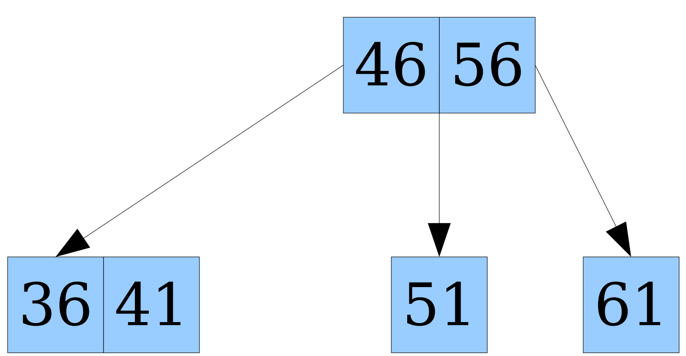


Deleting from a B-Tree

- If not in a leaf, replace the key with its successor from a leaf and delete out of a leaf.
- To delete a key from a node:
  - If the node has more than b – 1 keys, or if the node is the root, just remove the key.
  - Otherwise, find a sibling node whose shared parent is p.
  - If that sibling has more than b – 1 keys, move the max/min key from that sibling into p's place and p down into the current node, then remove the key.
  - Otherwise, fuse the node and its sibling into a single node by adding p into the block, then recursively remove p from the parent node.
- Work done is $O(b \log_b n)$: $O(b)$ work per level times $O(\log_b n)$ total levels. Requires $O(\log_b n)$ block reads/writes.

Time-Out for Announcements!

Amazingly, there aren't any announcements!

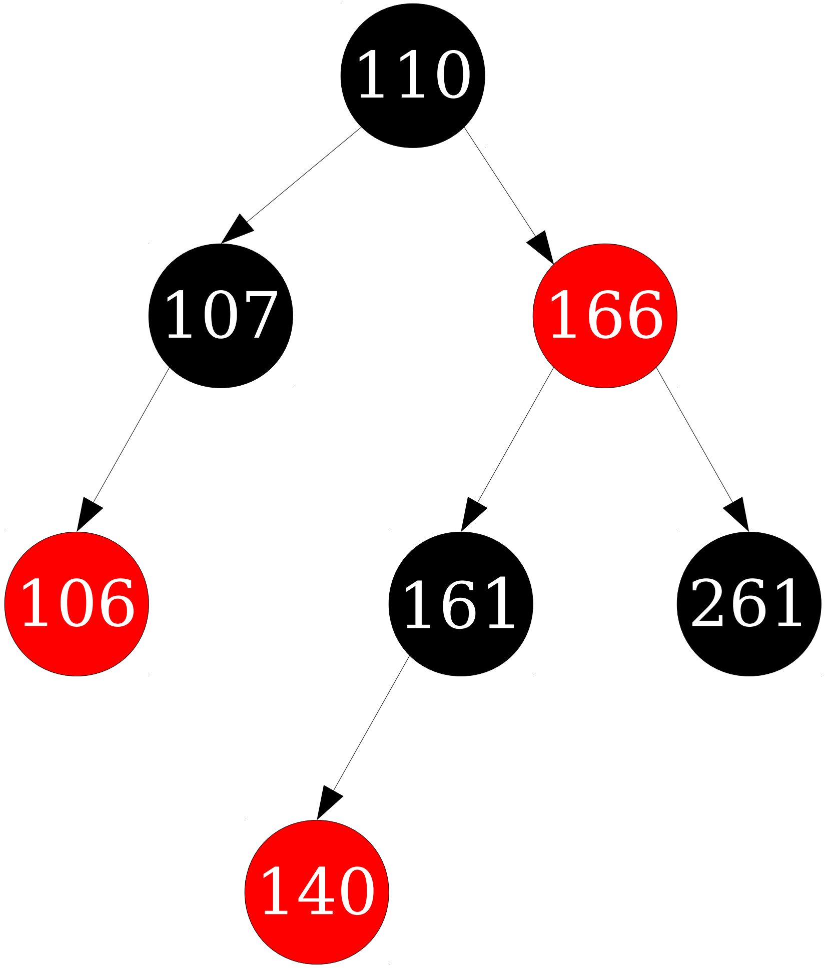
Let's just take a short break.

Back to CS166!

So… red/black trees?

Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



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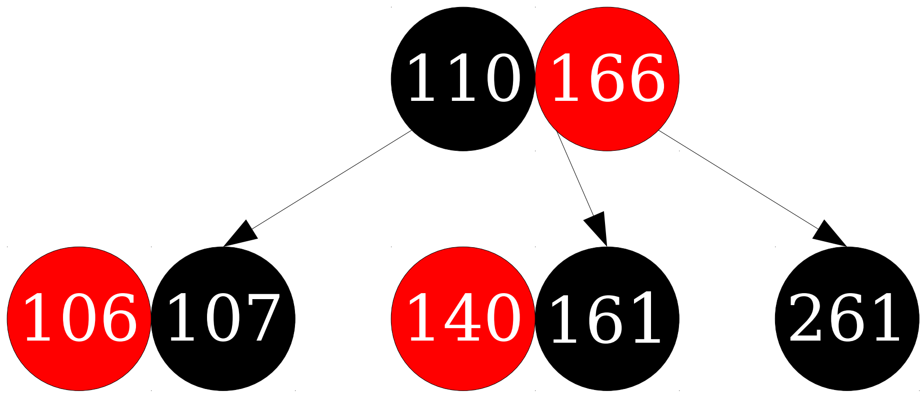

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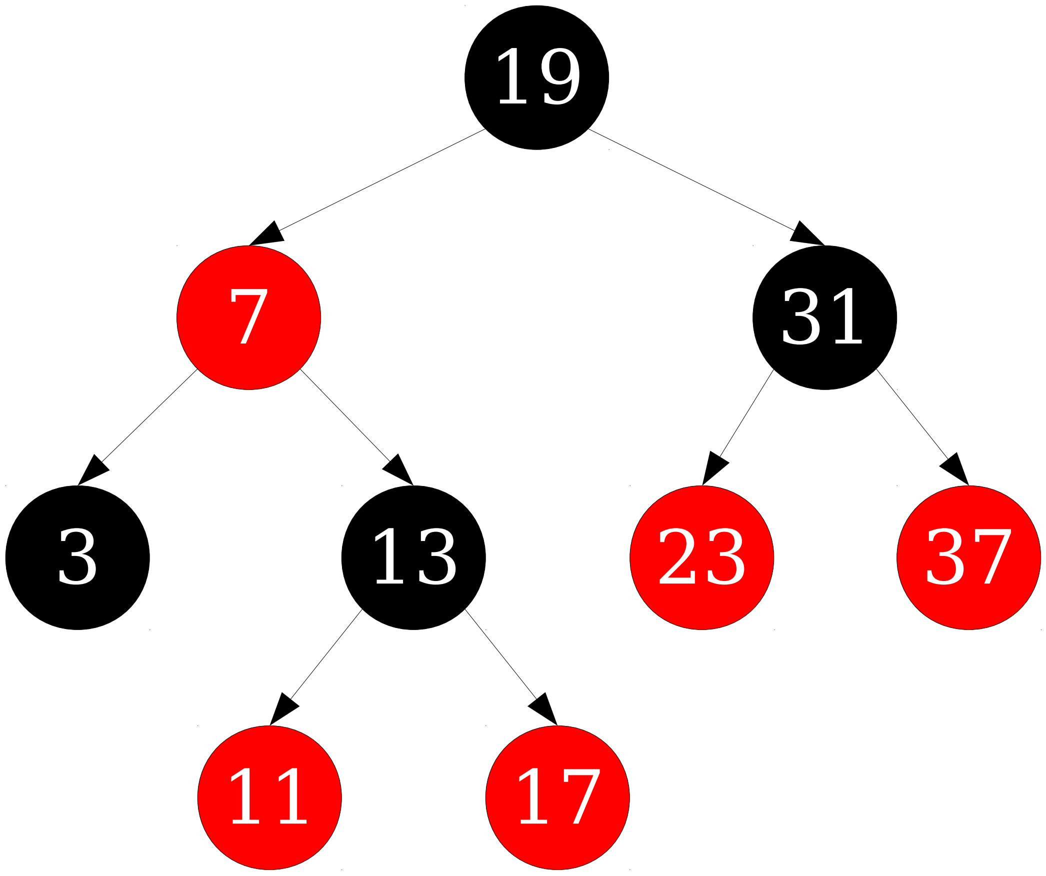

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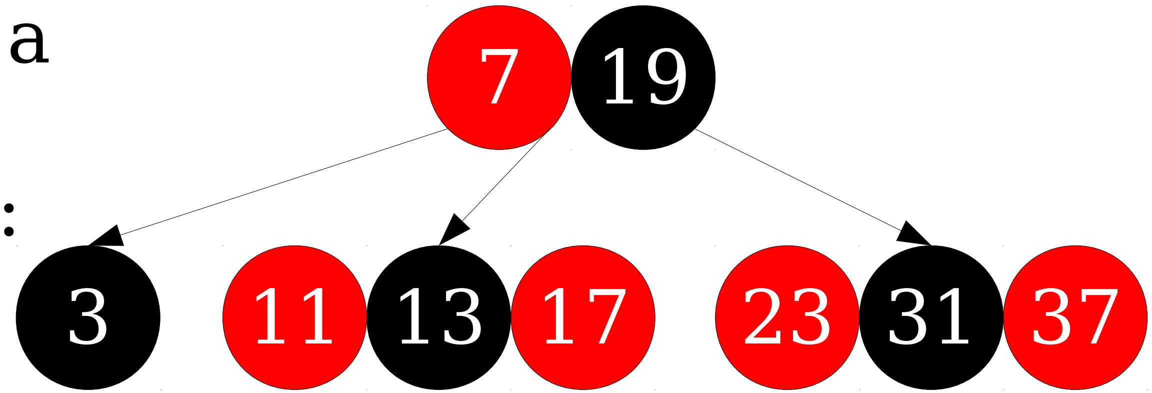


Red/Black Trees ≡ 2-3-4 Trees

- Red/black trees are an ***isometry*** of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- Many data structures can be designed and analyzed in the same way.
- ***Huge advantage:*** Rather than memorizing a complex list of red/black tree rules, just think about what the equivalent operation on the corresponding 2-3-4 tree would be and simulate it with color flips and rotations.

The Height of a Red/Black Tree

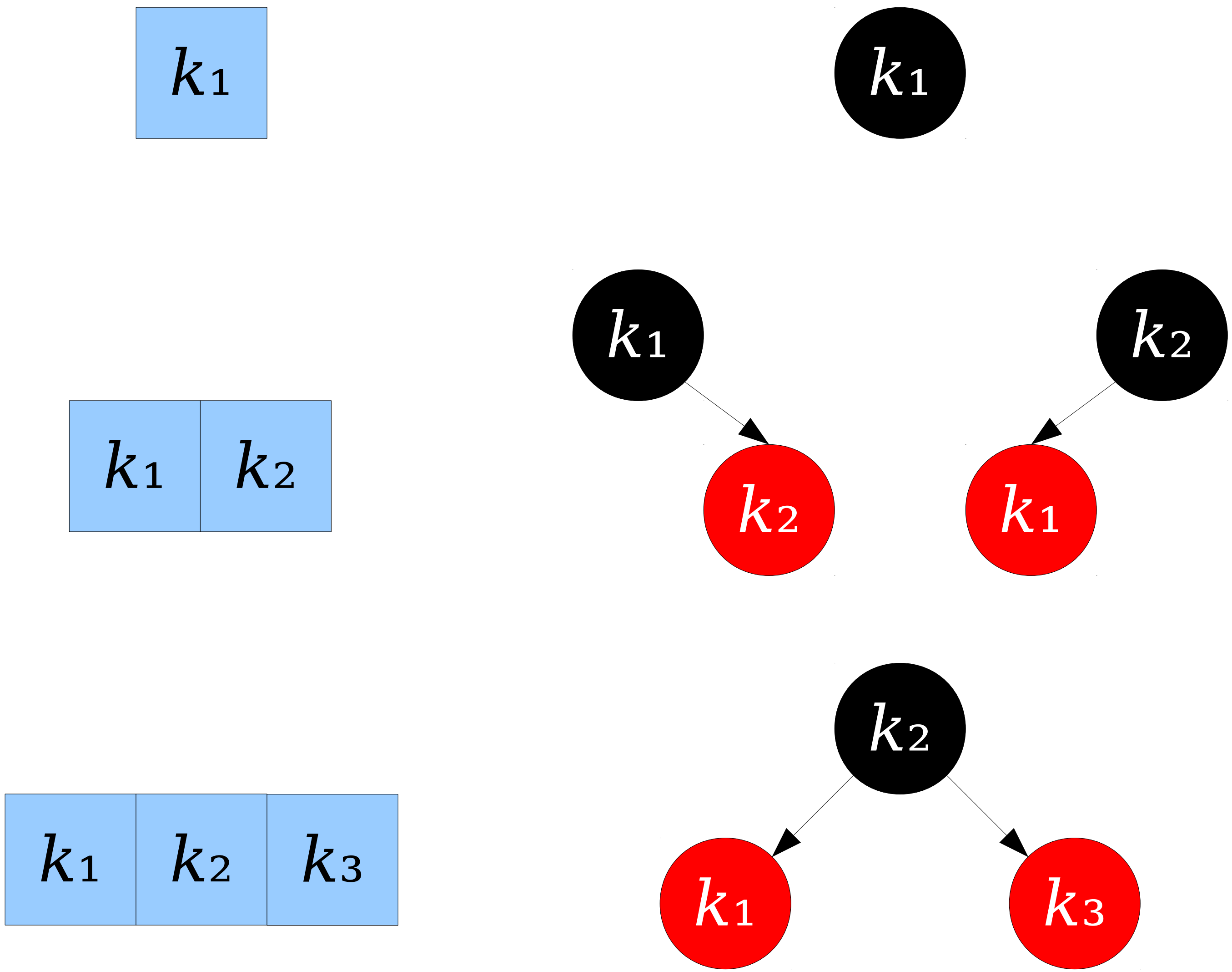
Theorem: Any red/black tree with n nodes has height $O(\log n)$.

Proof: Contract all red nodes into their parent nodes to convert the red/black tree into a 2-3-4 tree. This decreases the height of the tree by at most a factor of two. The resulting 2-3-4 tree has height $O(\log n)$, so the original red/black tree has height $2 \cdot O(\log n) = O(\log n)$. ■

Exploring the Isometry

- Nodes in a 2-3-4 tree are classified into types based on the number of children they can have.
  - ***2-nodes*** have one key (two children).
  - ***3-nodes*** have two keys (three children).
  - ***4-nodes*** have three keys (four children).
- How might these nodes be represented?

Exploring the Isometry



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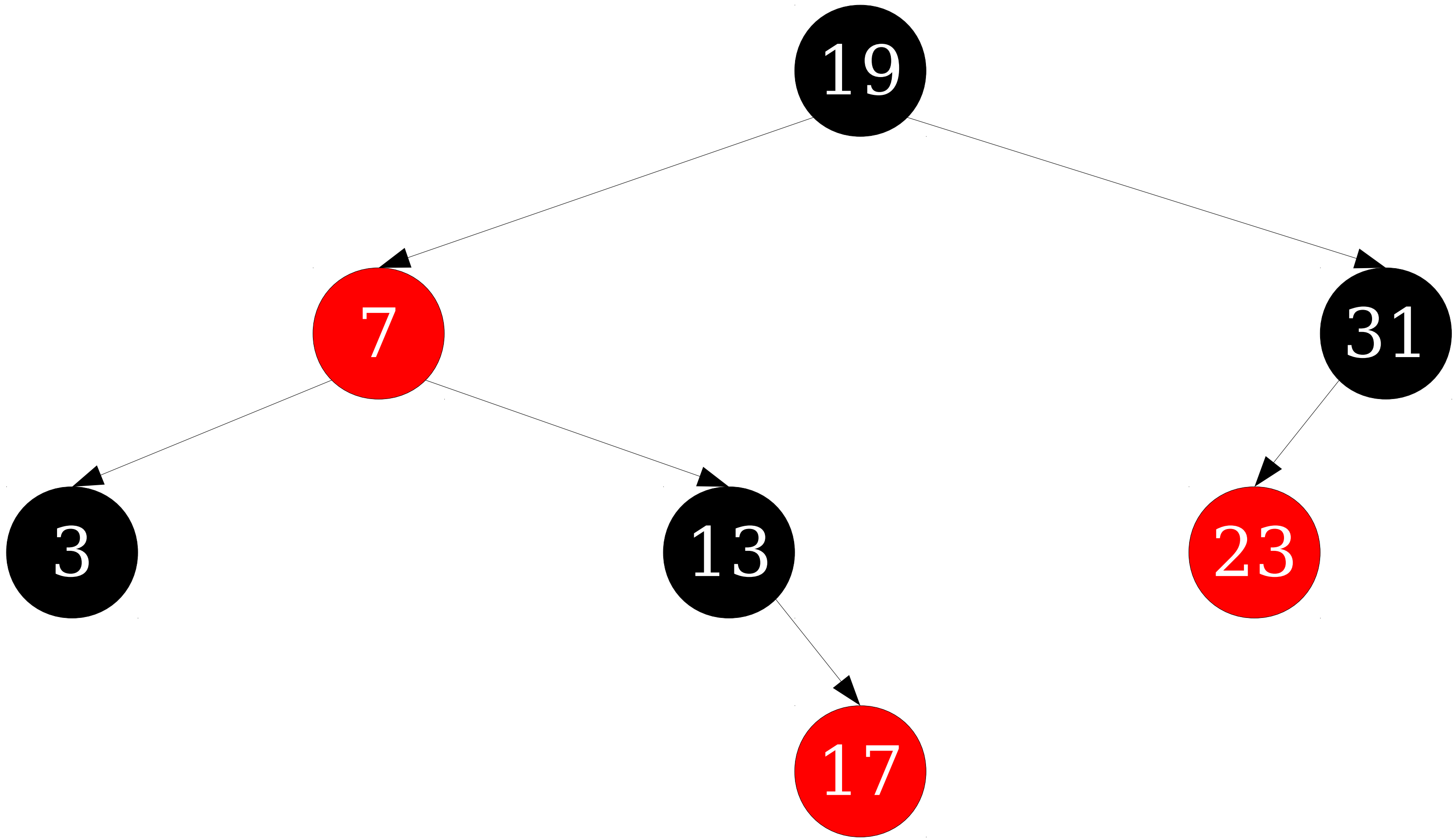

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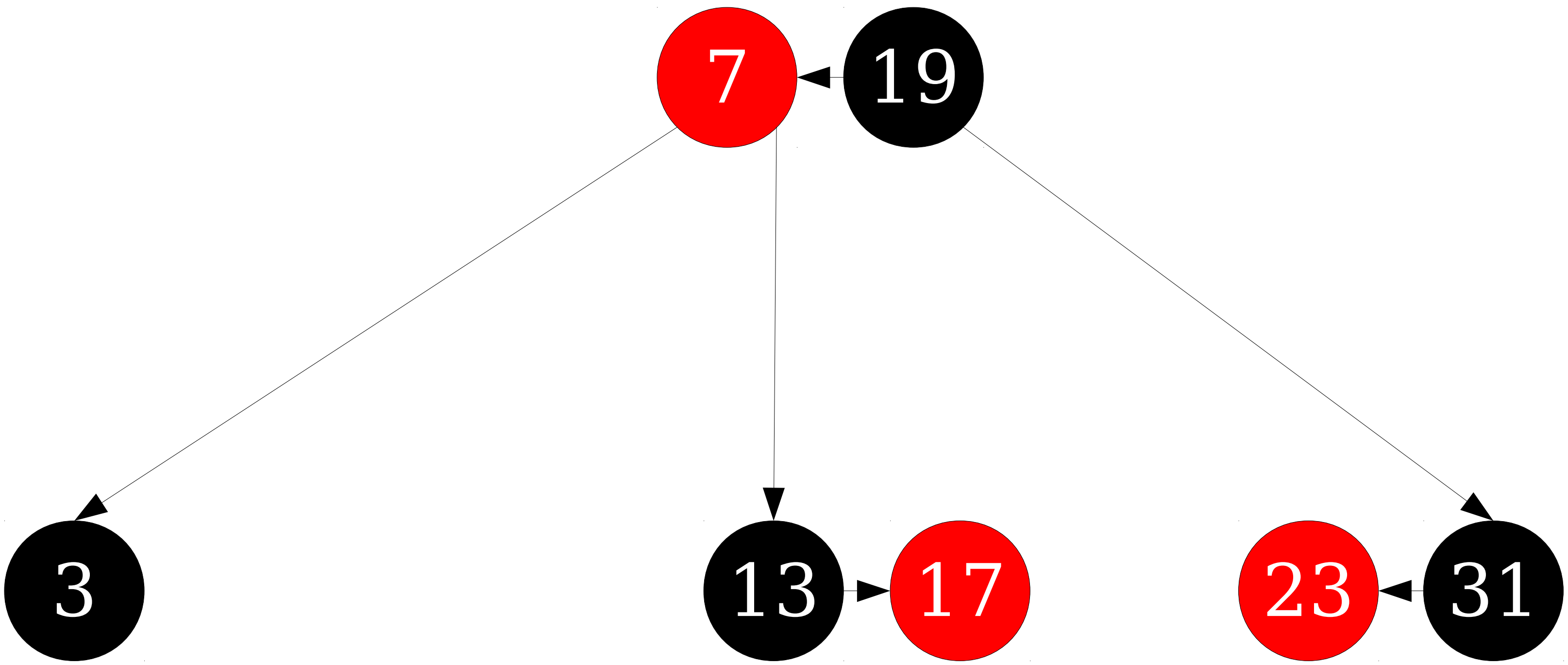

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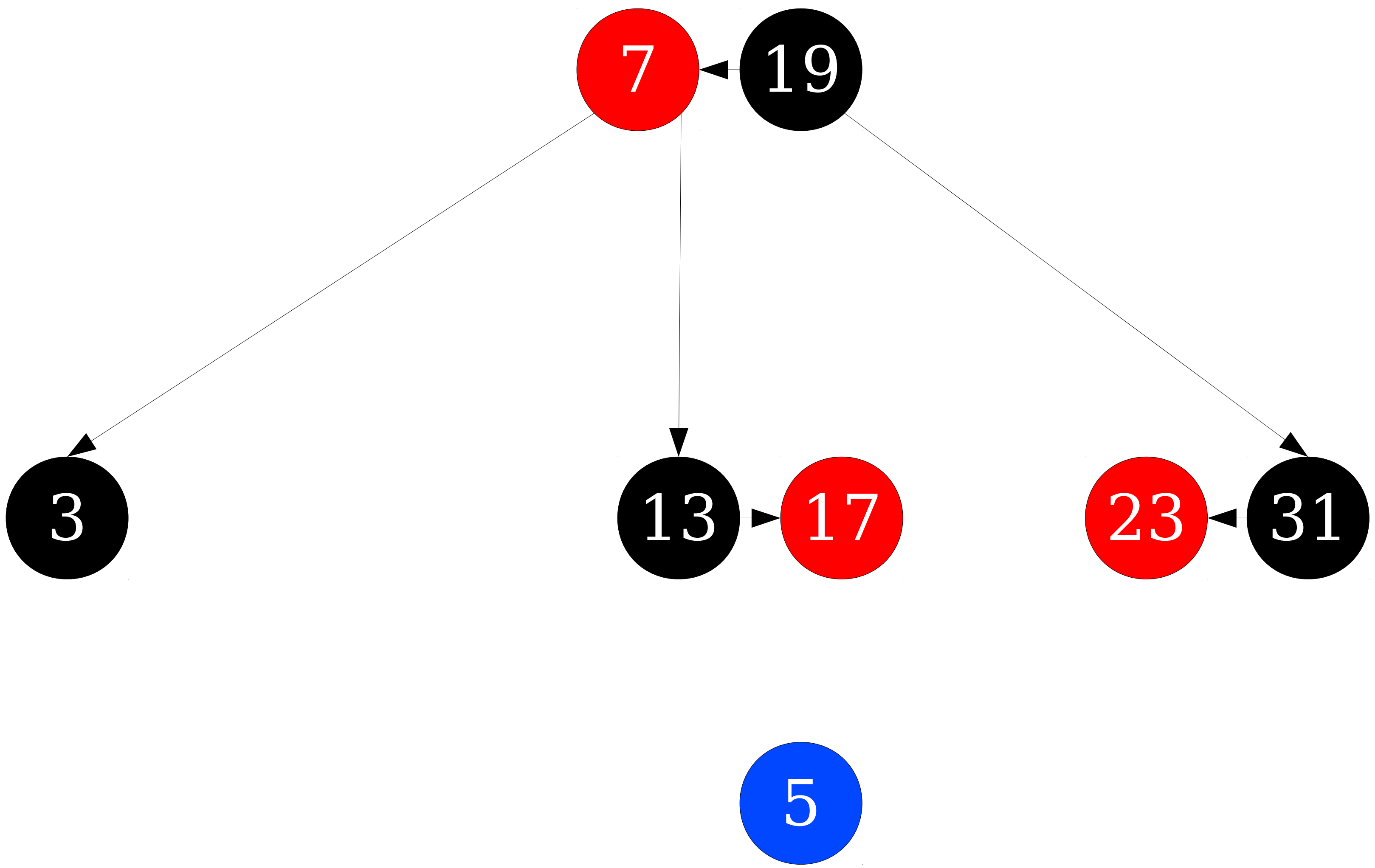

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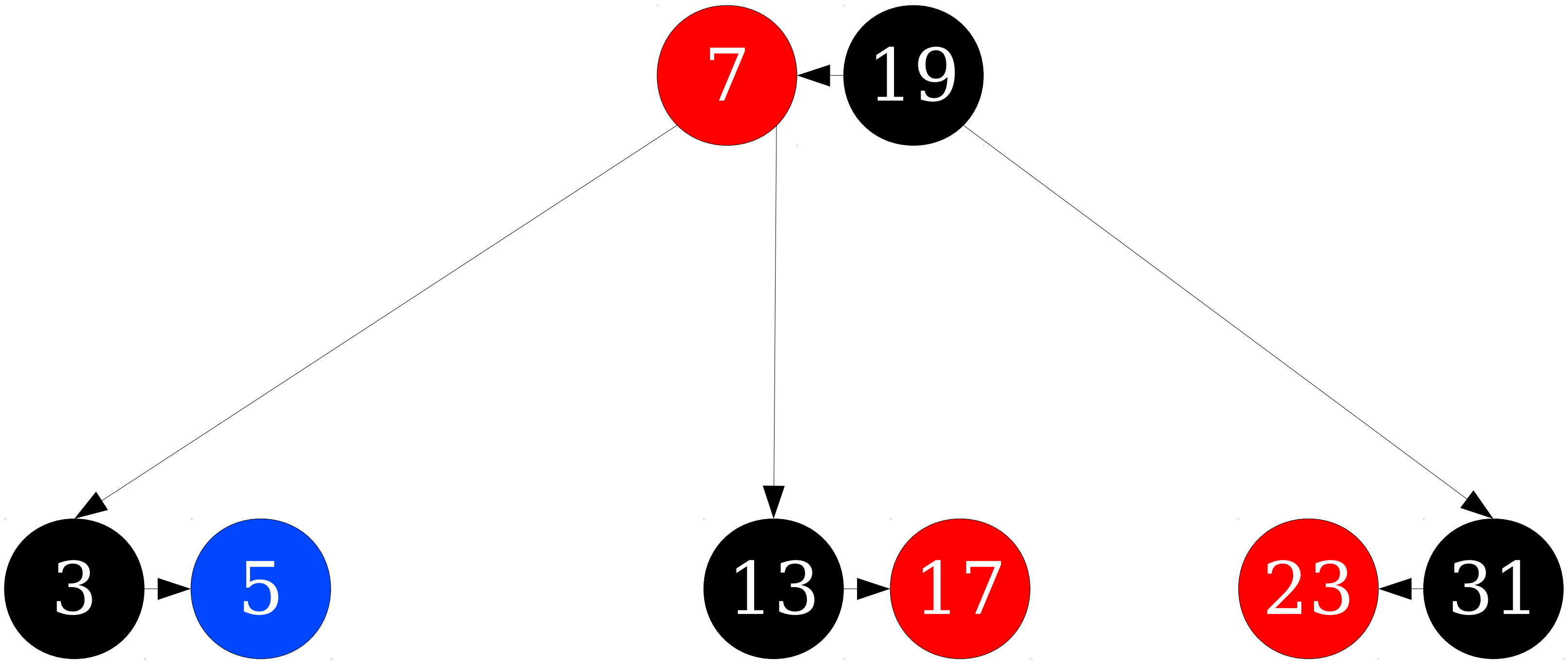

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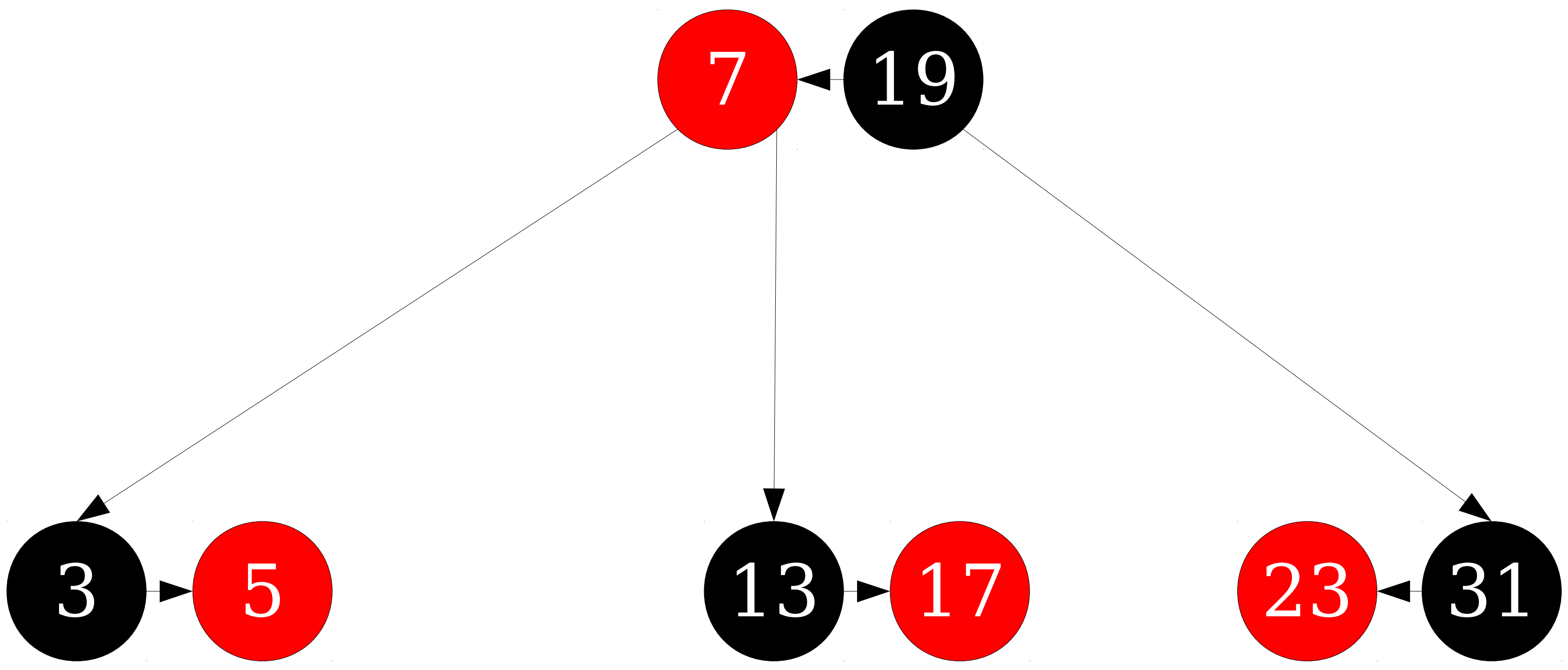

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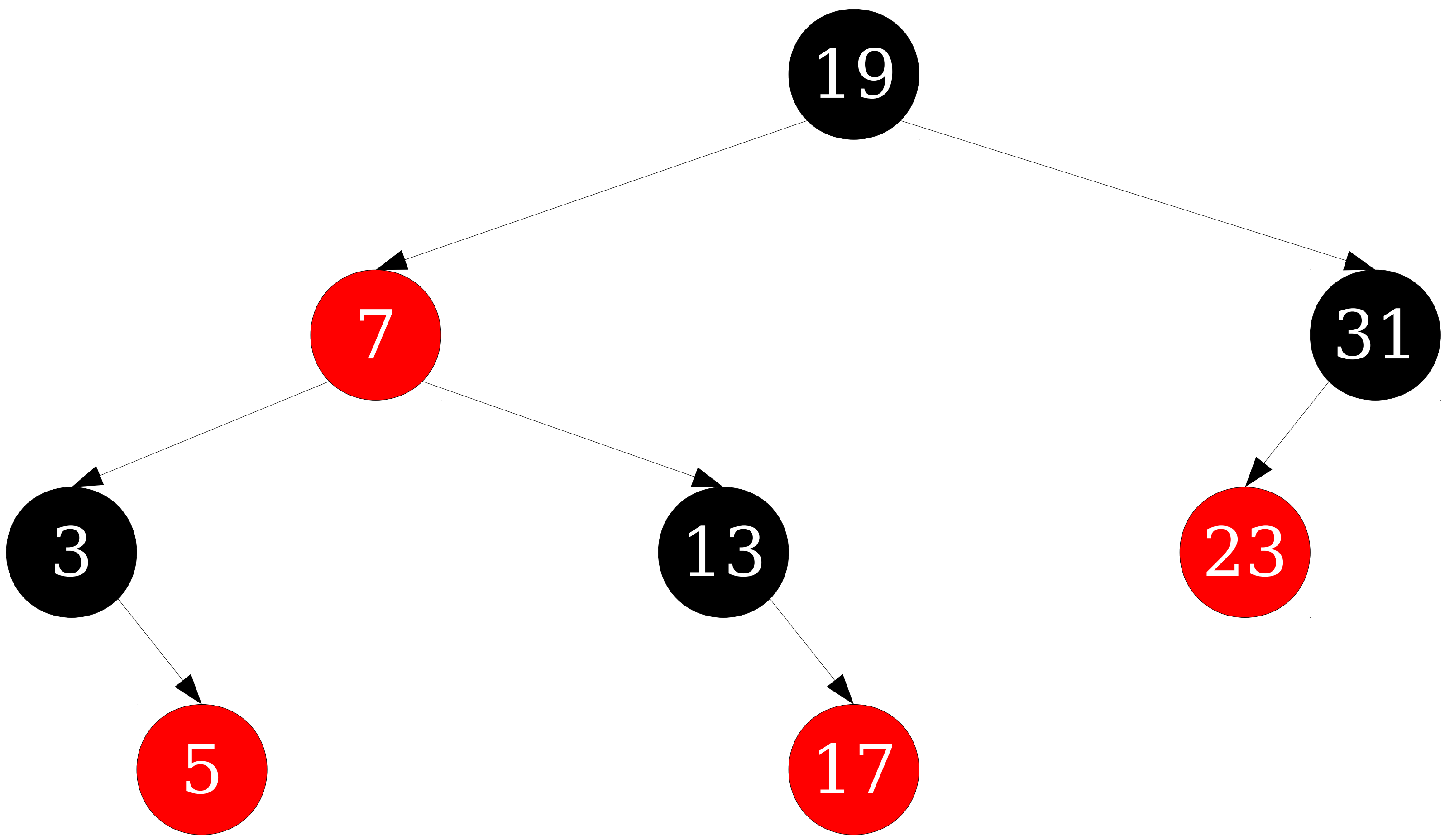

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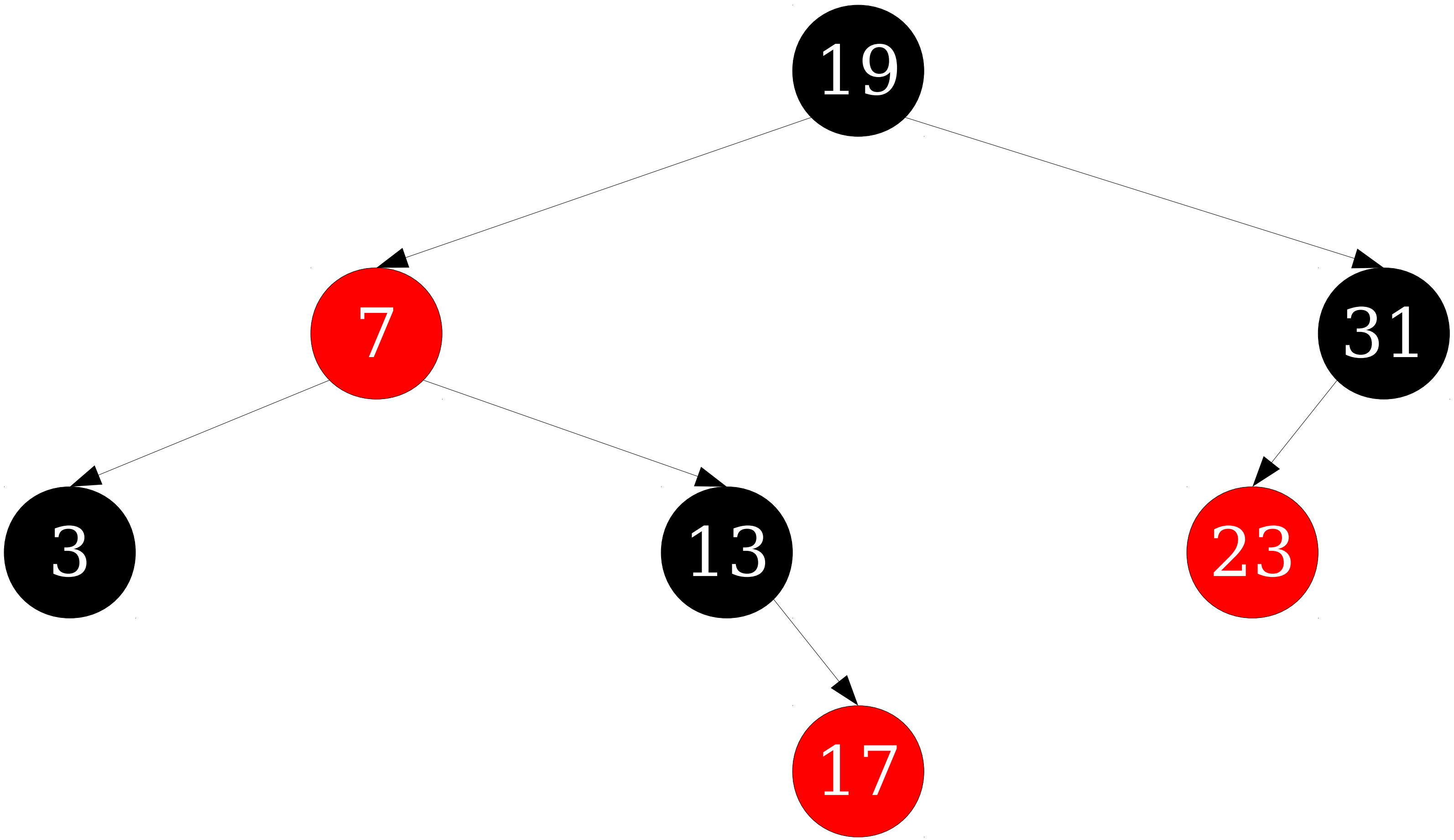

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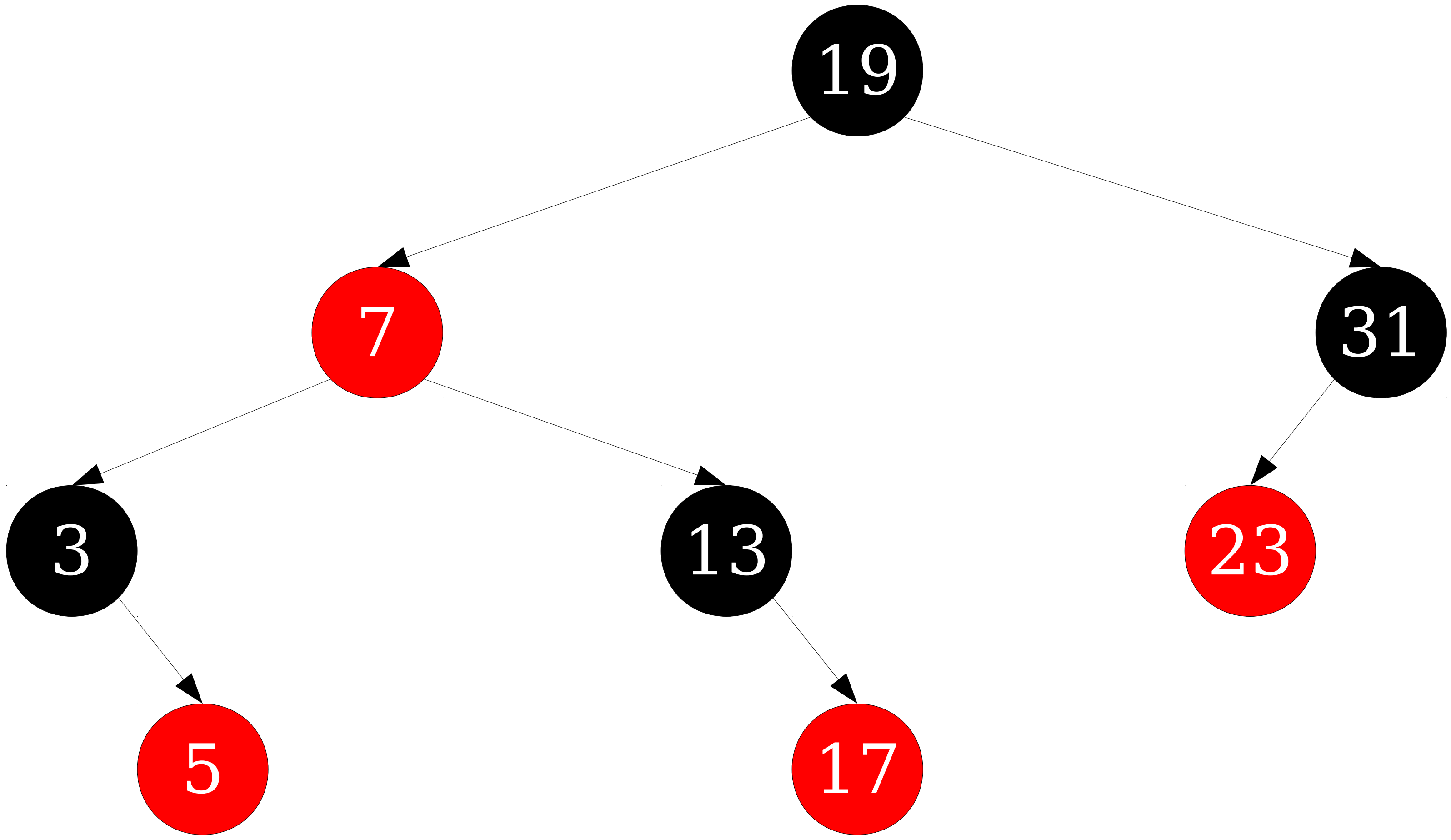

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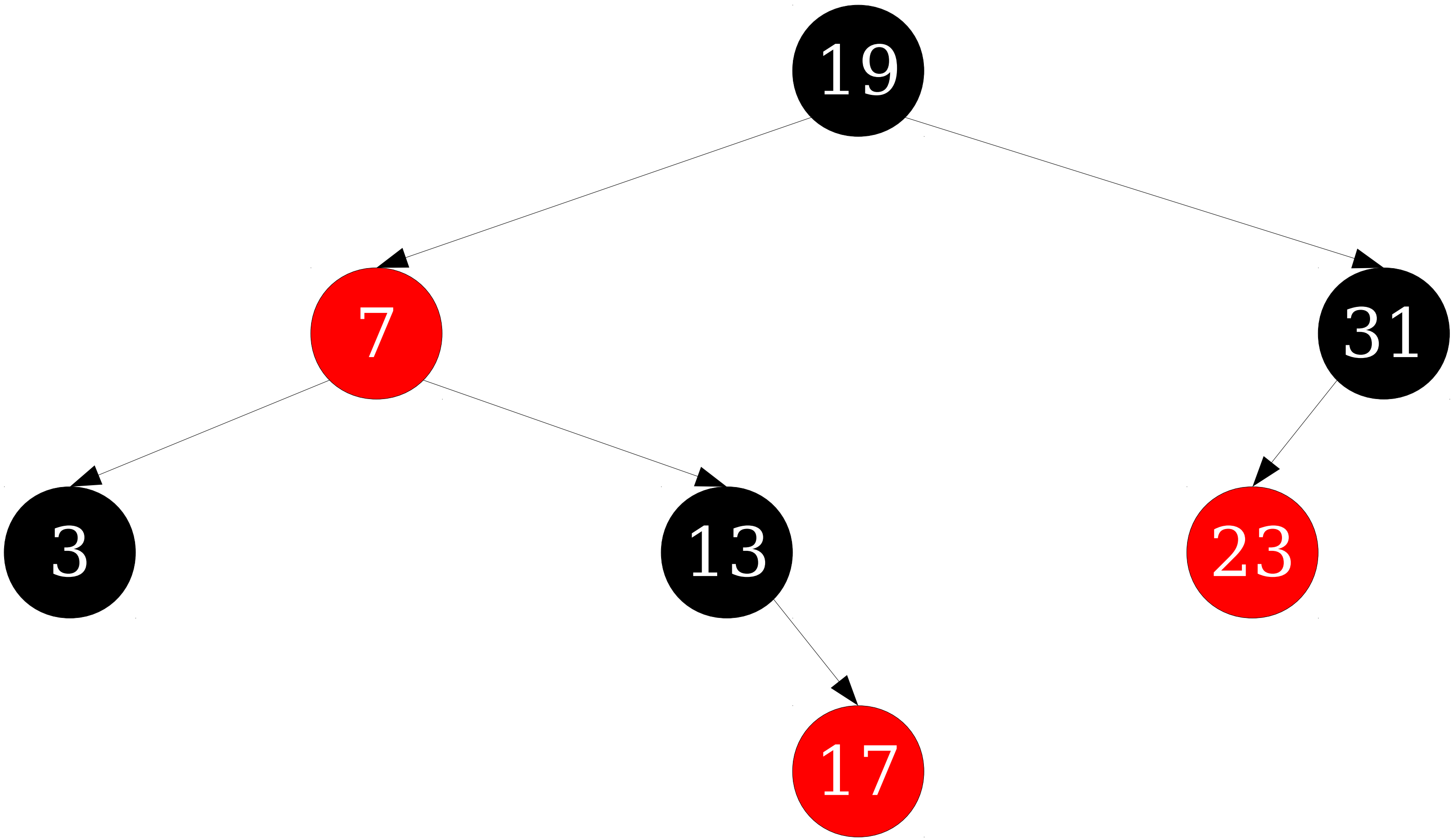

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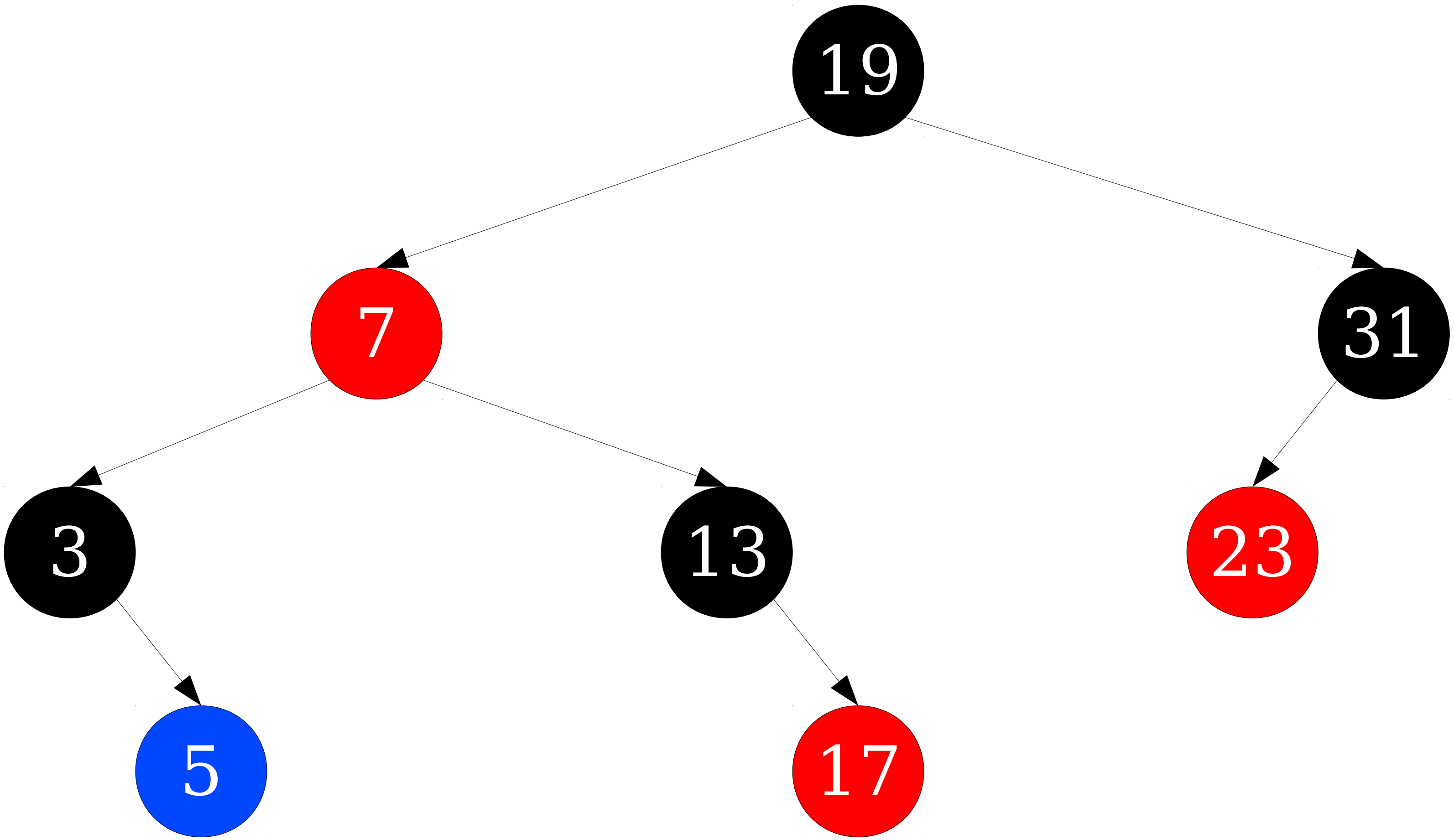

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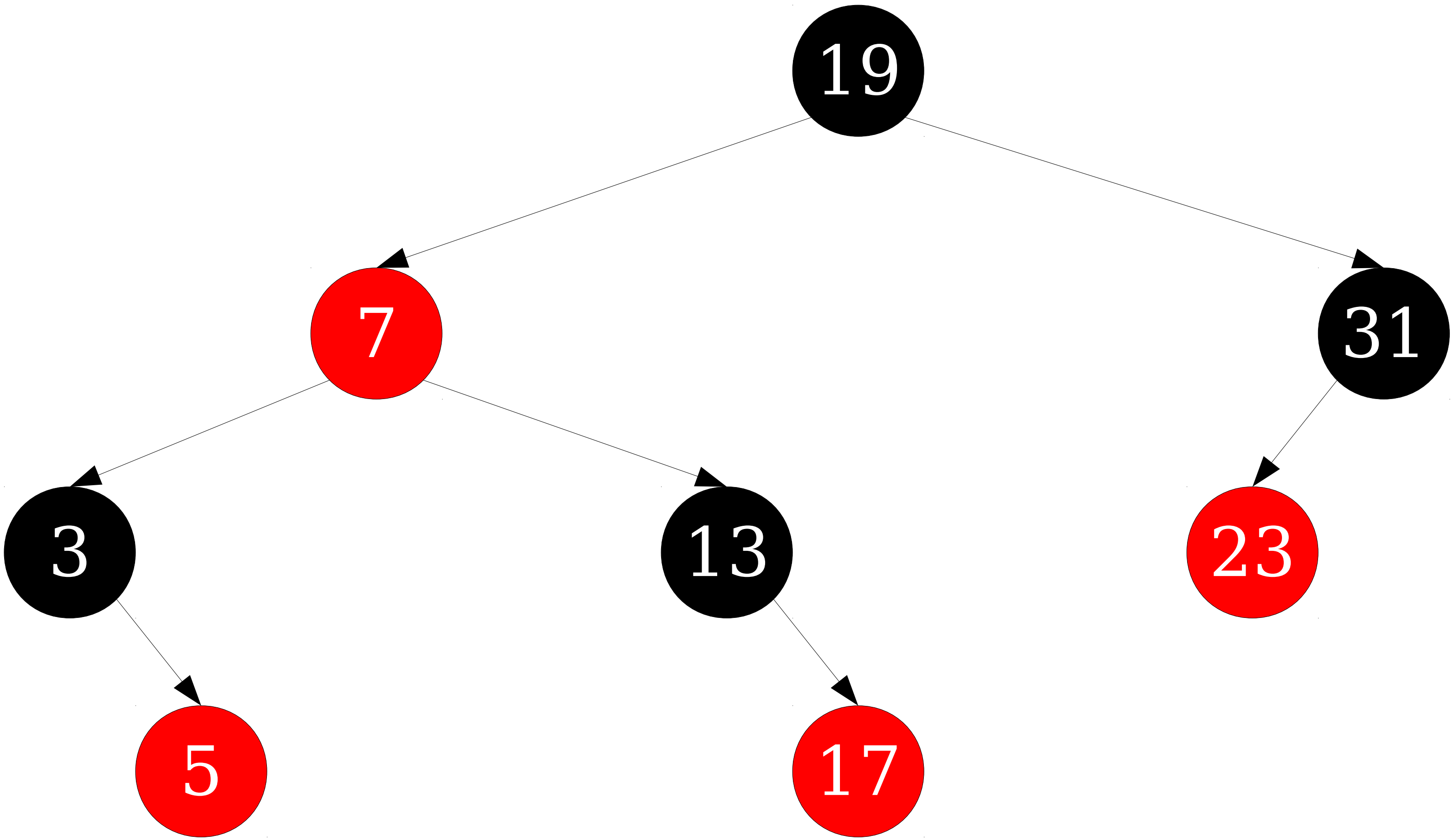

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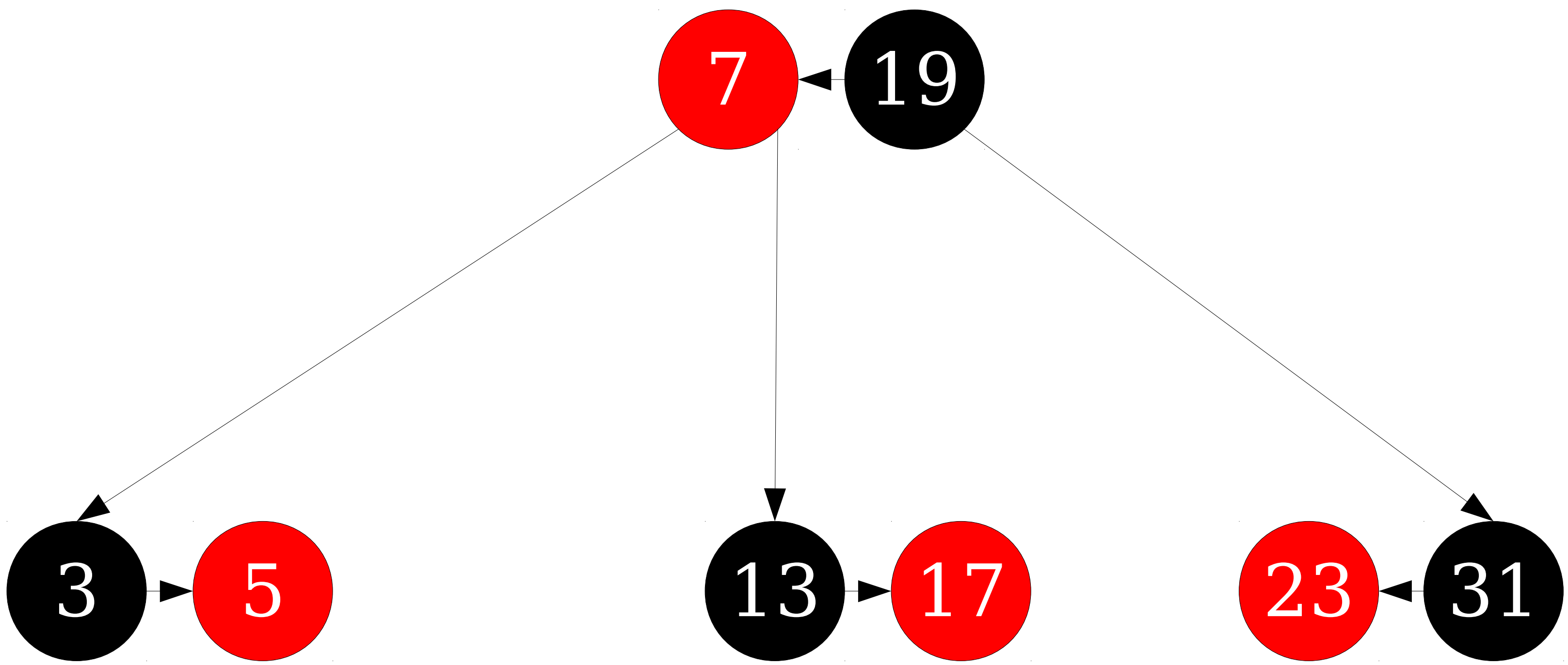

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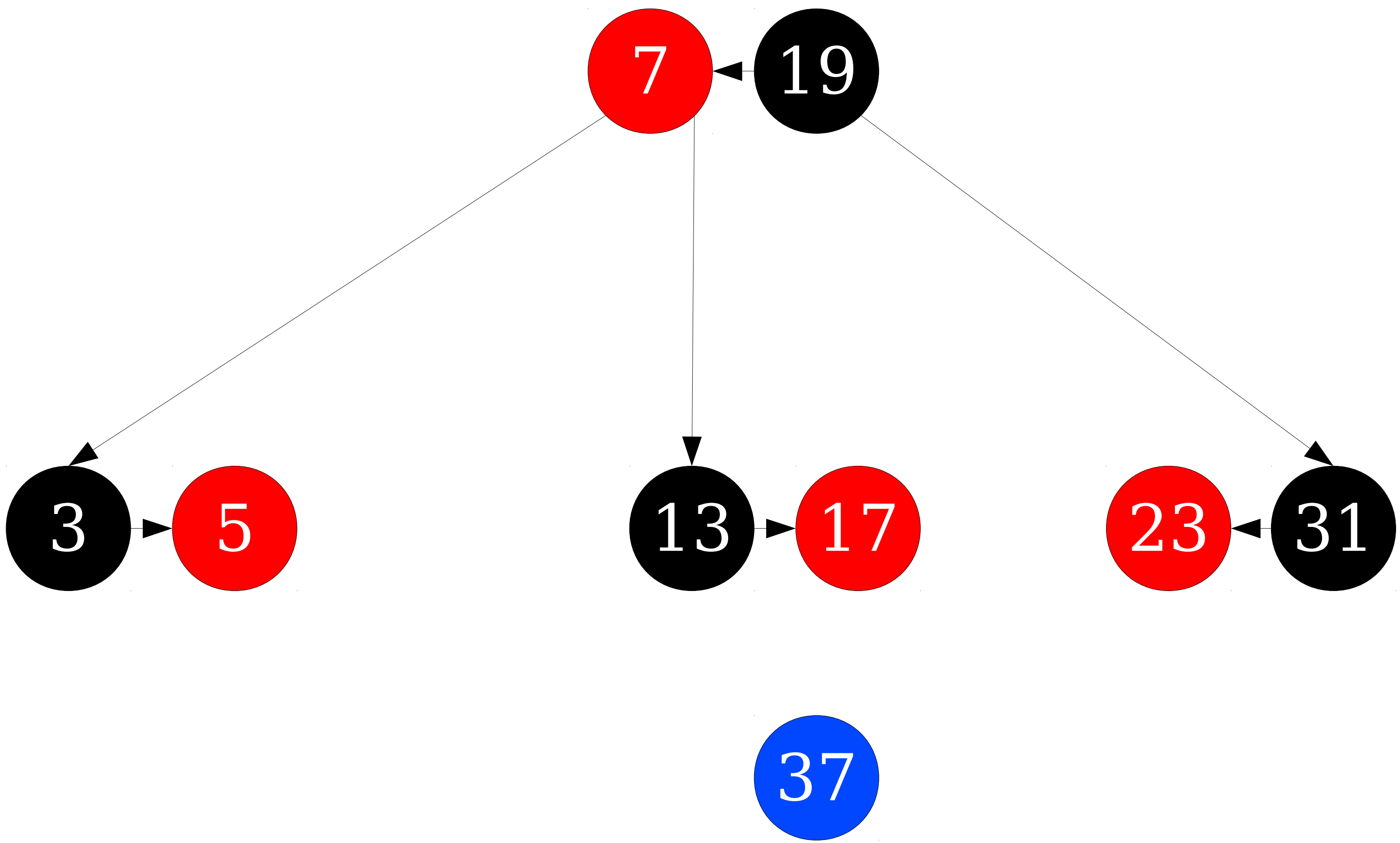
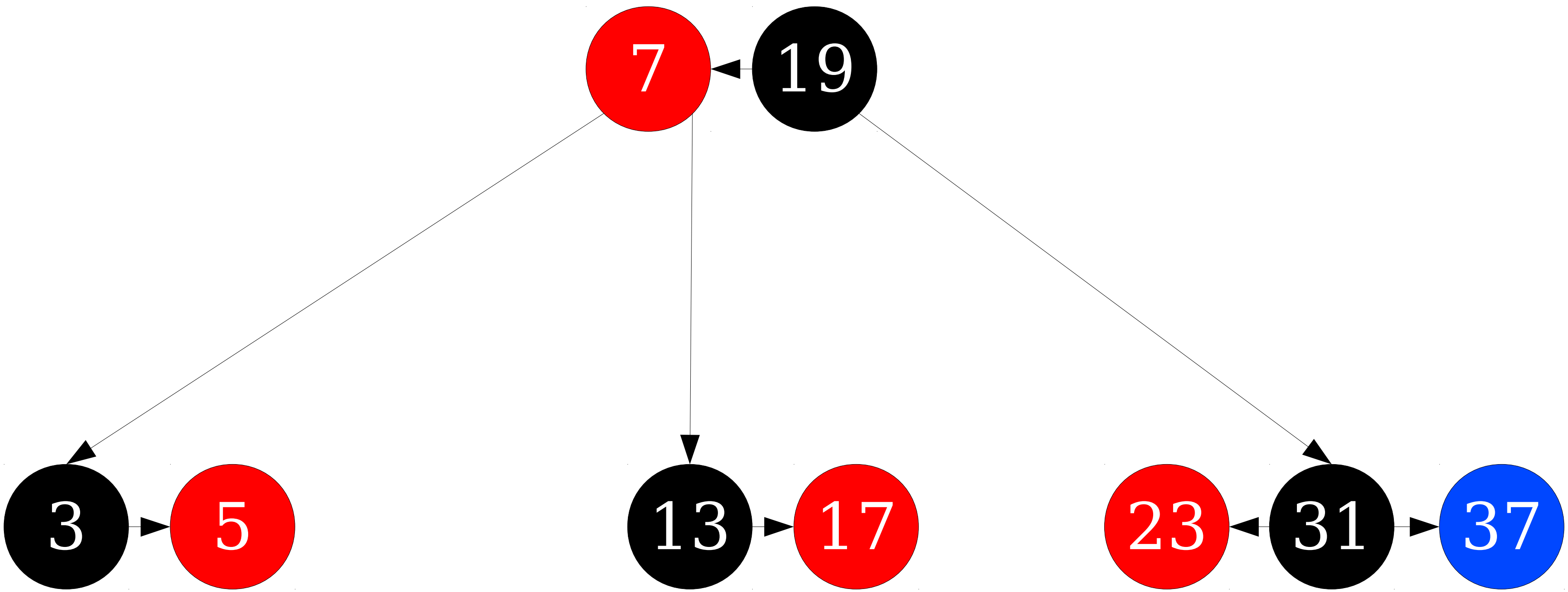


Using the Isometry



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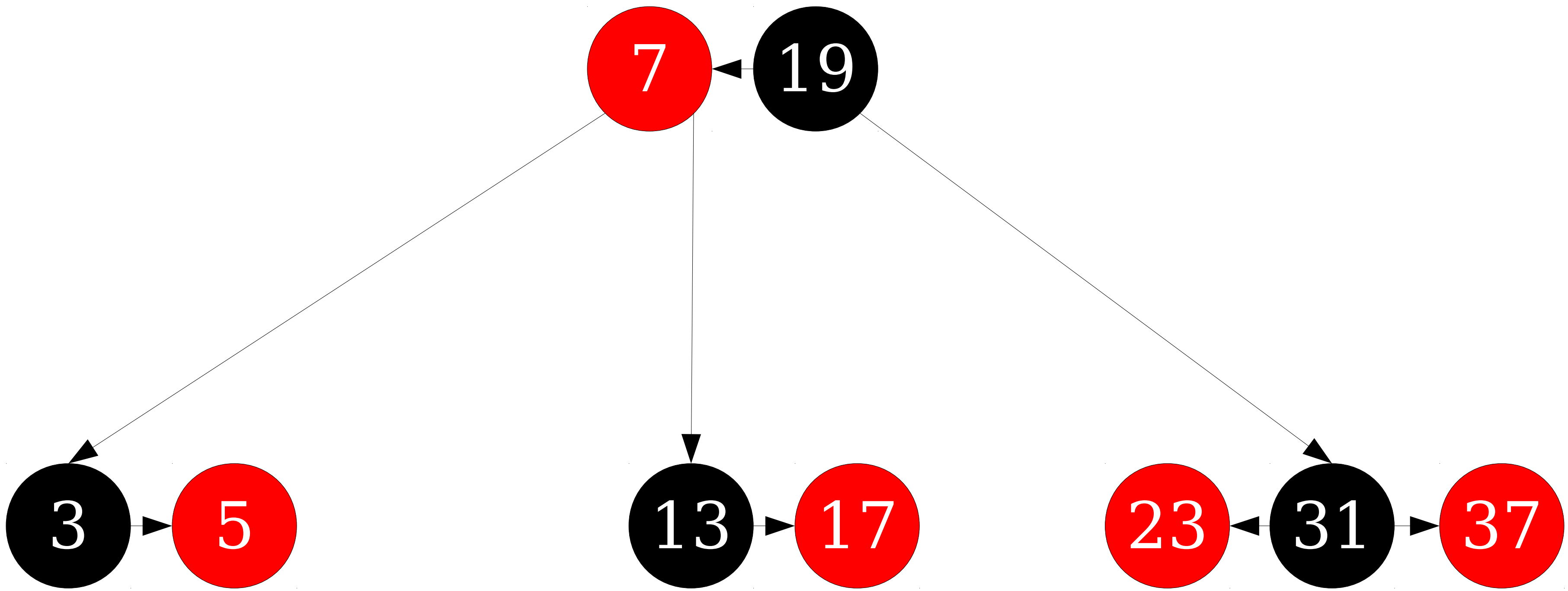

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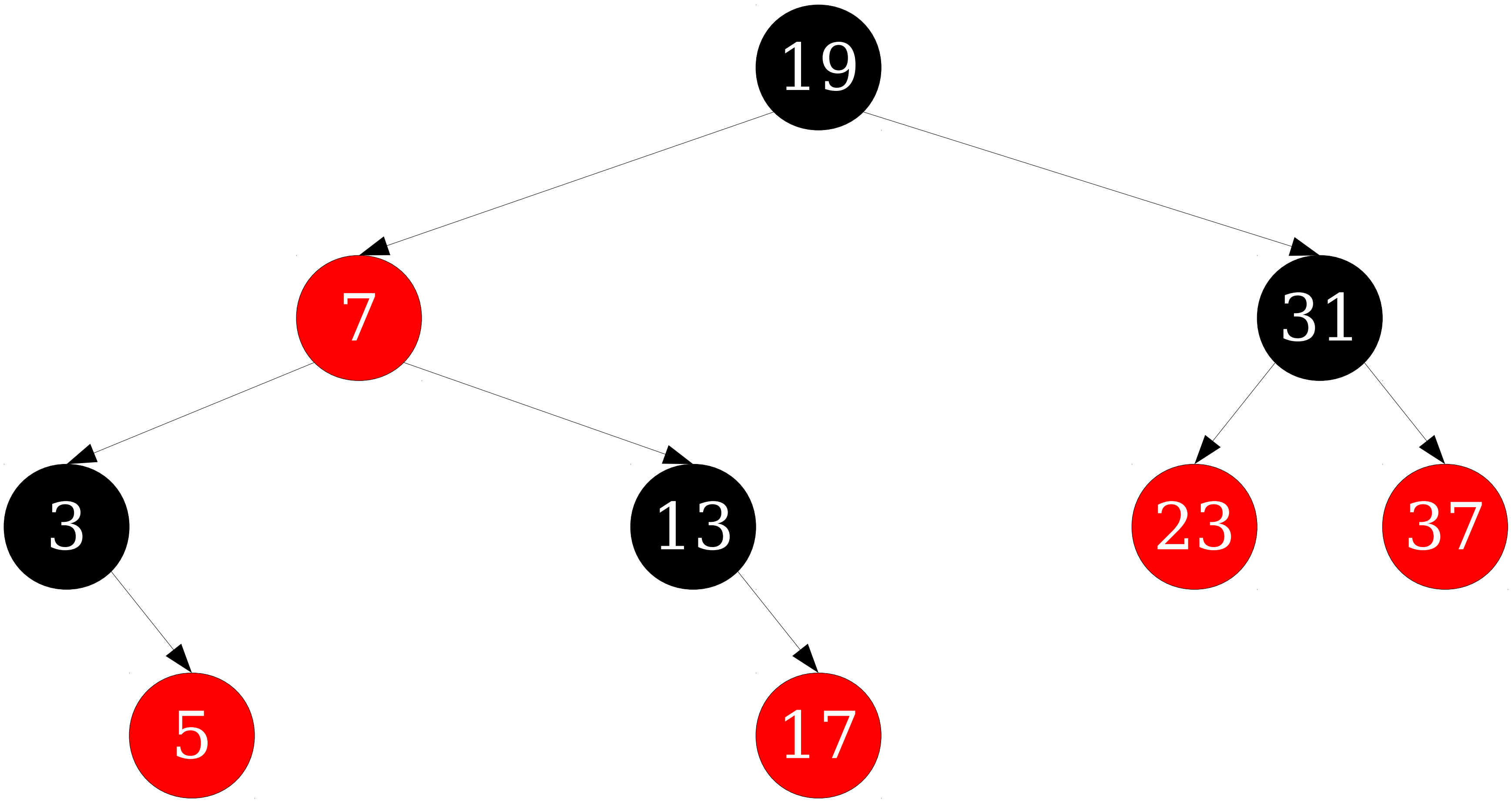

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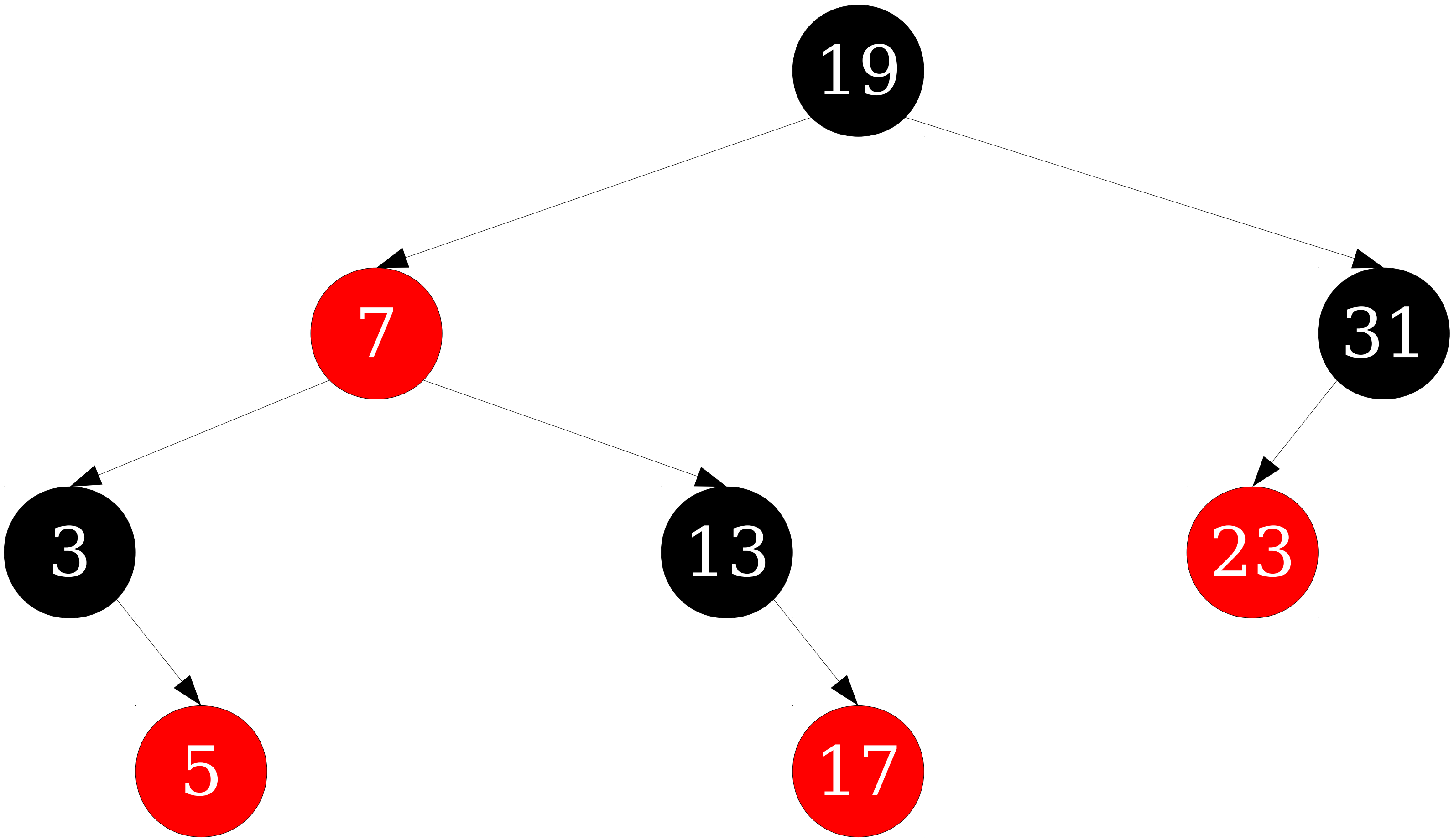

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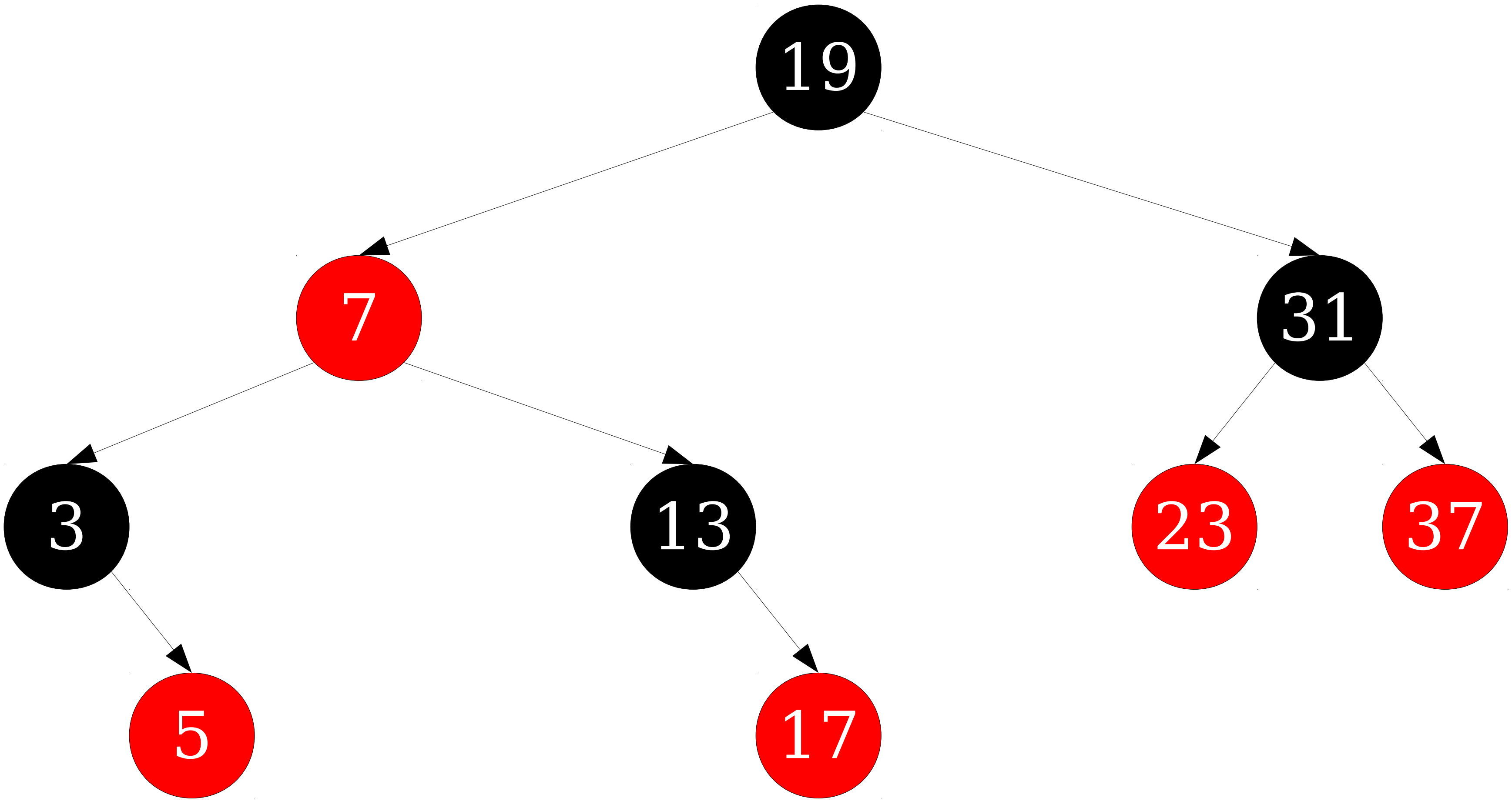
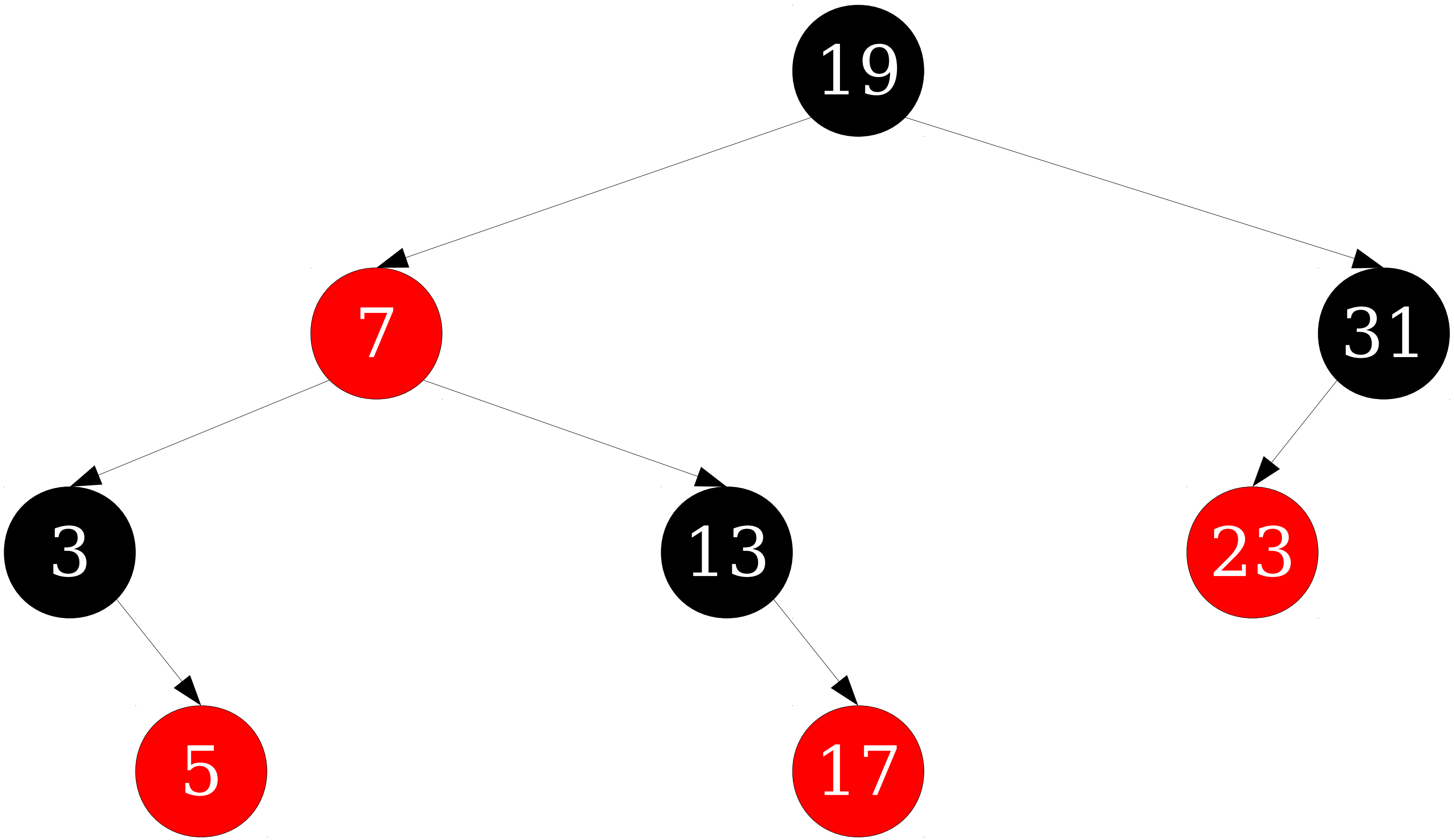


Using the Isometry



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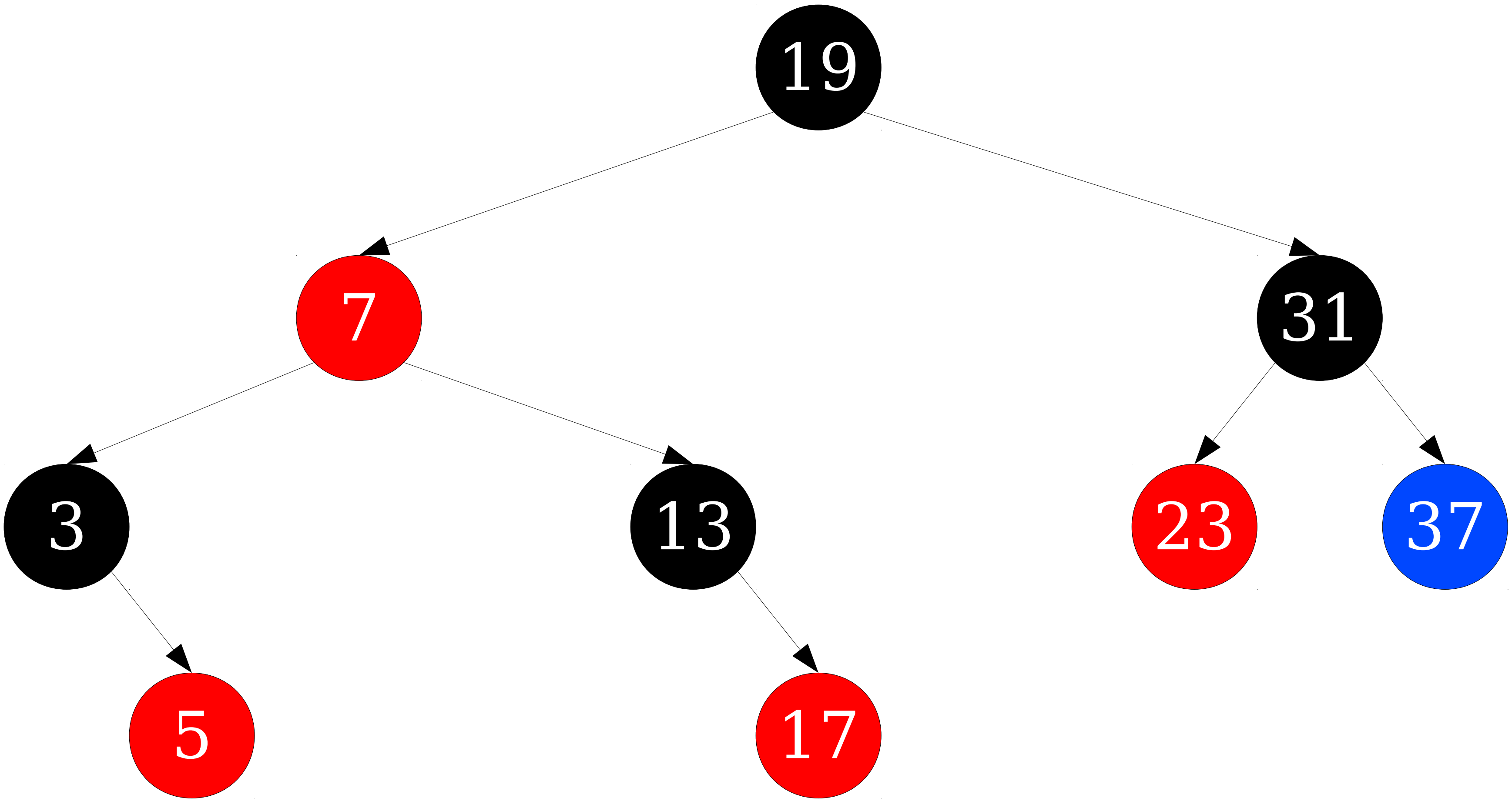
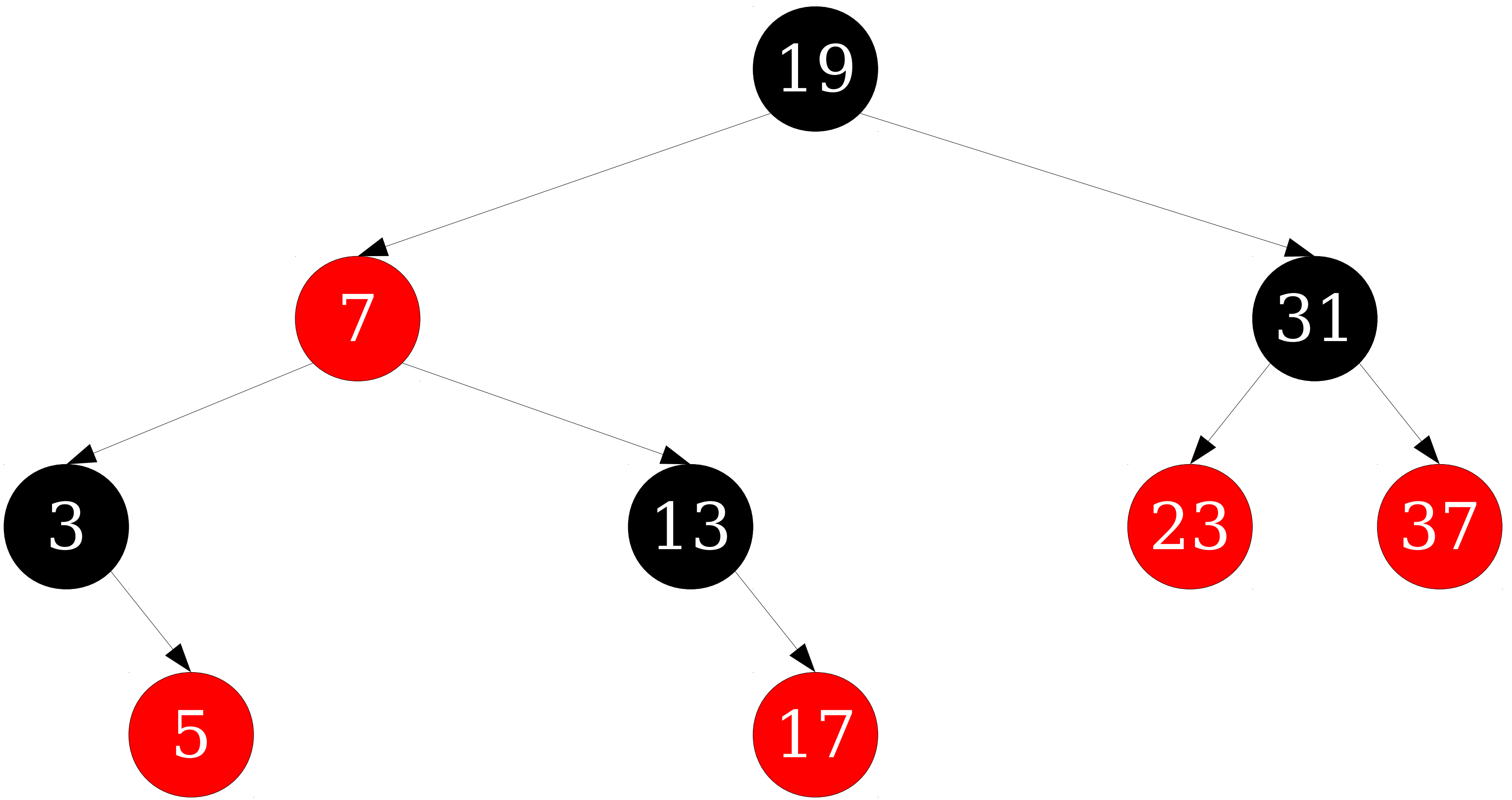
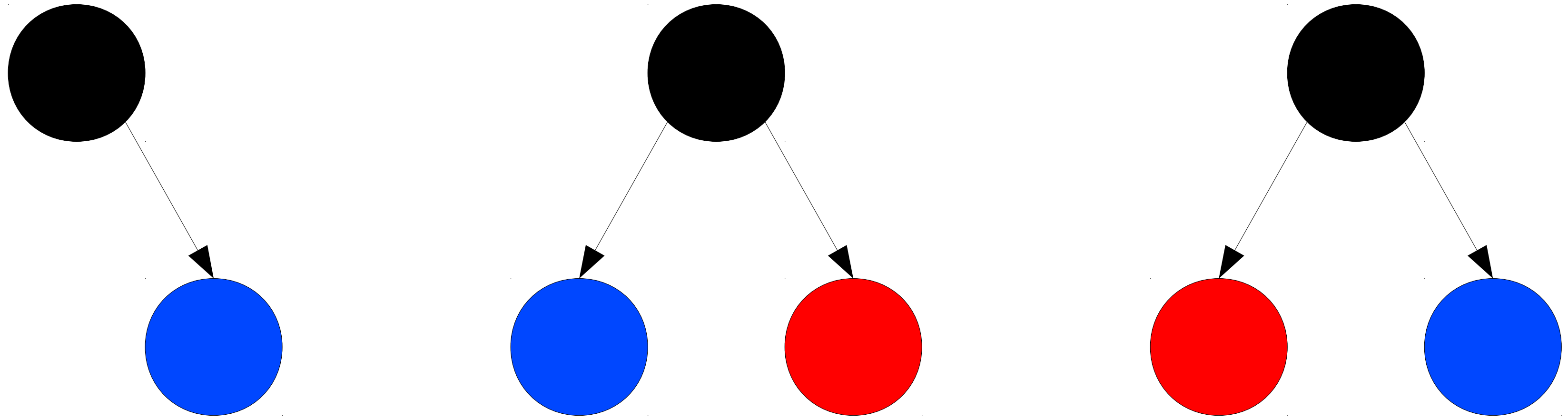


Using the Isometry



Red/Black Tree Insertion

- ***Rule #1:*** When inserting a node, if its parent is black, make the node red and stop.
- ***Justification:*** This simulates inserting a key into an existing 2-node or 3-node.



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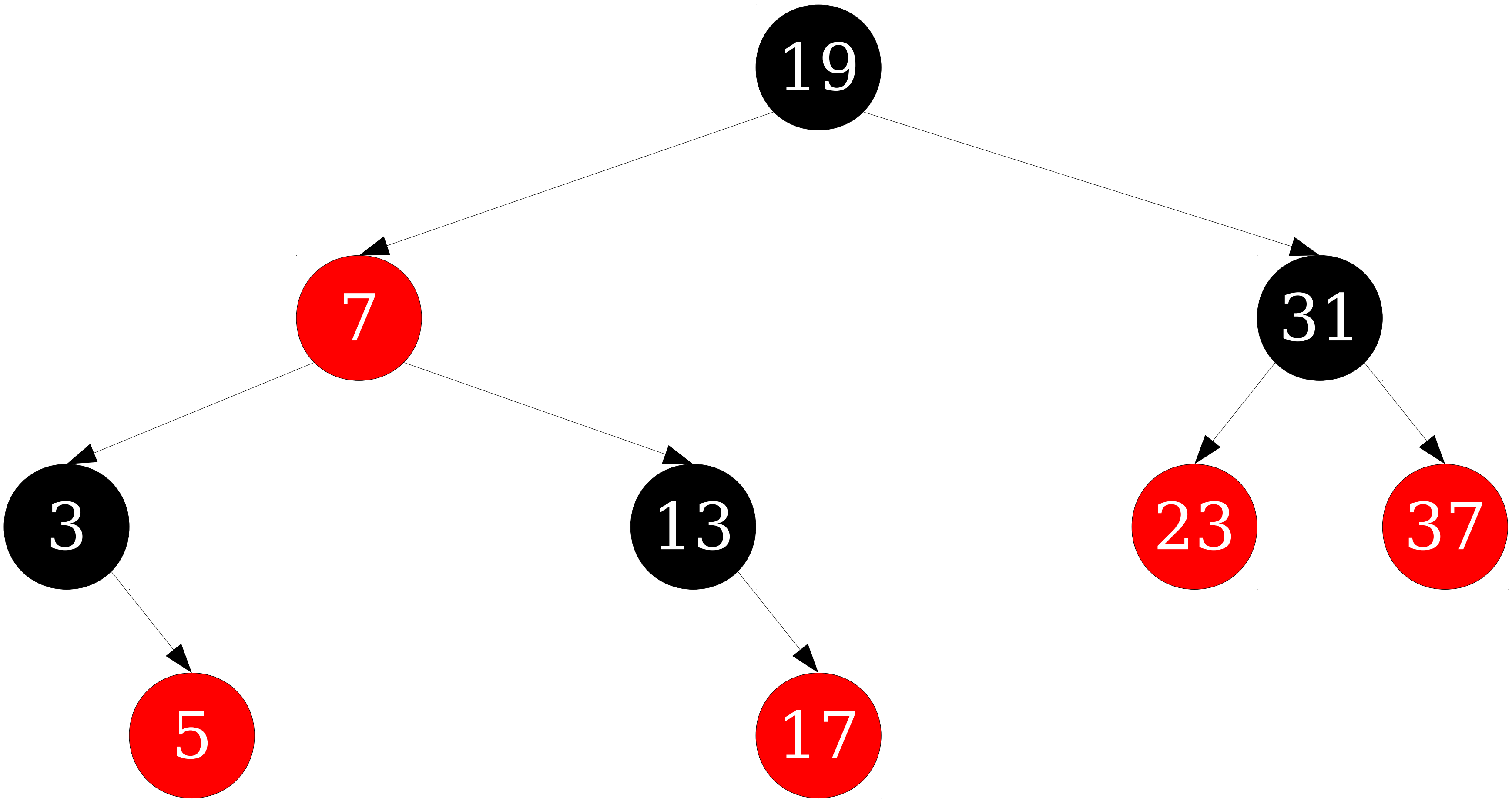

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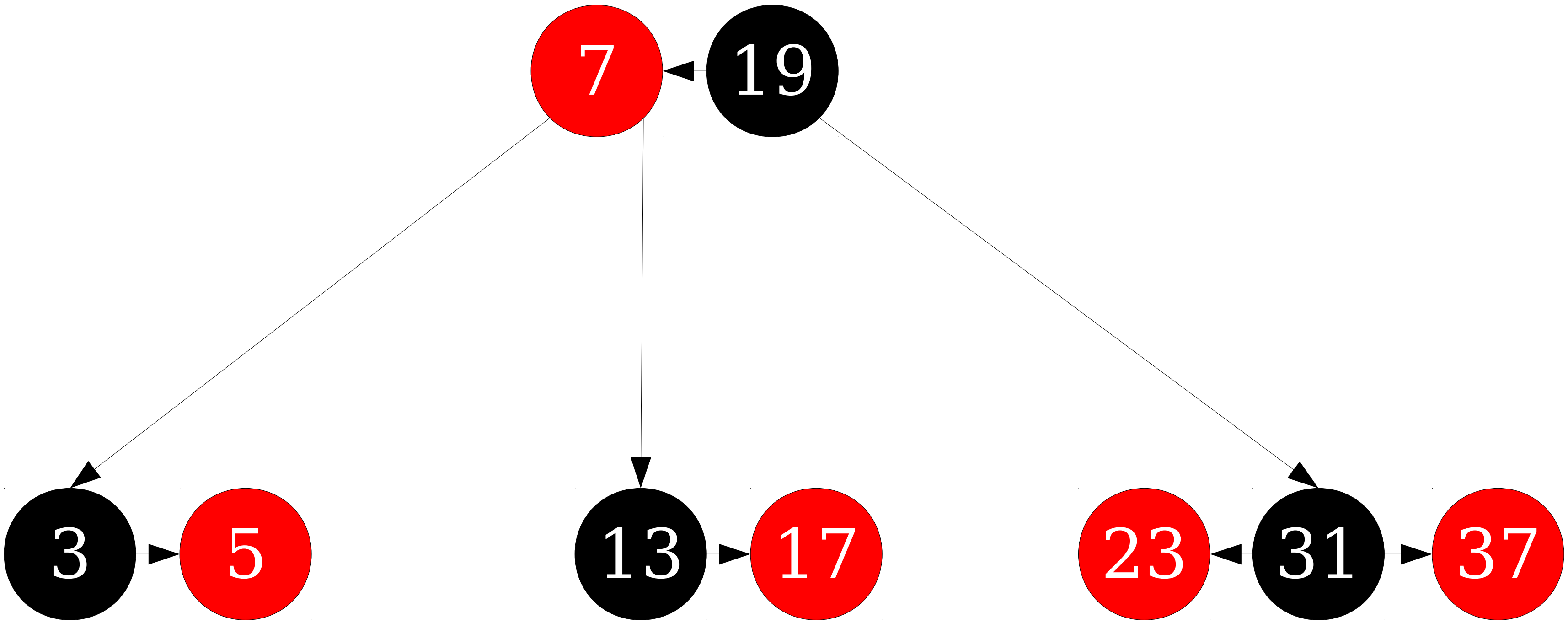

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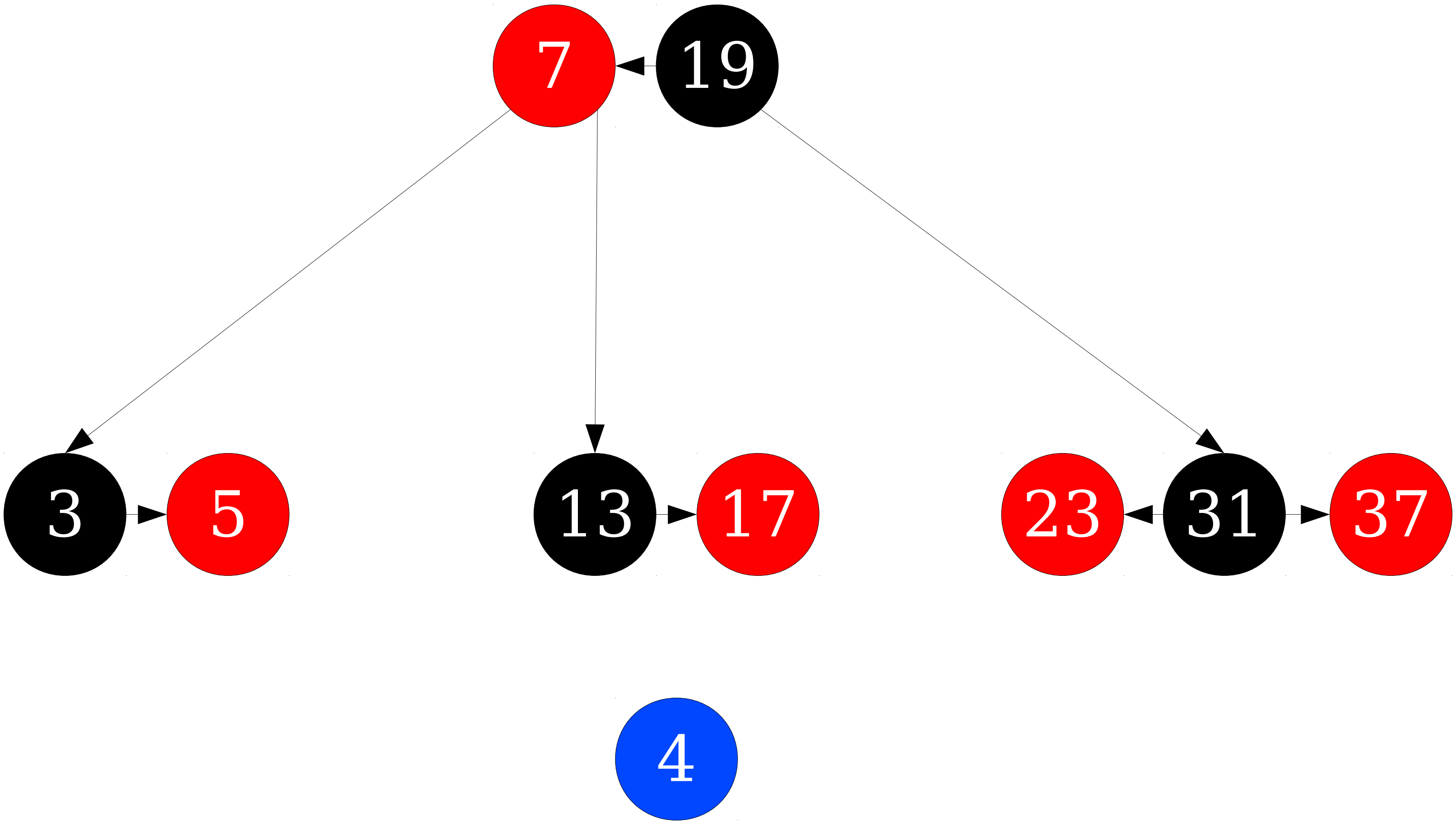

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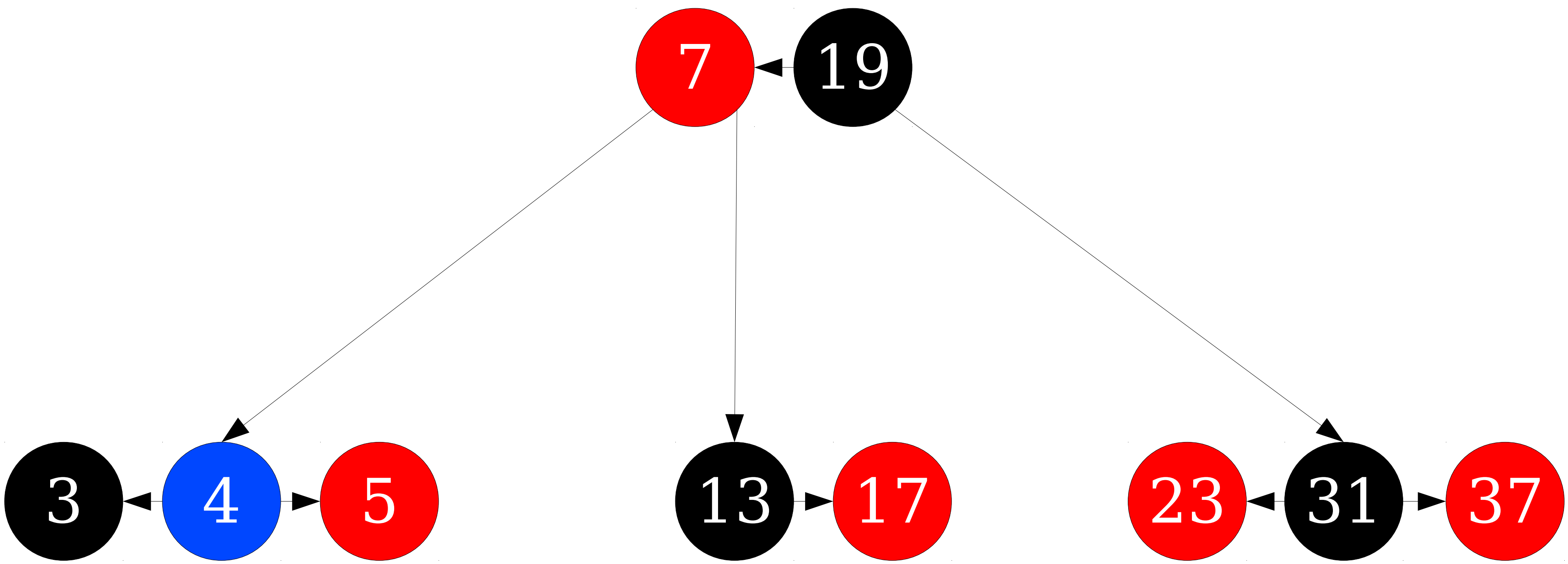

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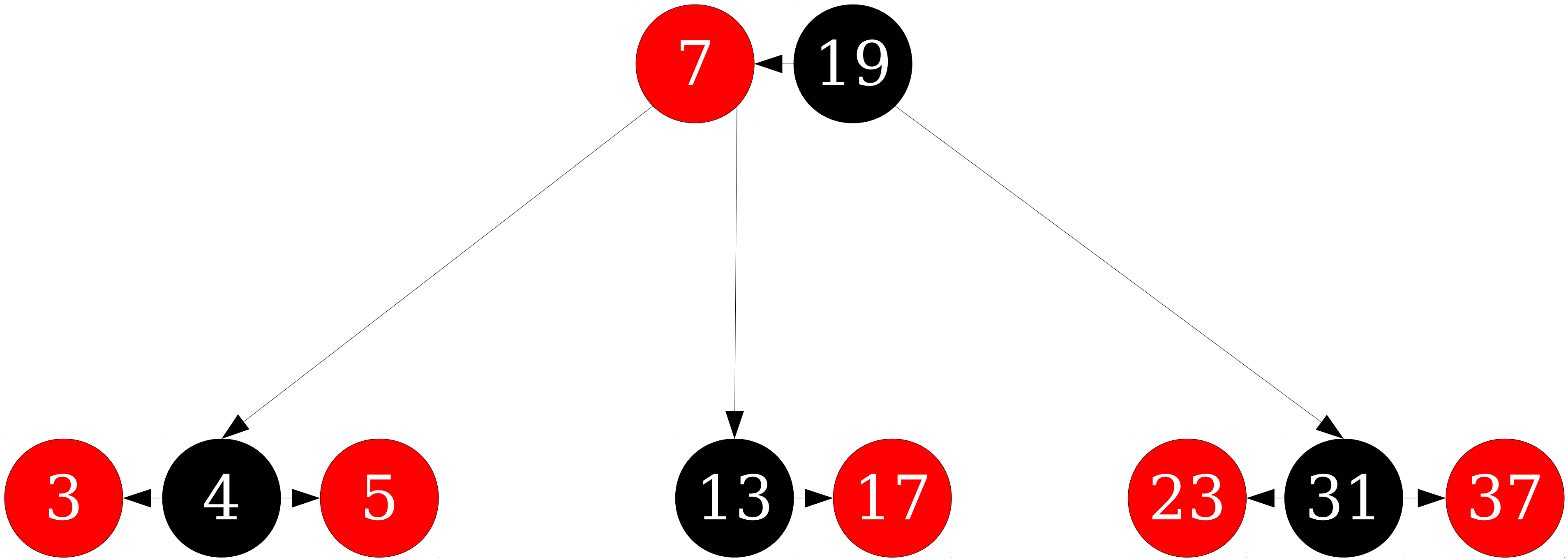

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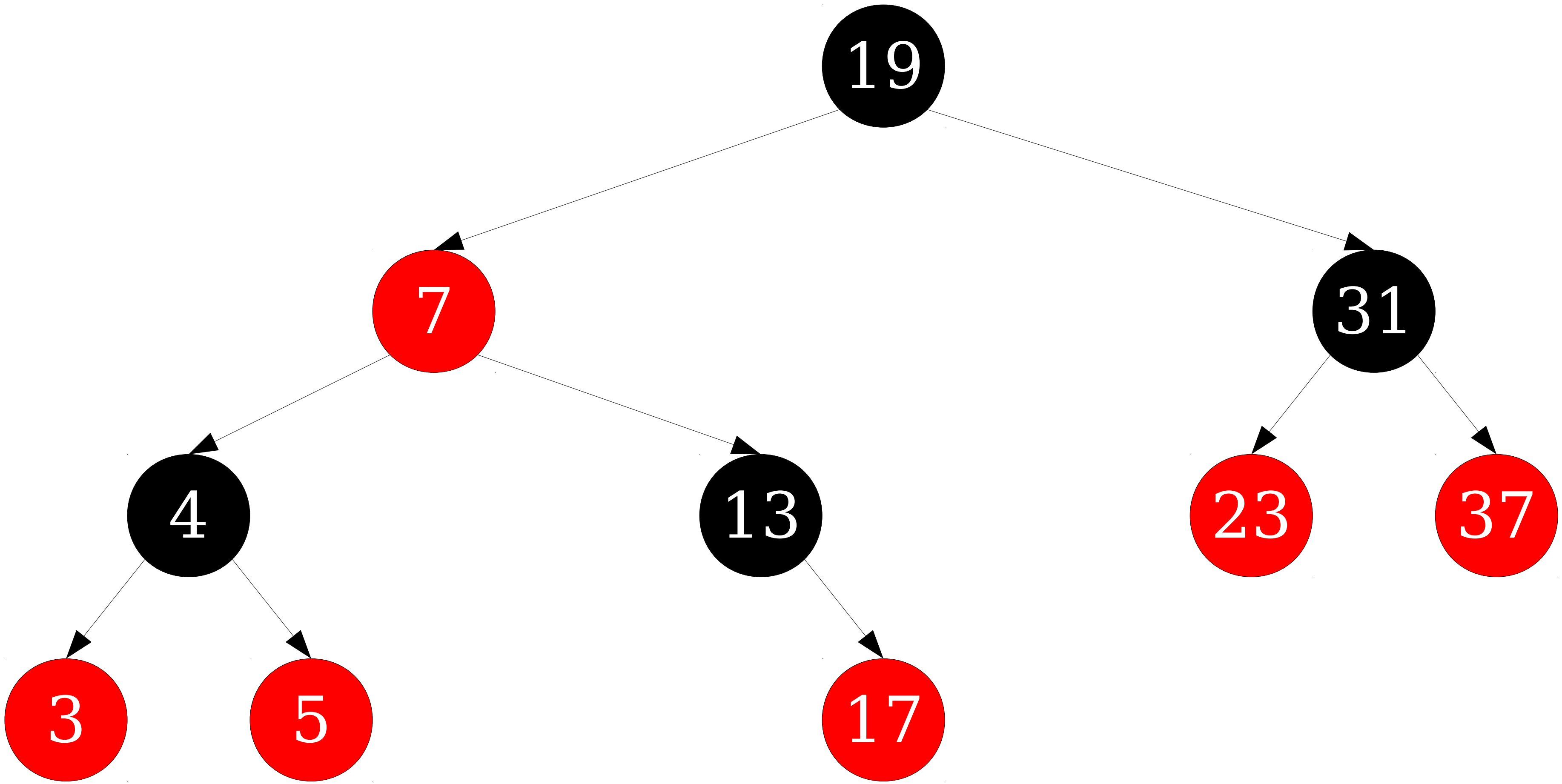

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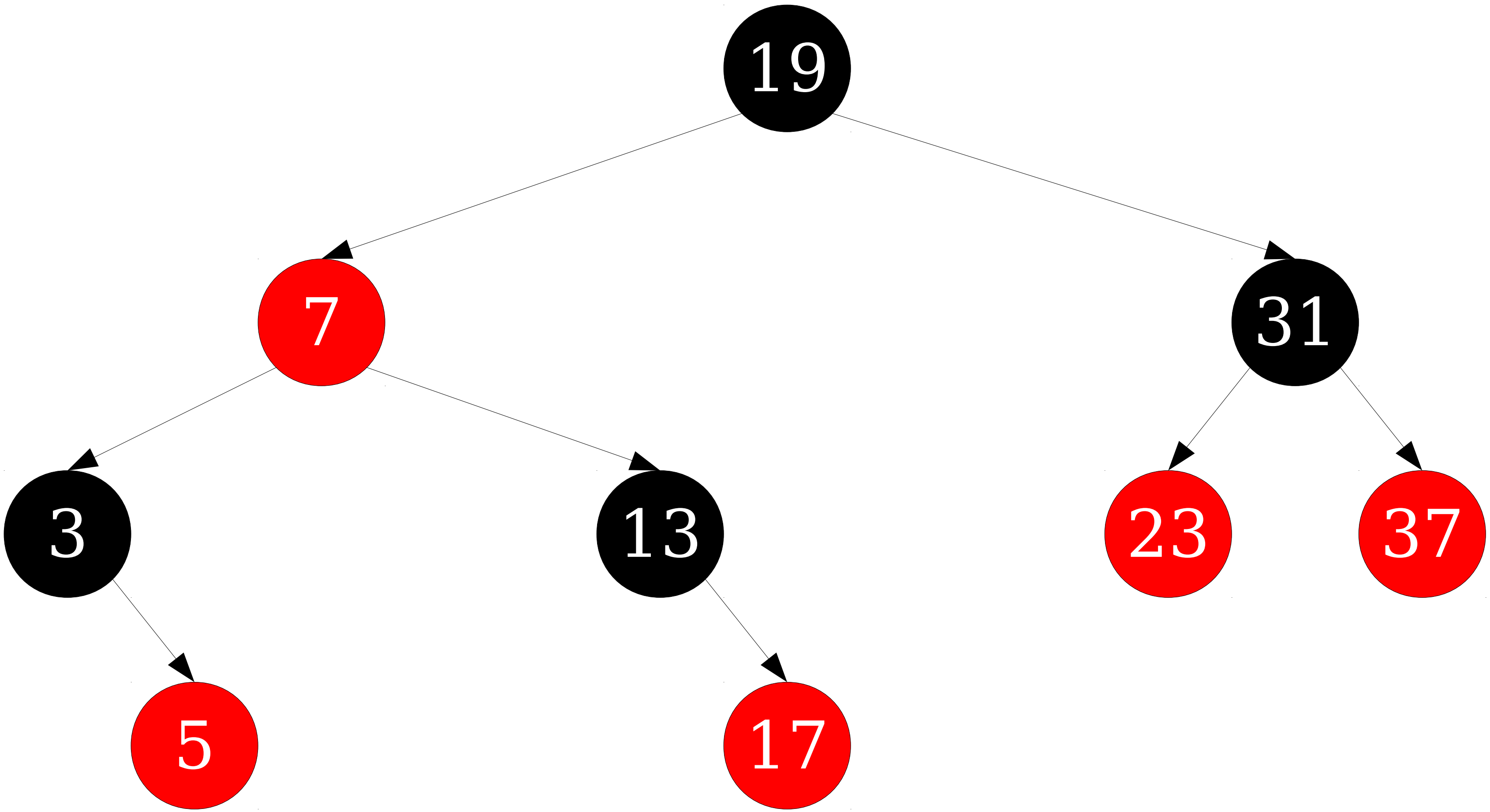
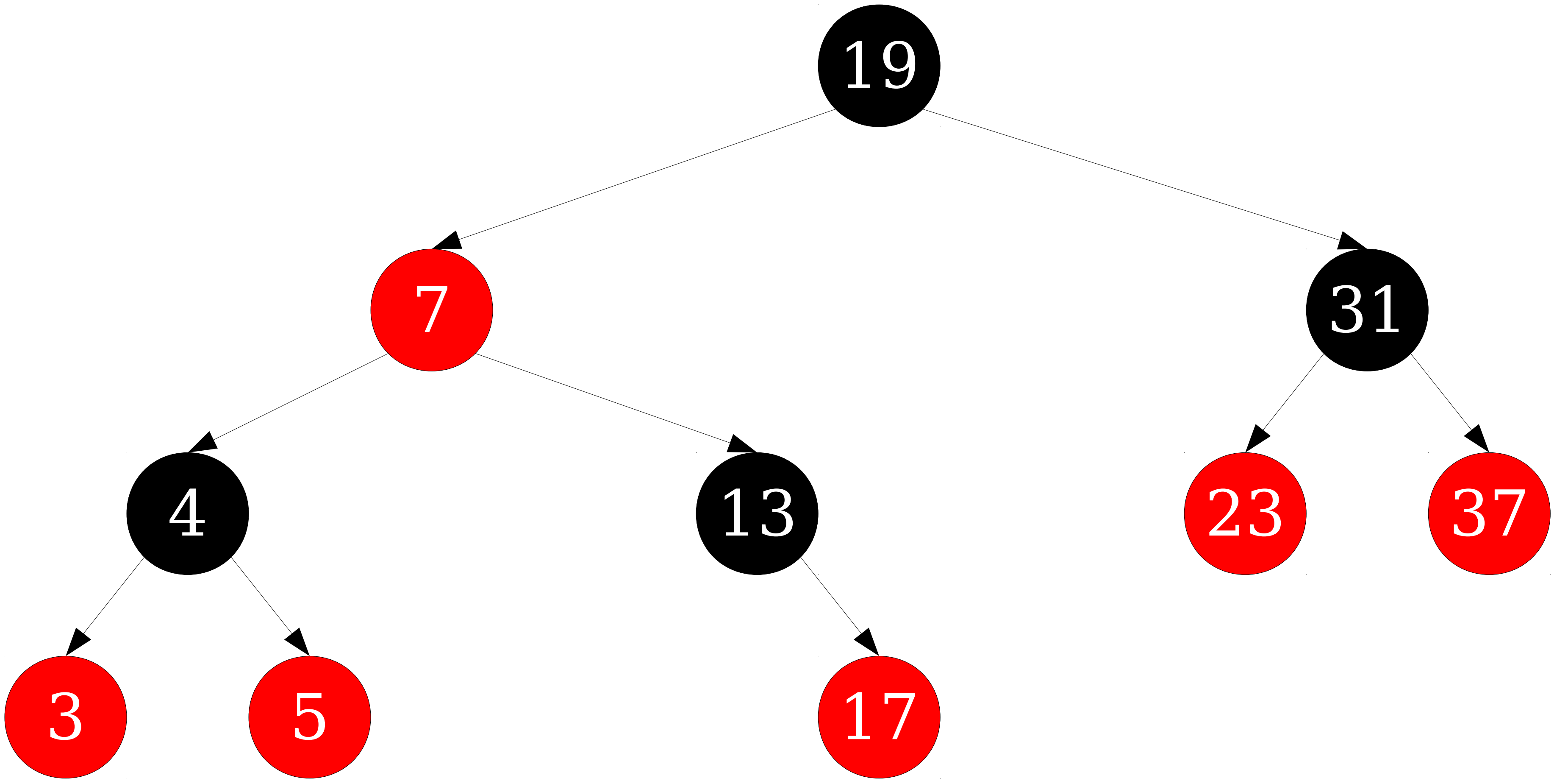
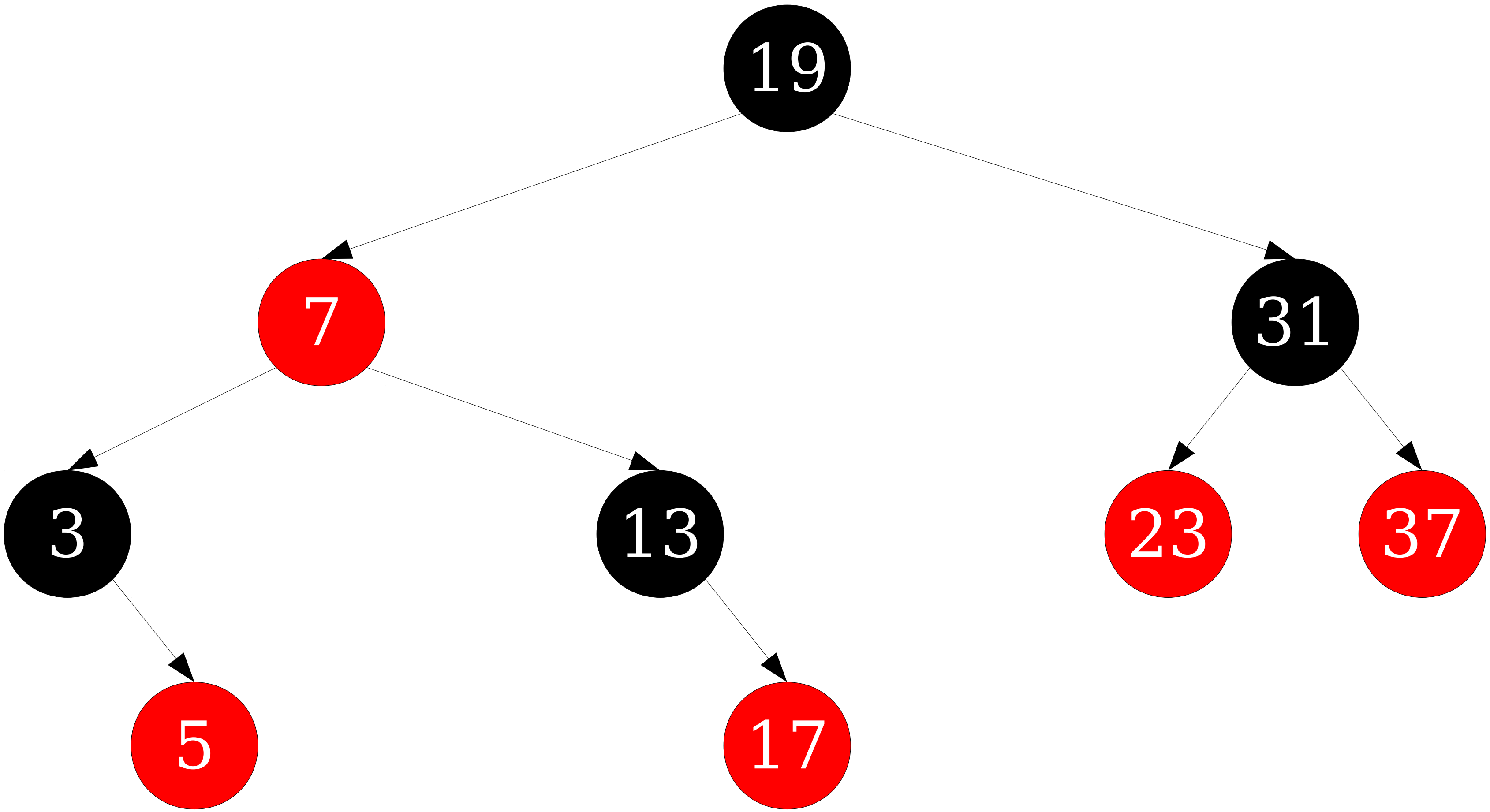


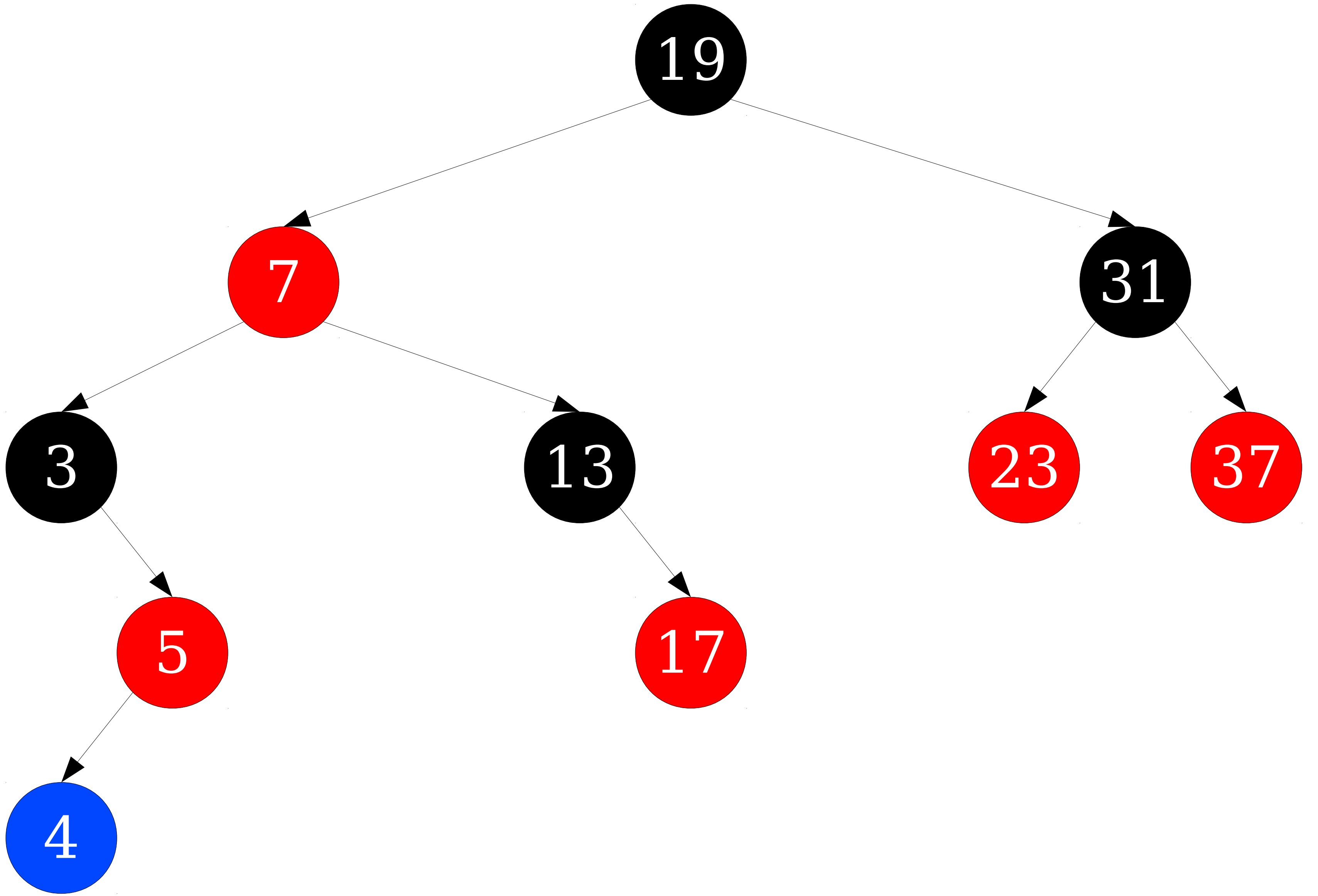
Using the Isometry



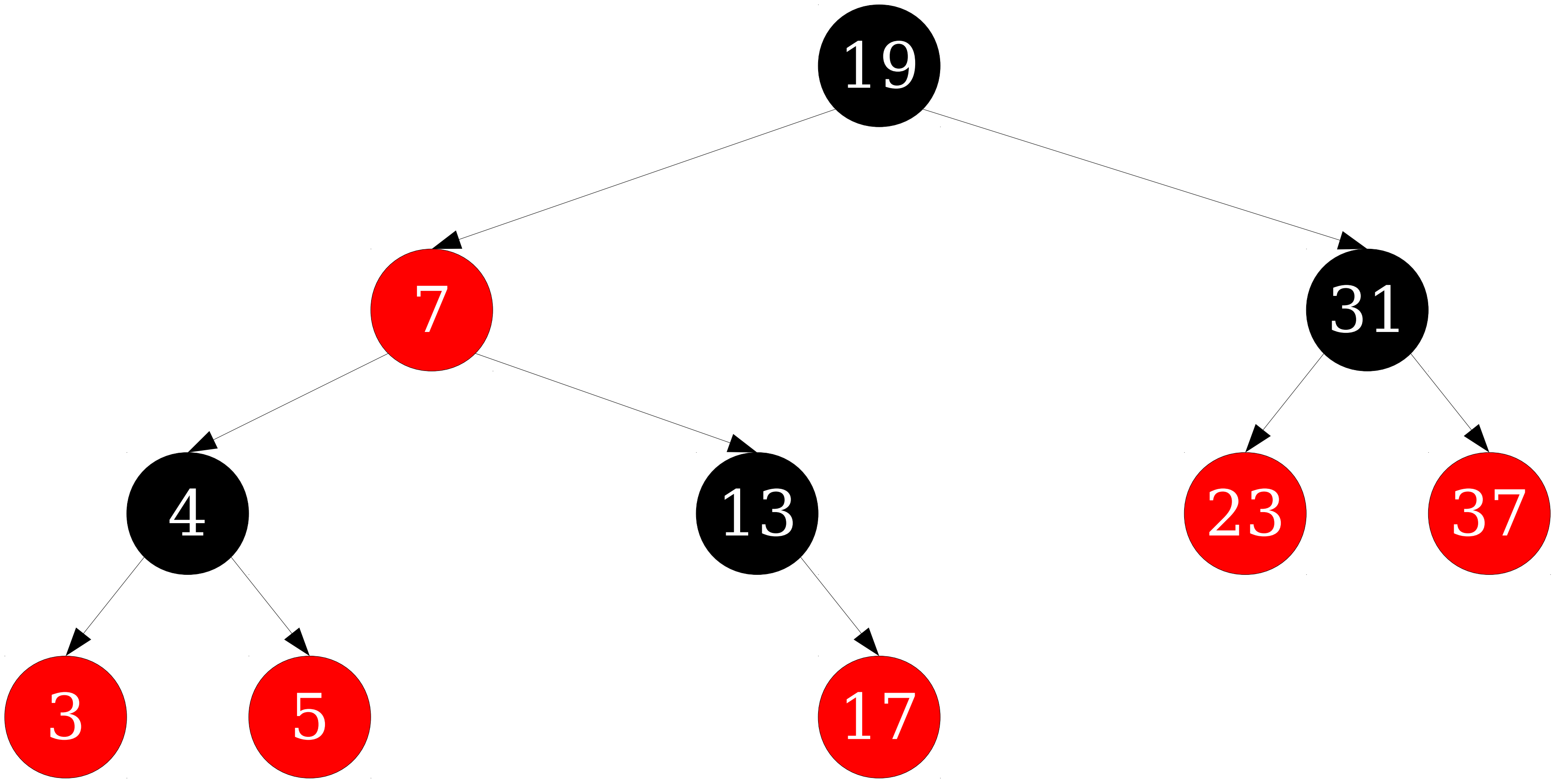
Using the Isometry



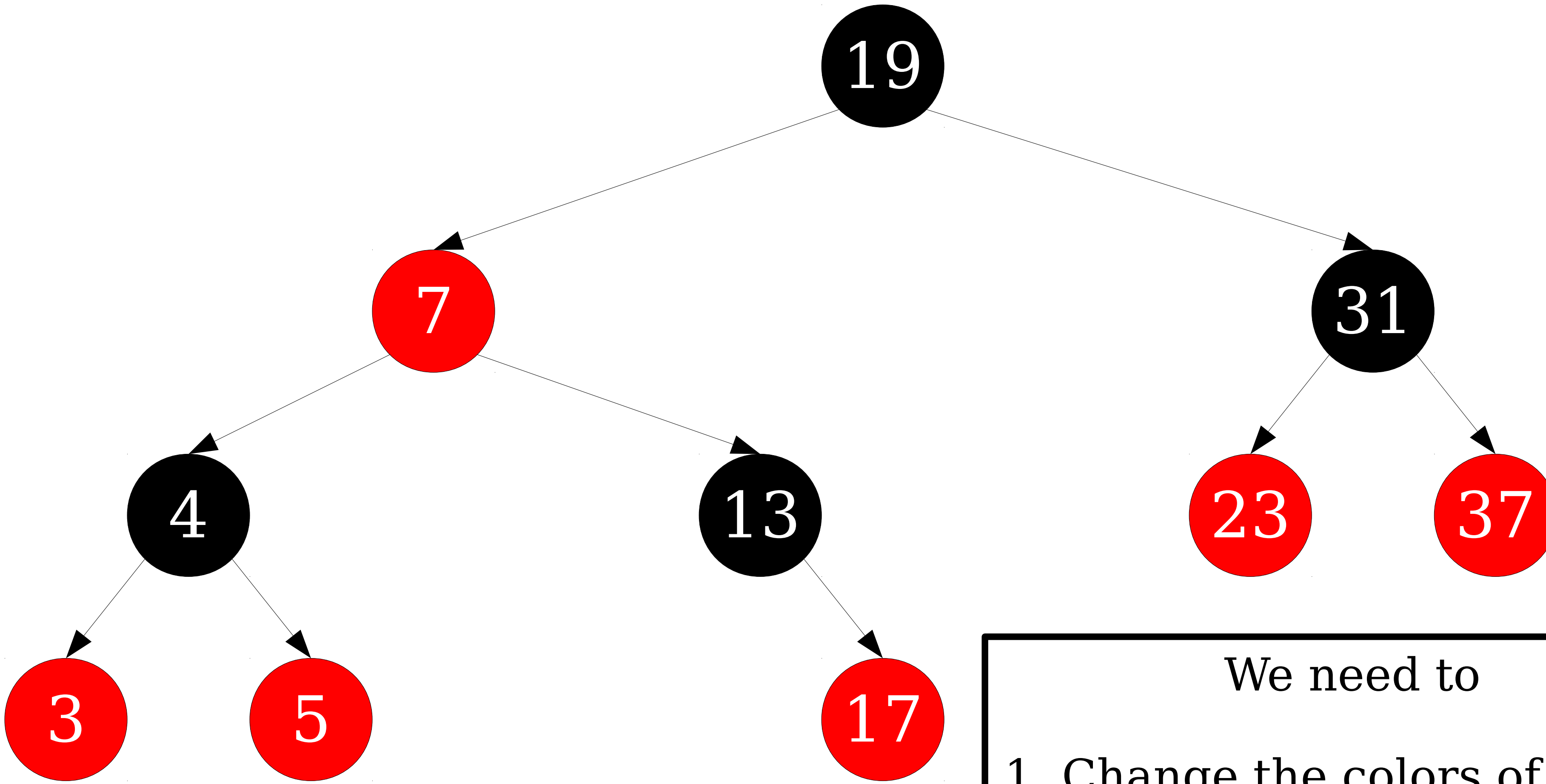
Using the Isometry



Using the Isometry



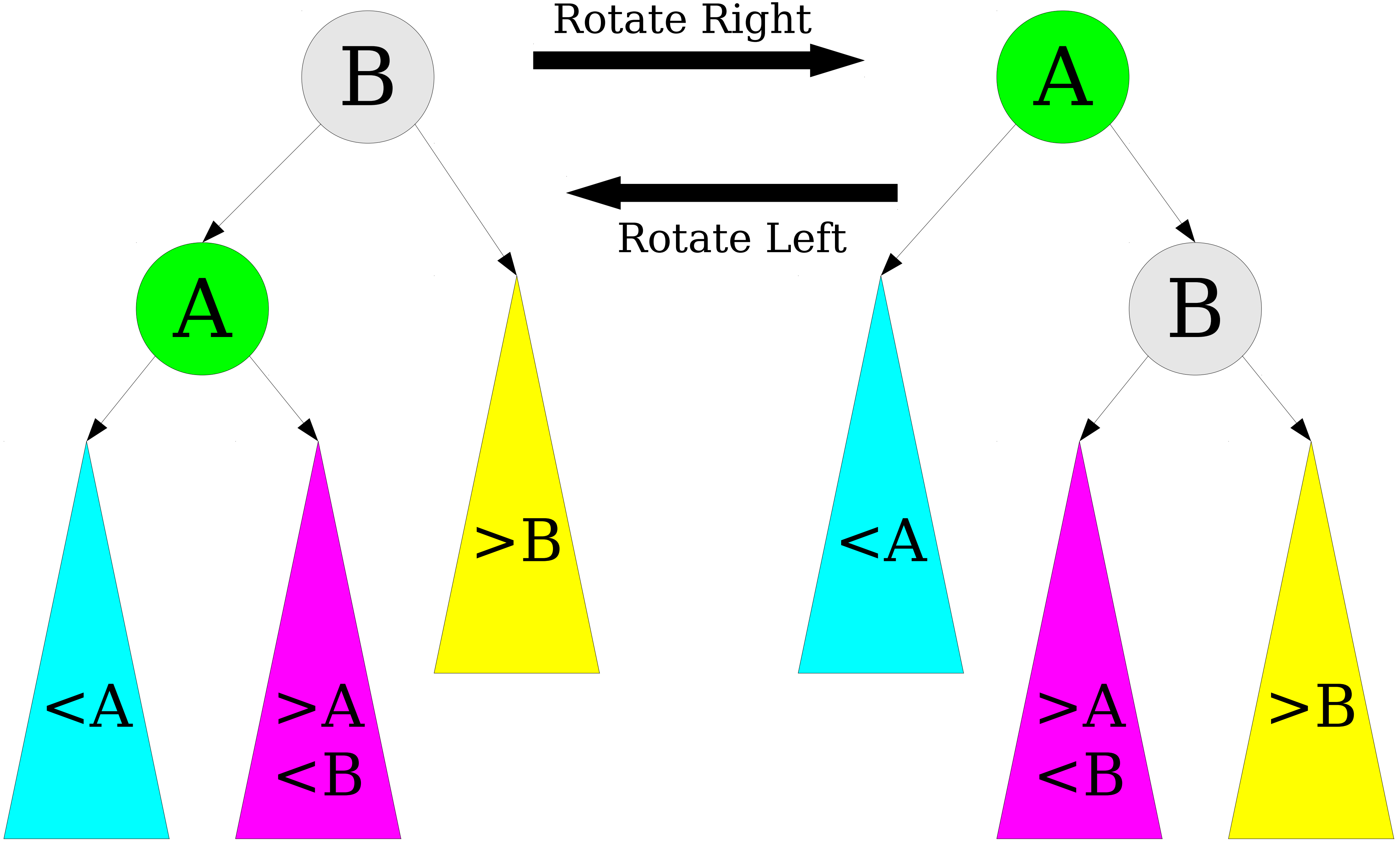
Using the Isometry



We need to

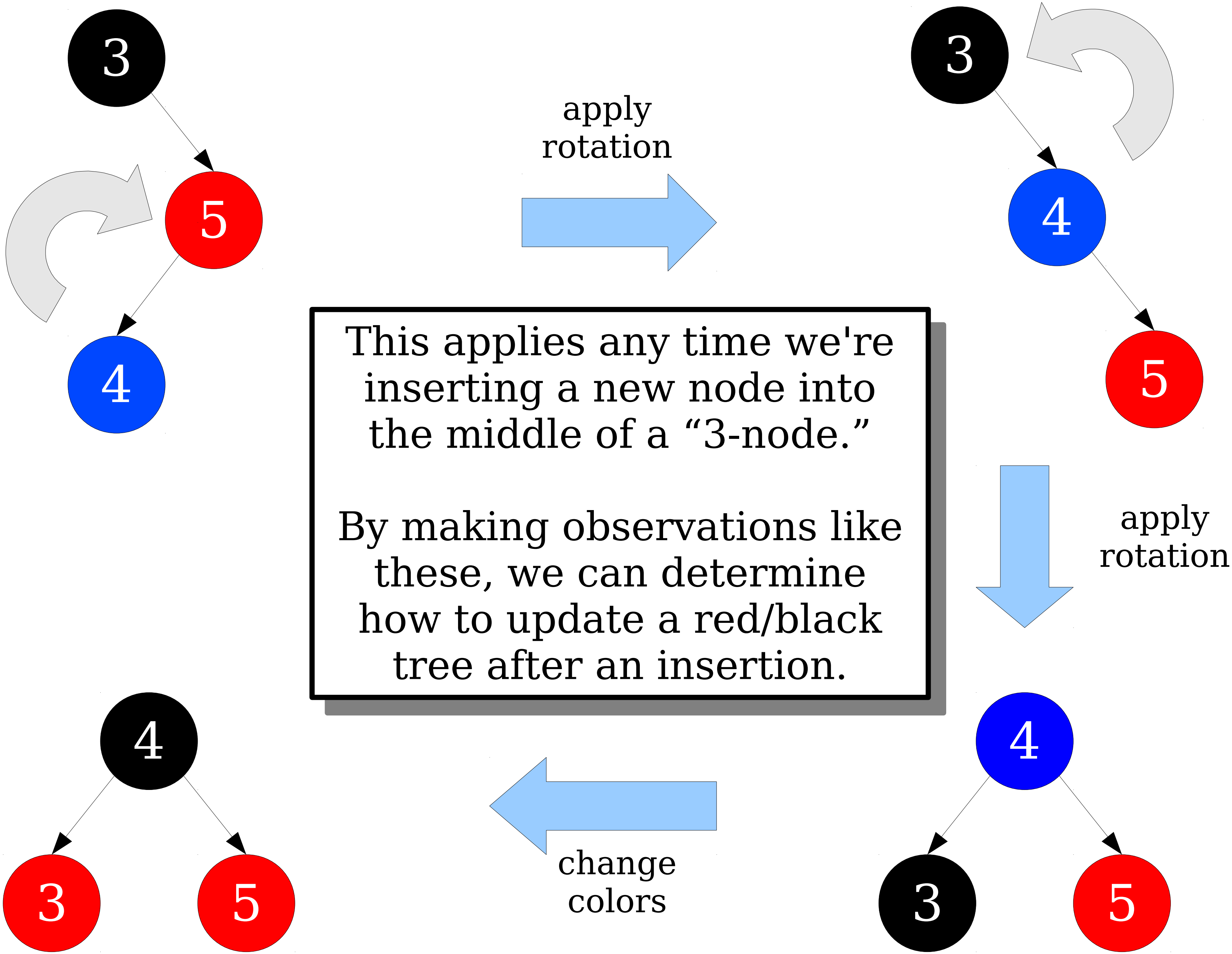
1. Change the colors of the nodes, and
2. Move the nodes around in the tree.

Tree Rotations



Tree Rotations

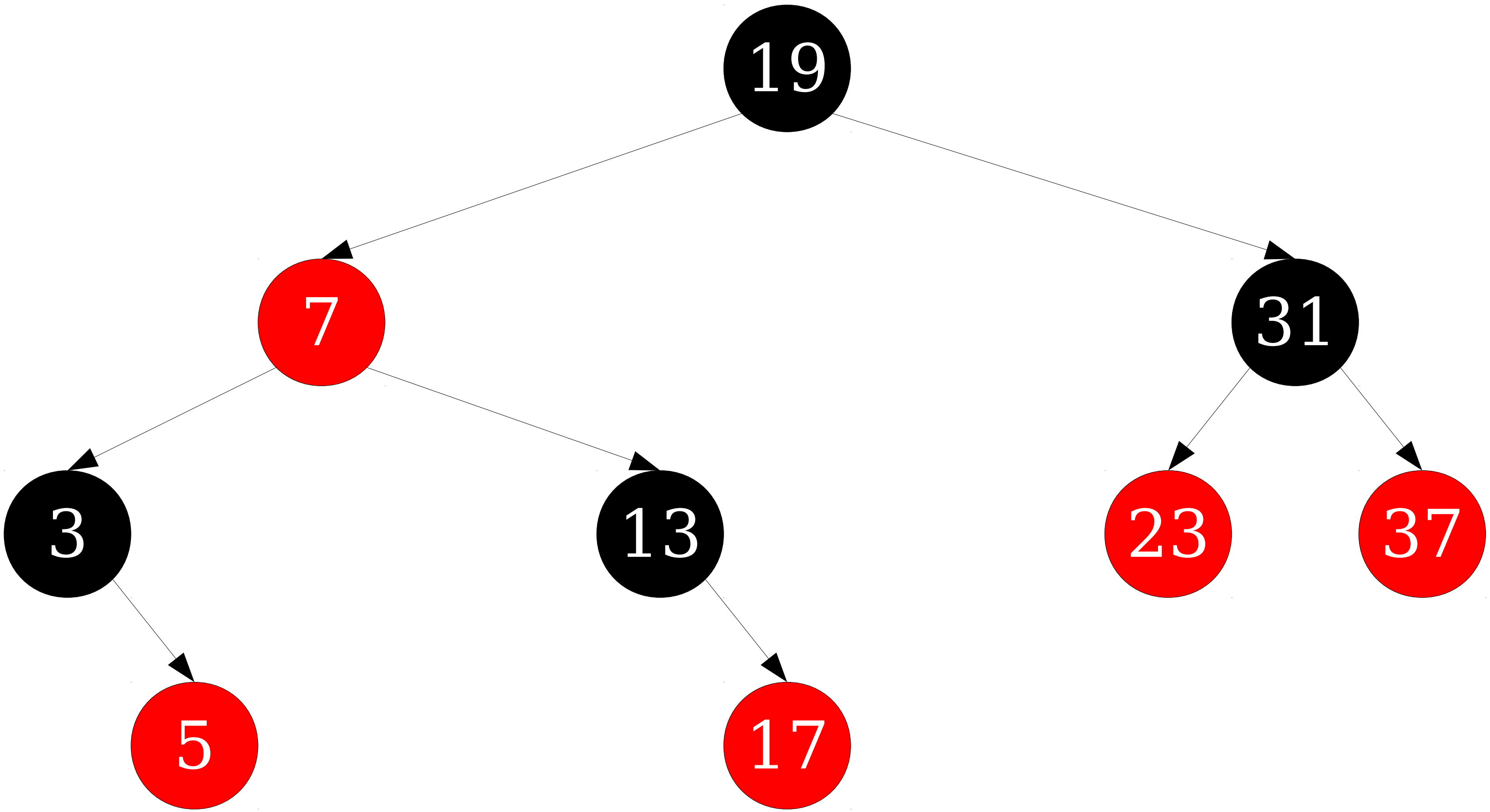
- Tree rotations are a fundamental primitive on binary search trees.
- Rotations locally reorder nodes while preserving the binary search property.
- Most balanced trees use tree rotations at some point.



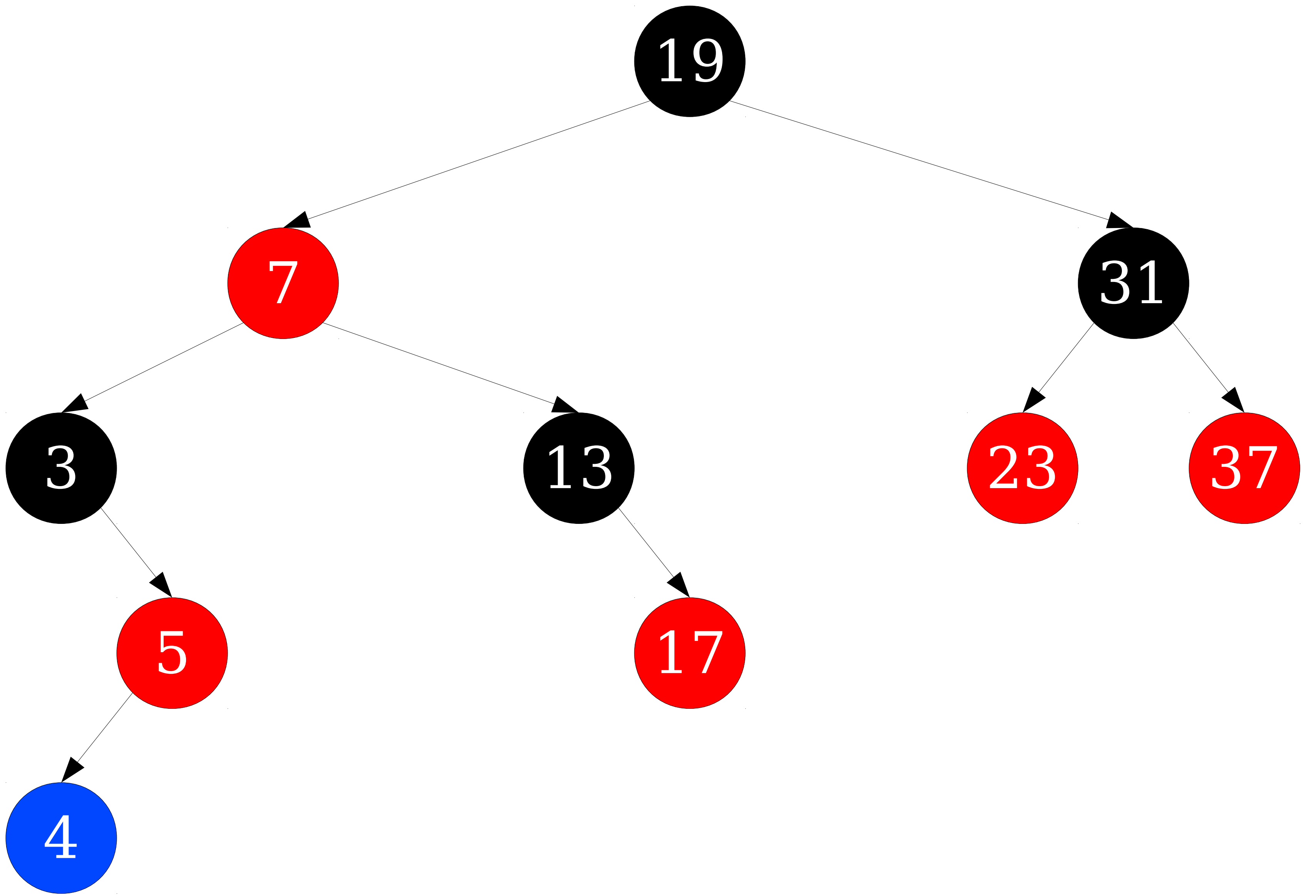
This applies any time we're inserting a new node into the middle of a "3-node."

By making observations like these, we can determine how to update a red/black tree after an insertion.

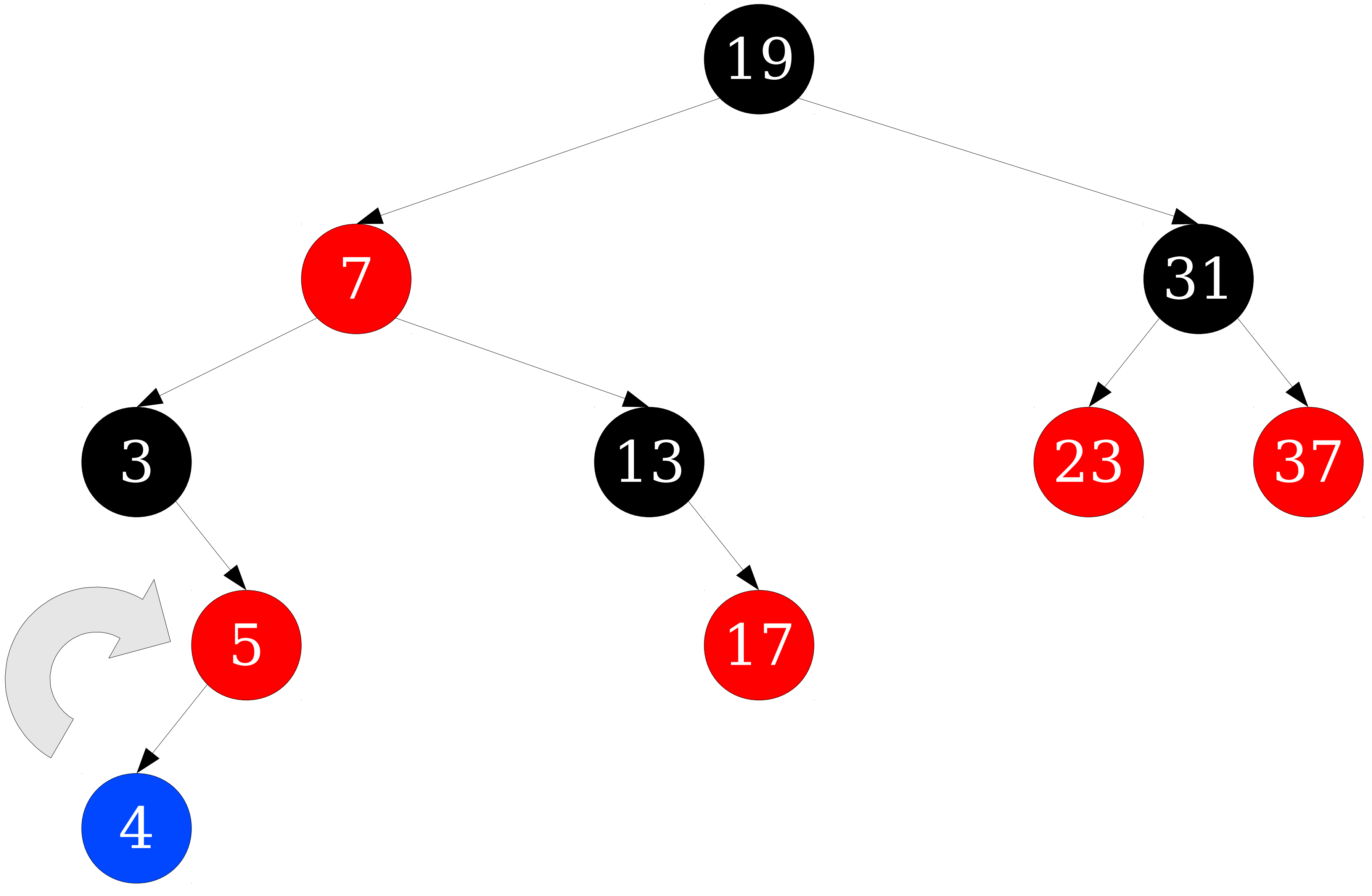
Using the Isometry



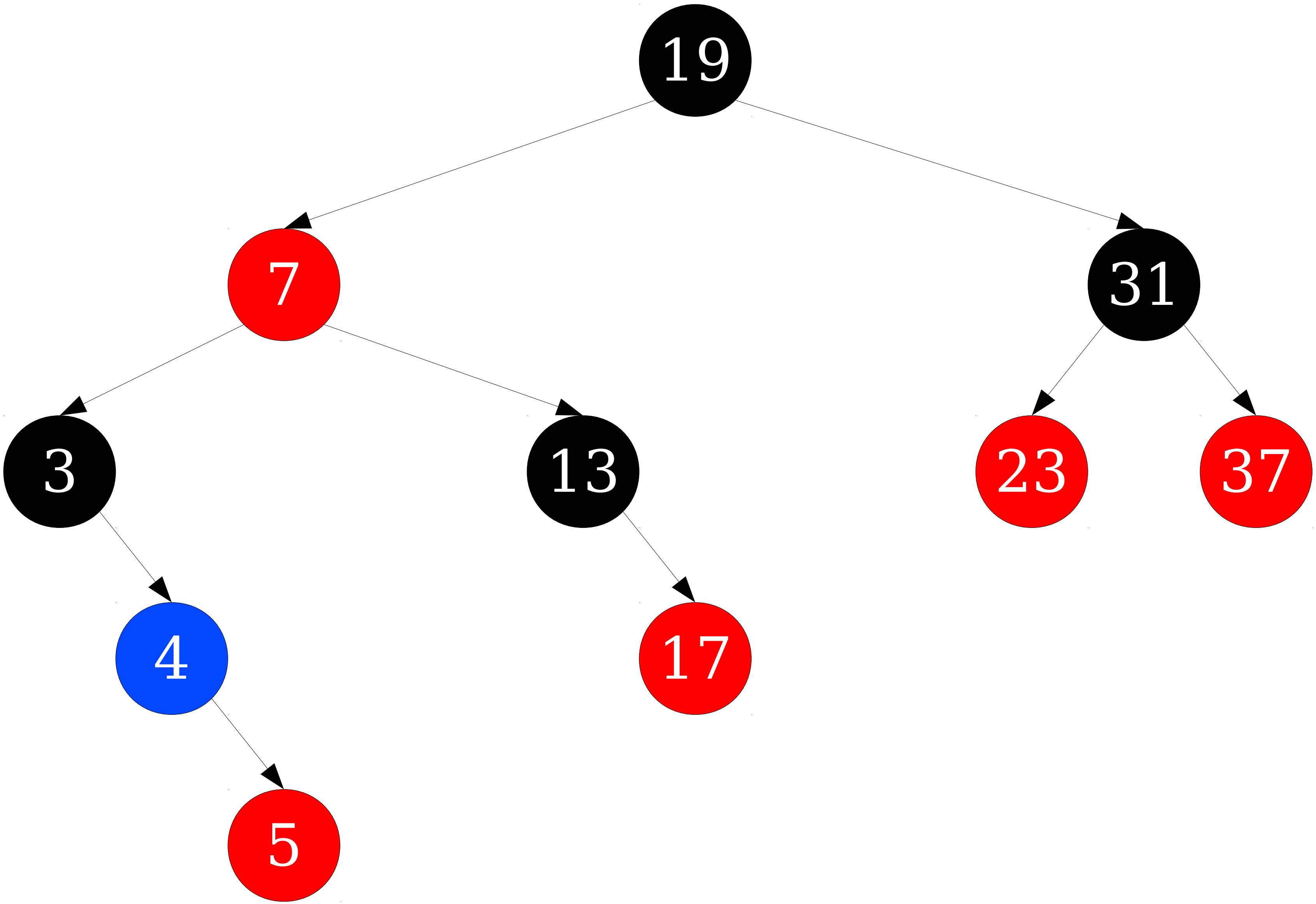
Using the Isometry



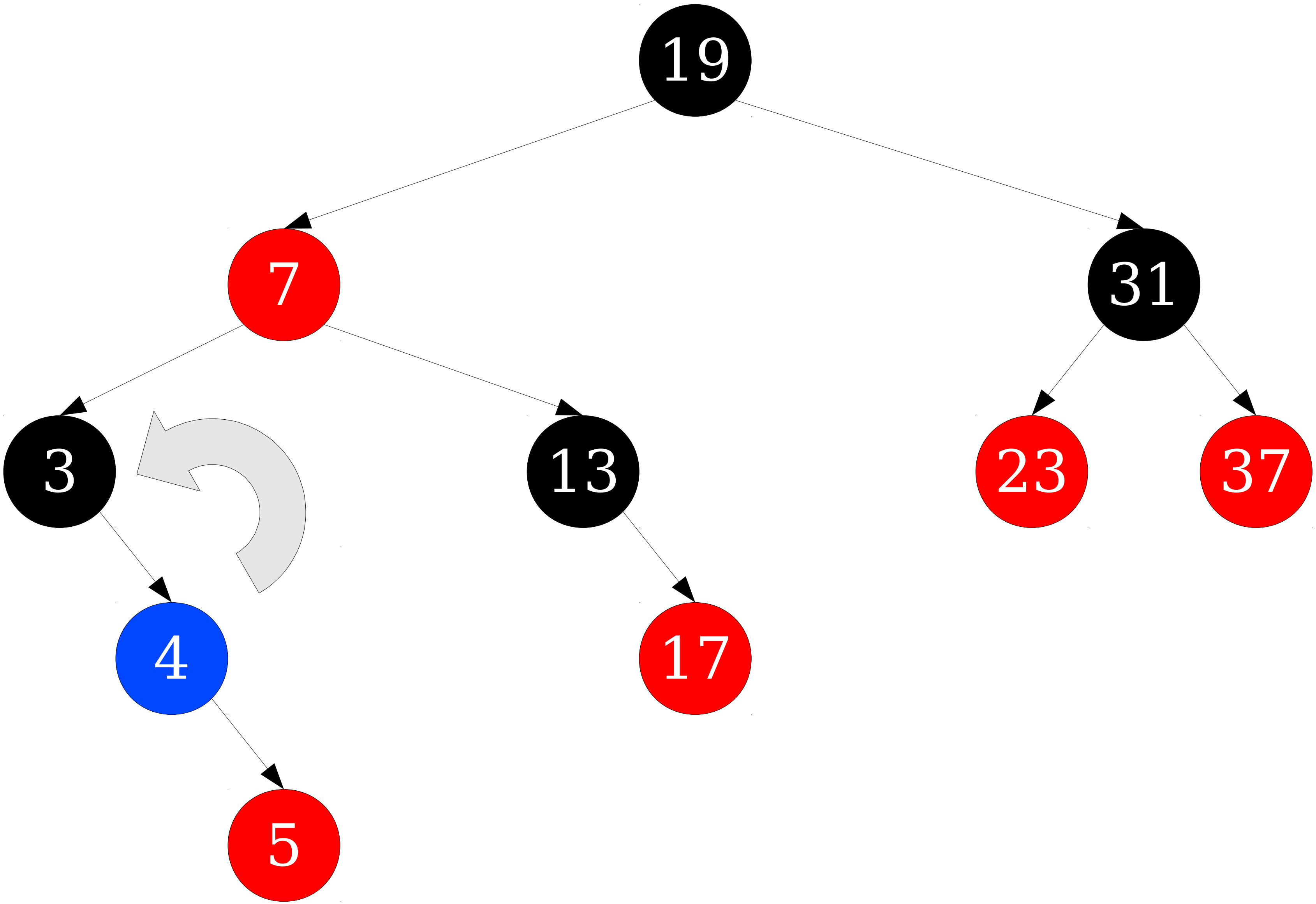
Using the Isometry



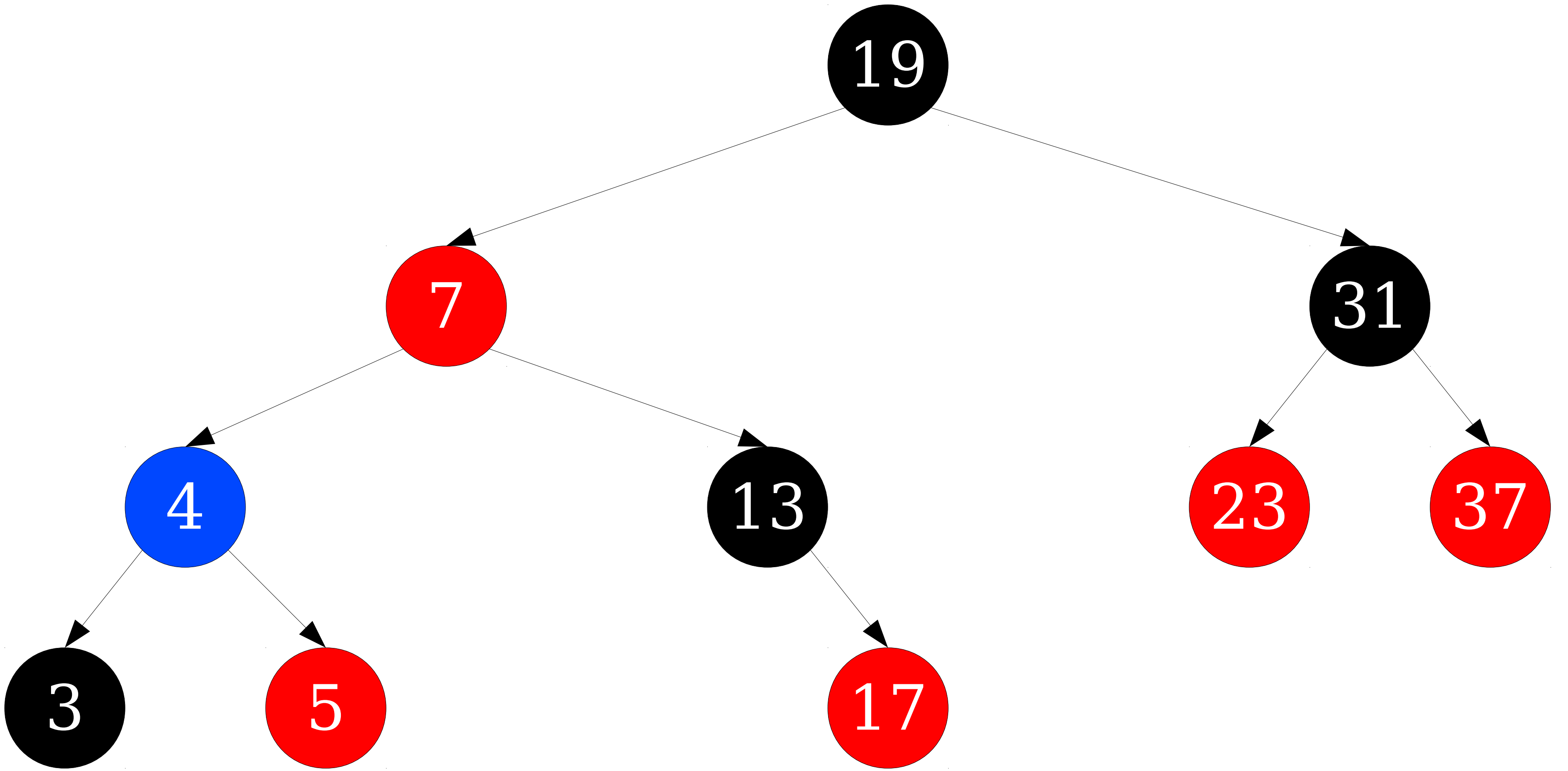
Using the Isometry



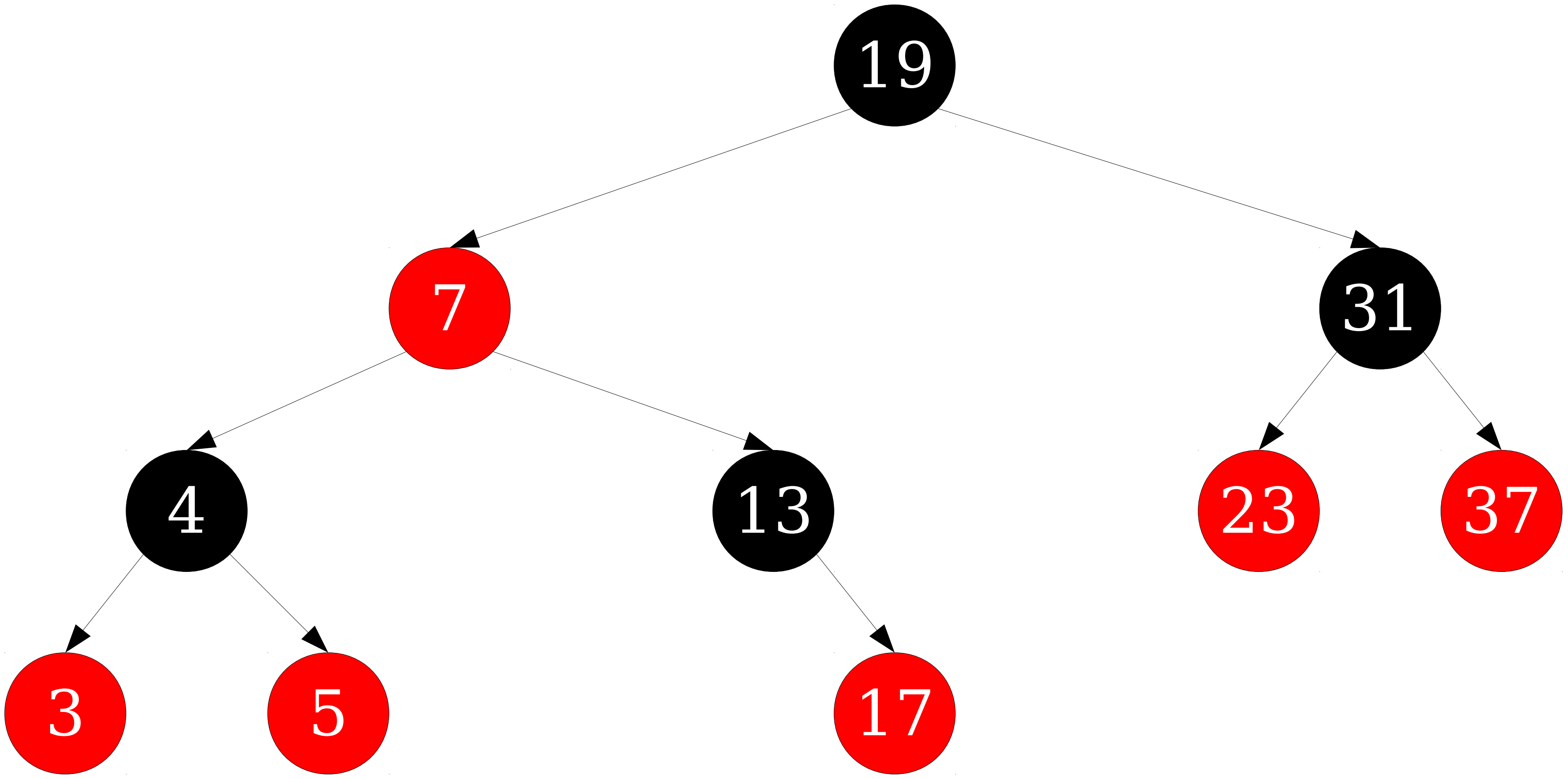
Using the Isometry



Using the Isometry



Using the Isometry



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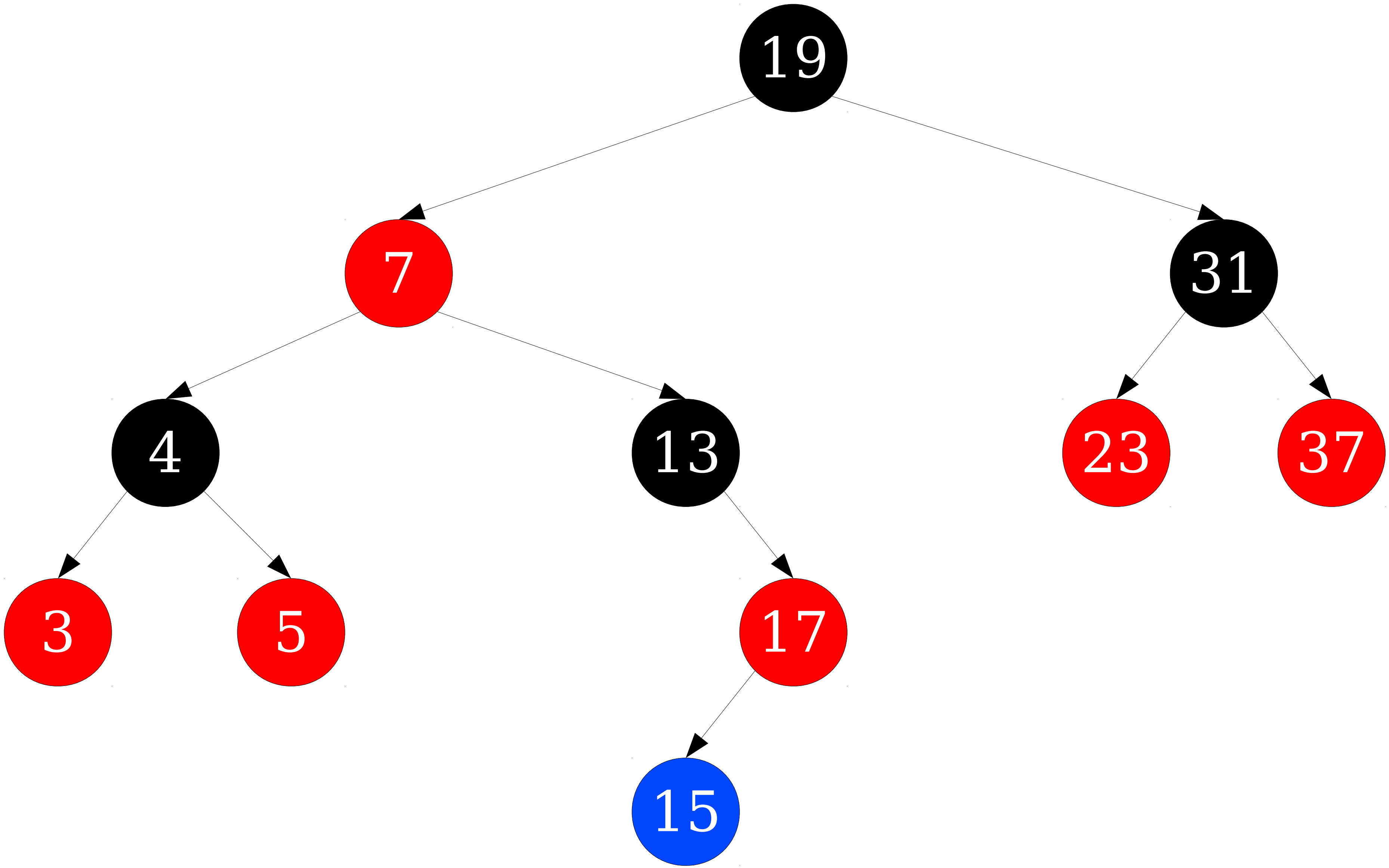
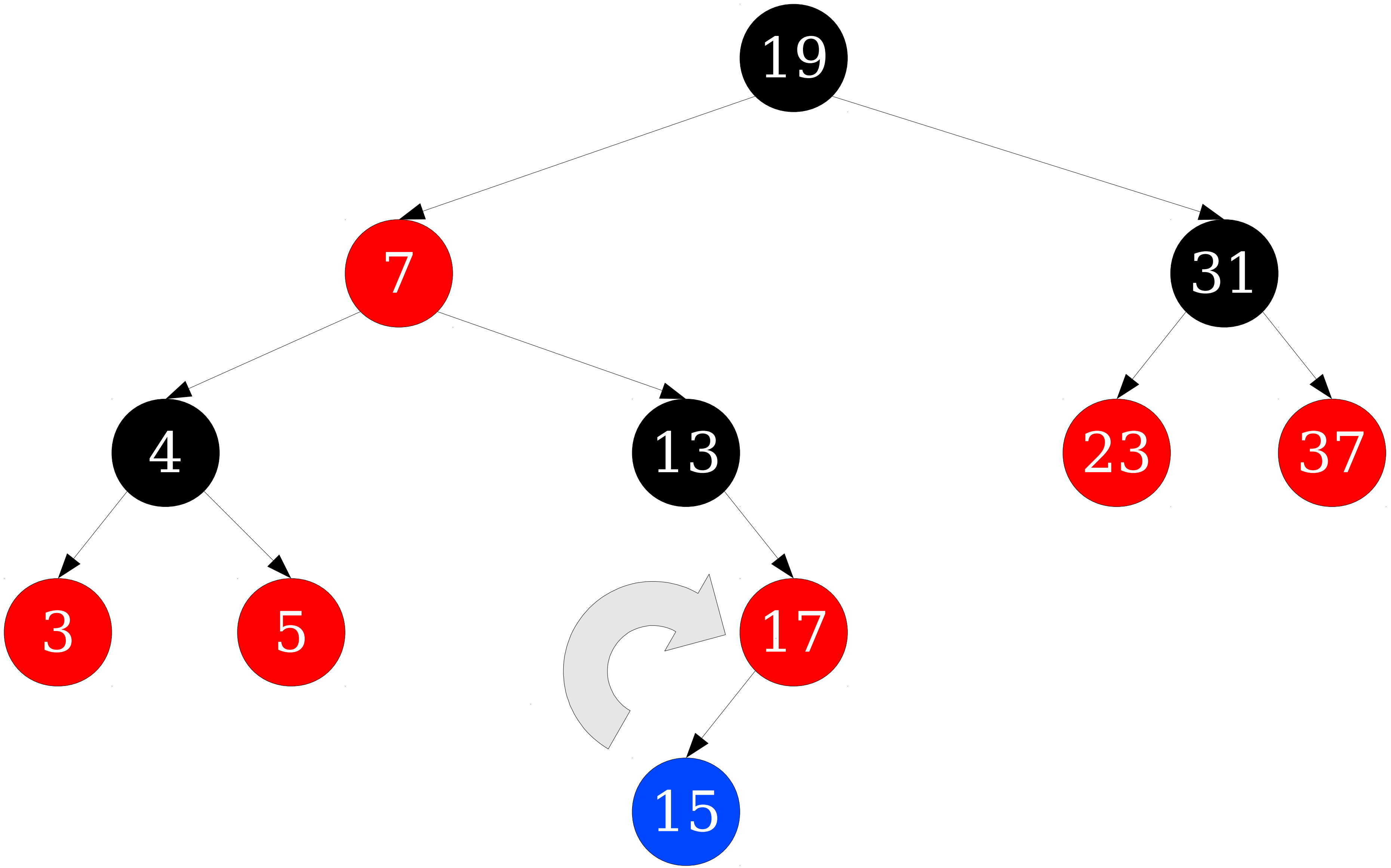


Using the Isometry



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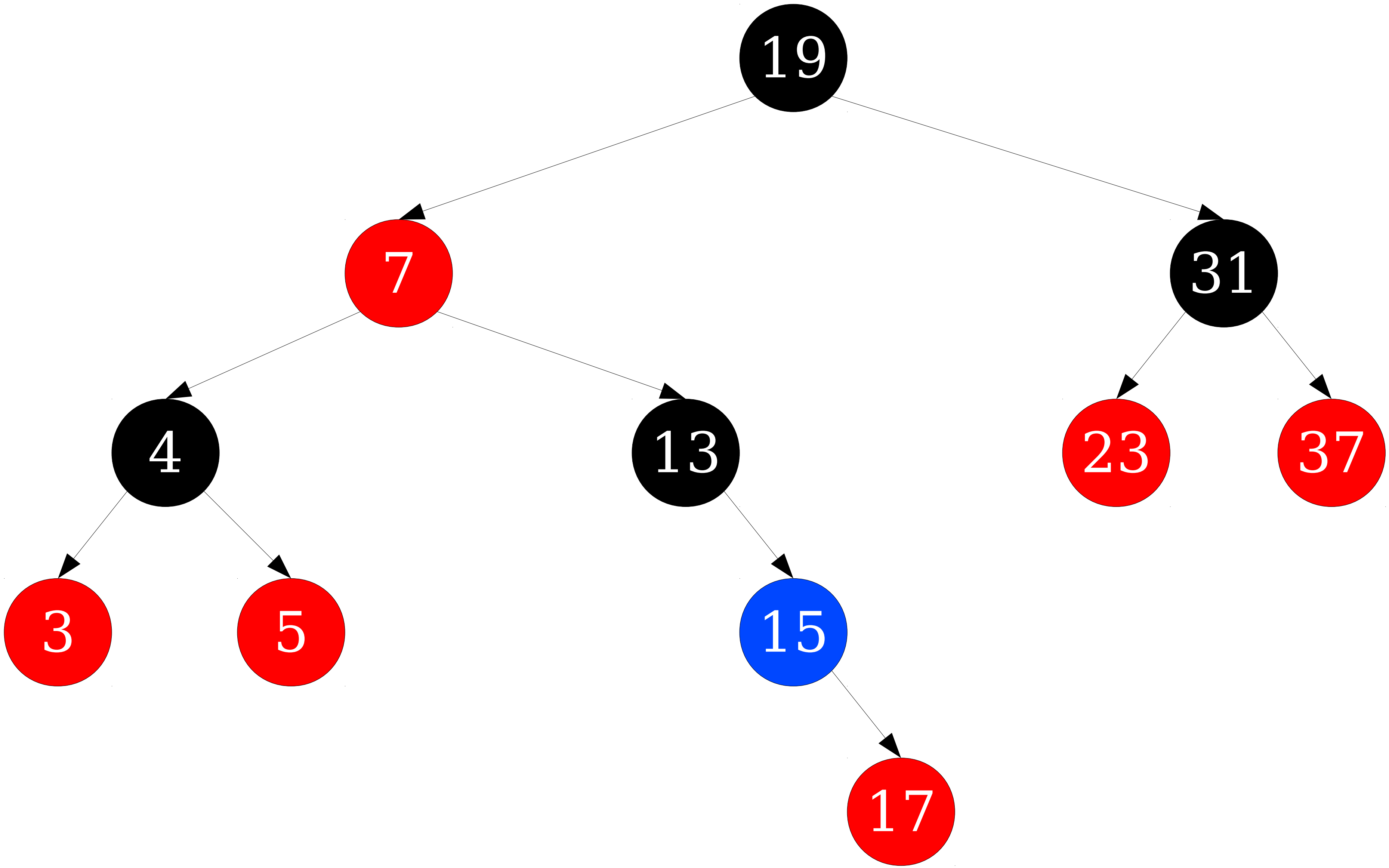
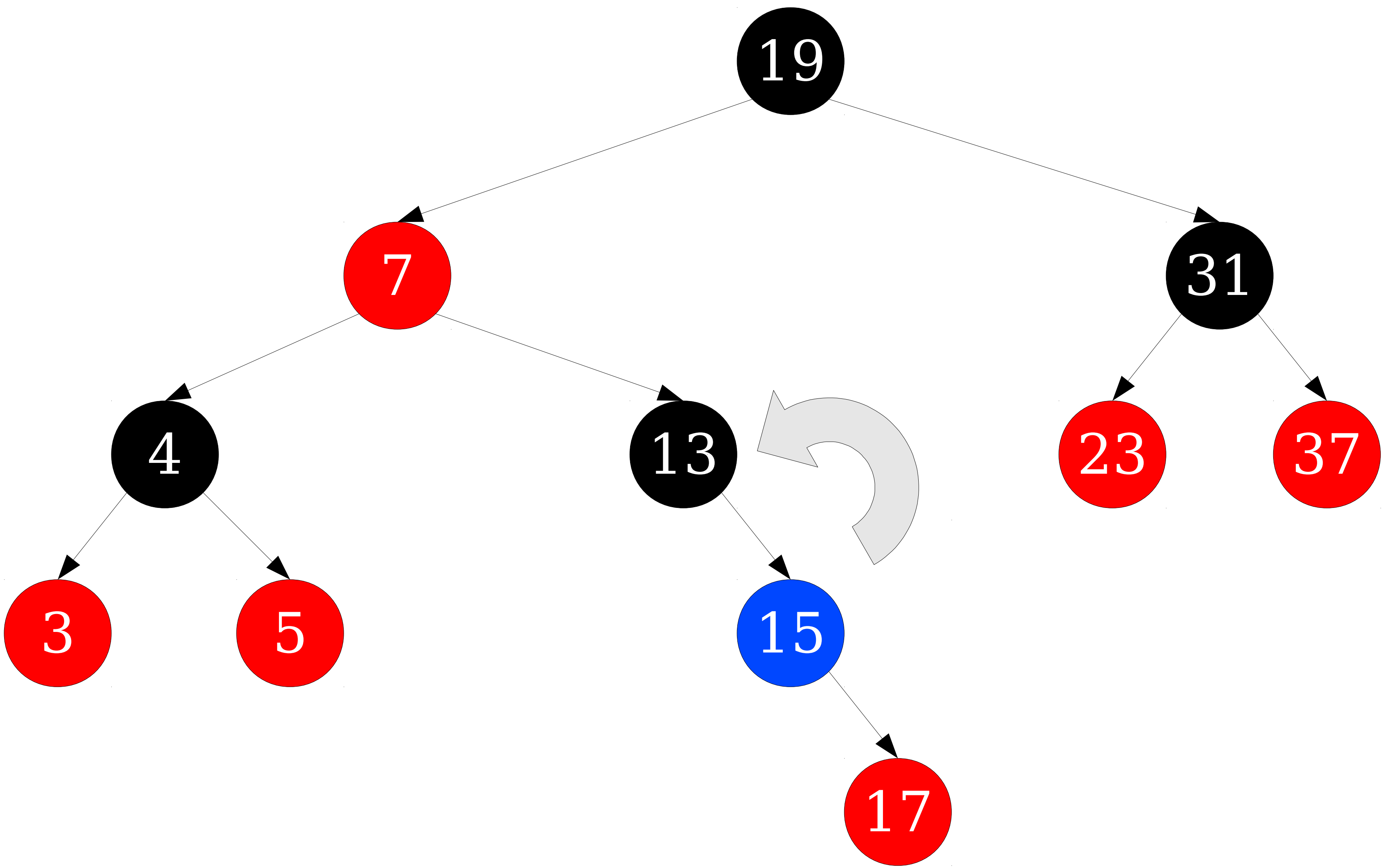
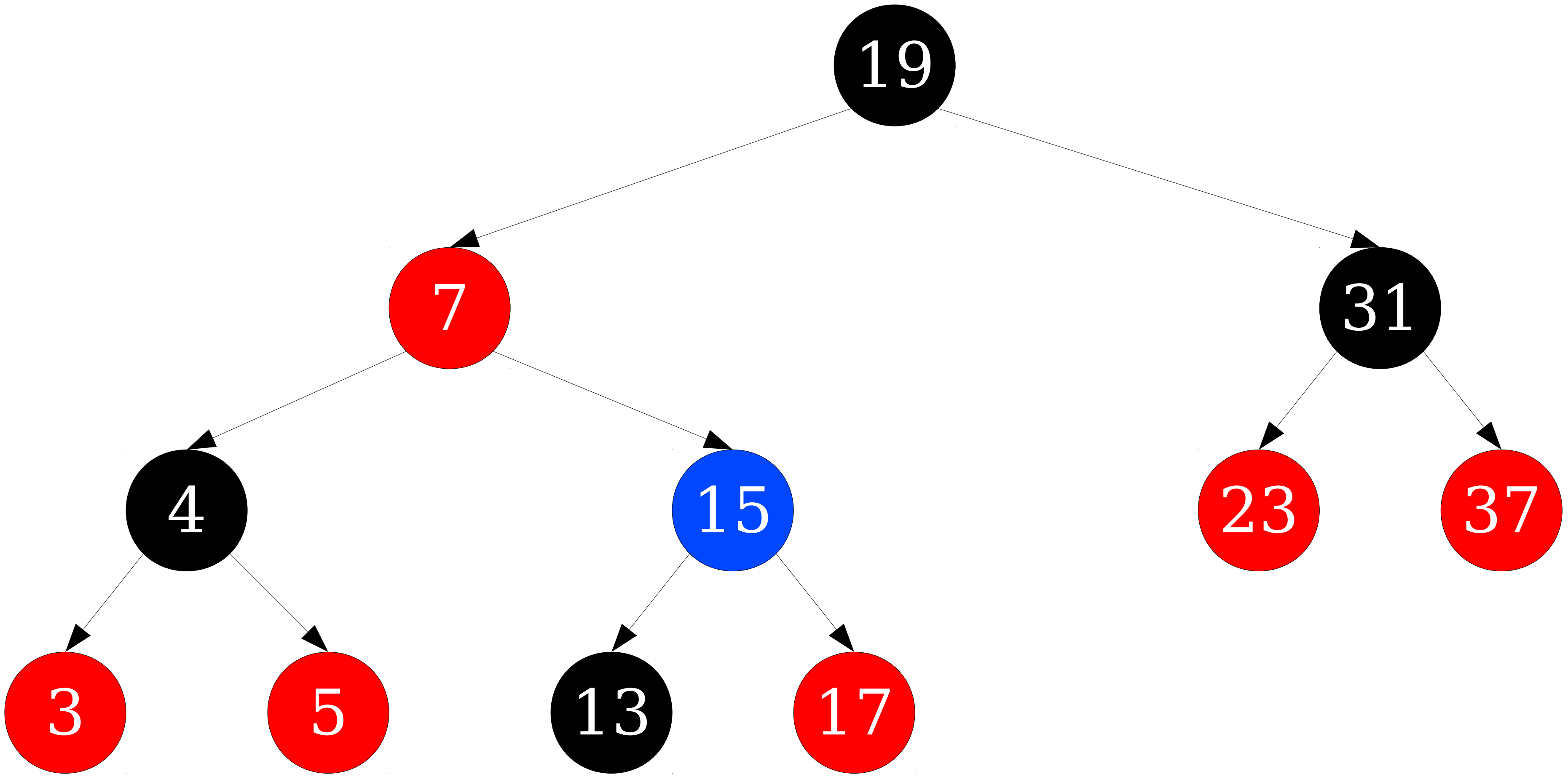


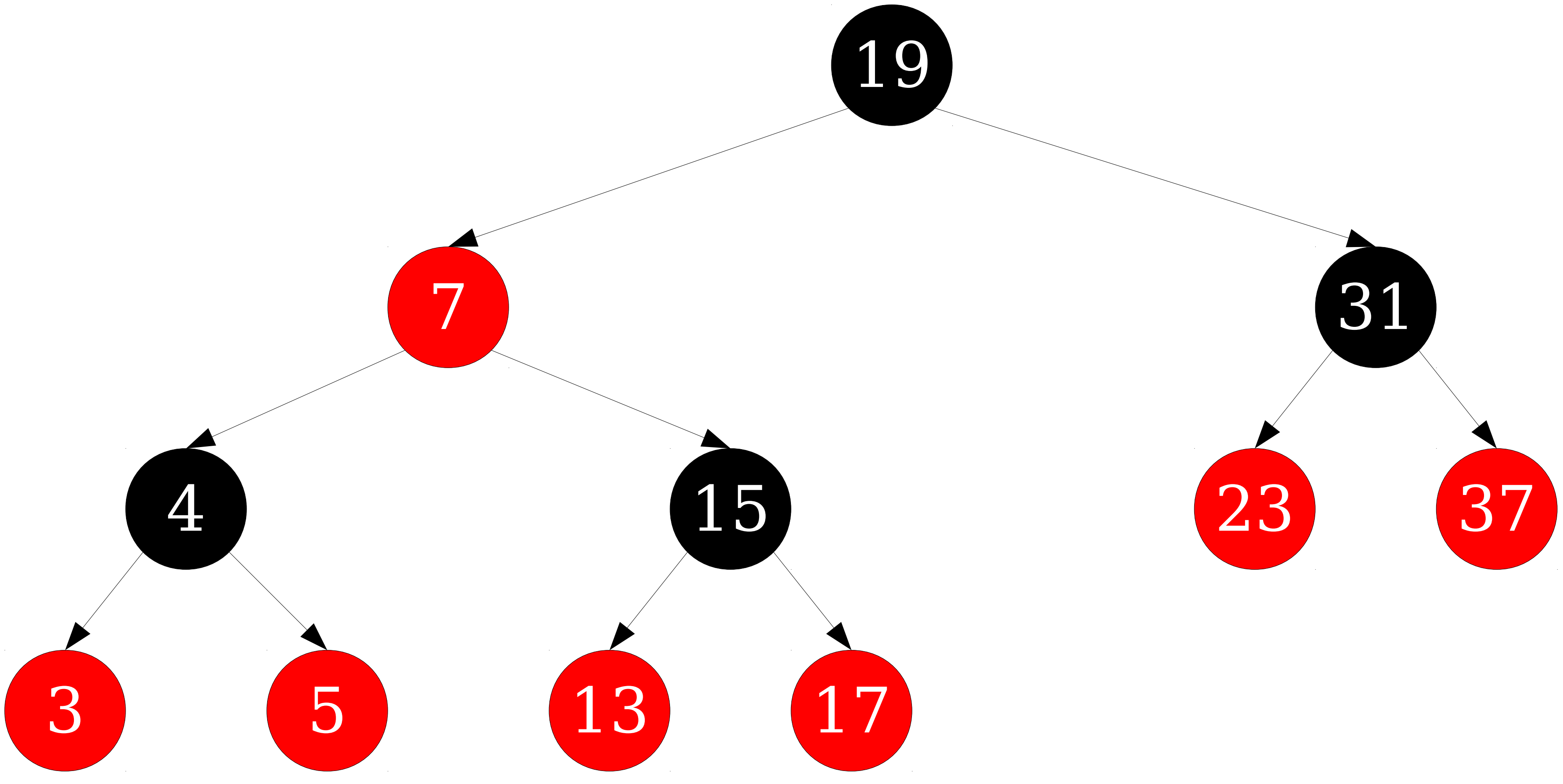
Using the Isometry



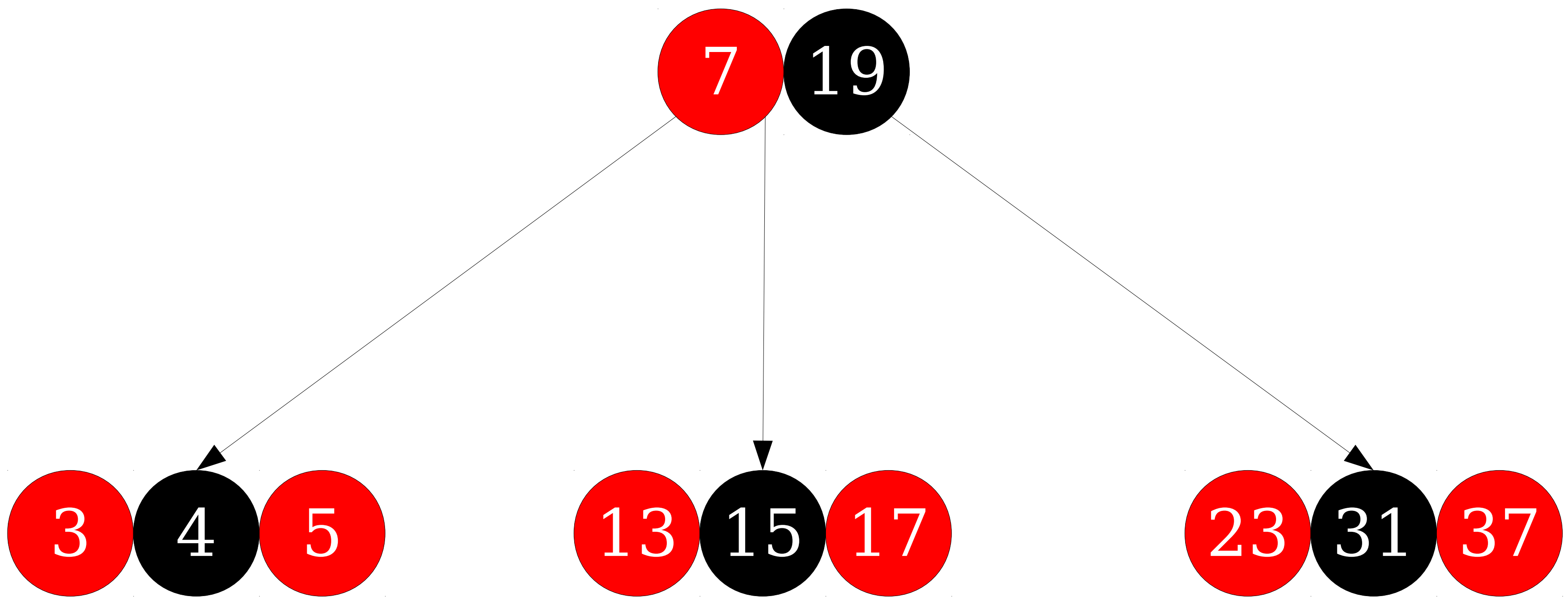
Using the Isometry



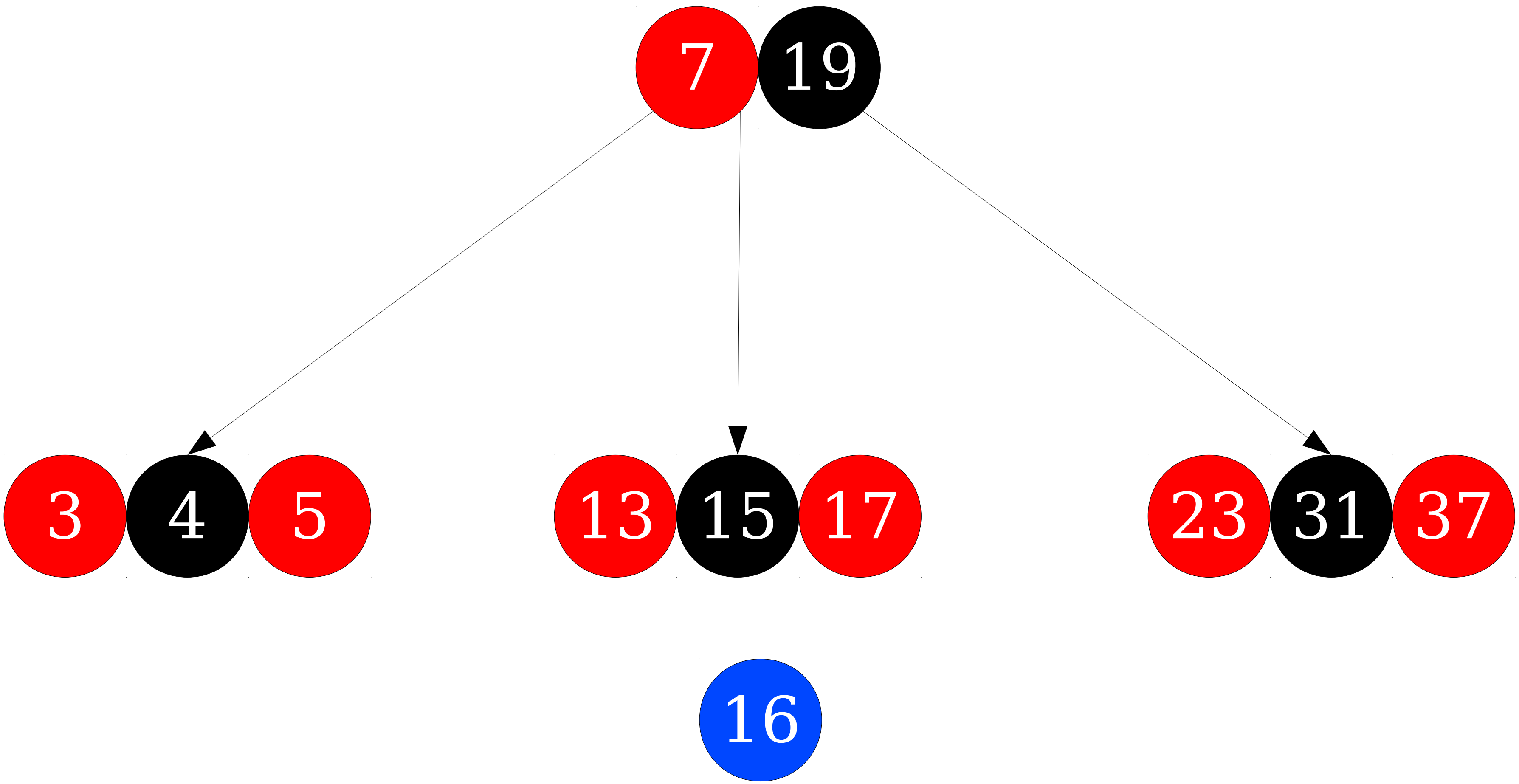
Using the Isometry



Using the Isometry



Using the Isometry



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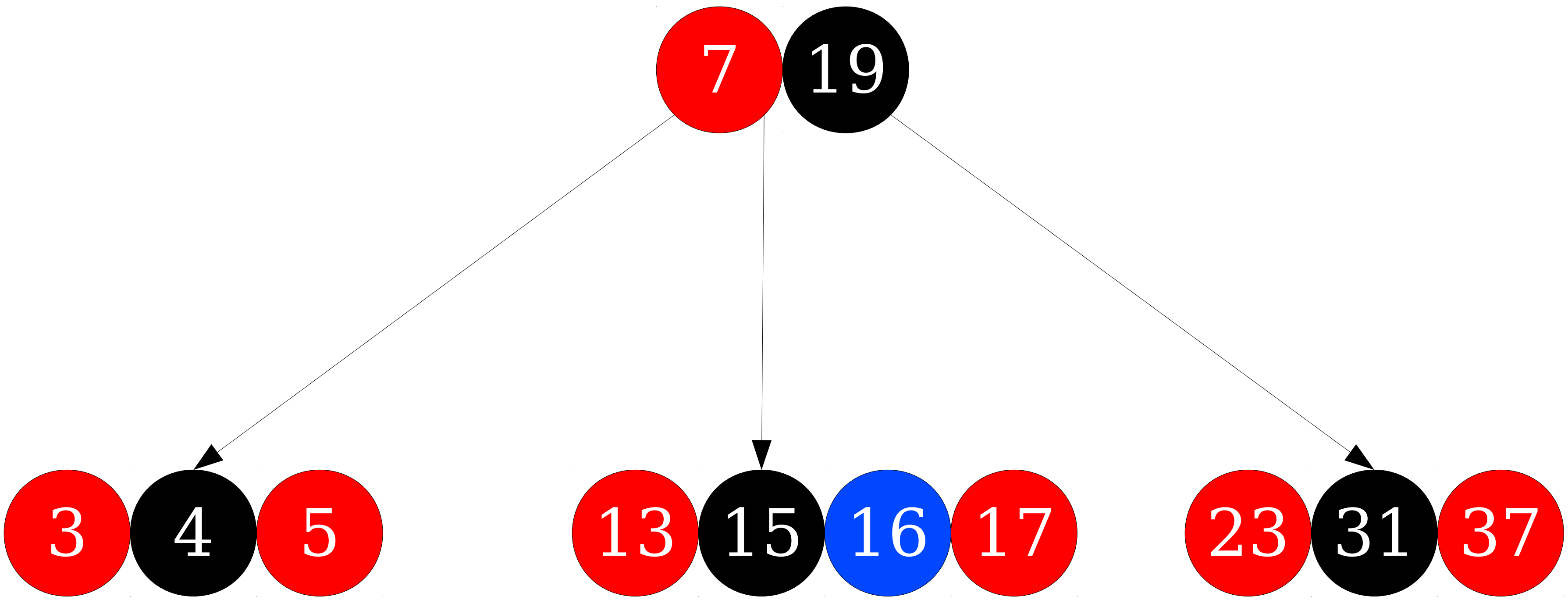

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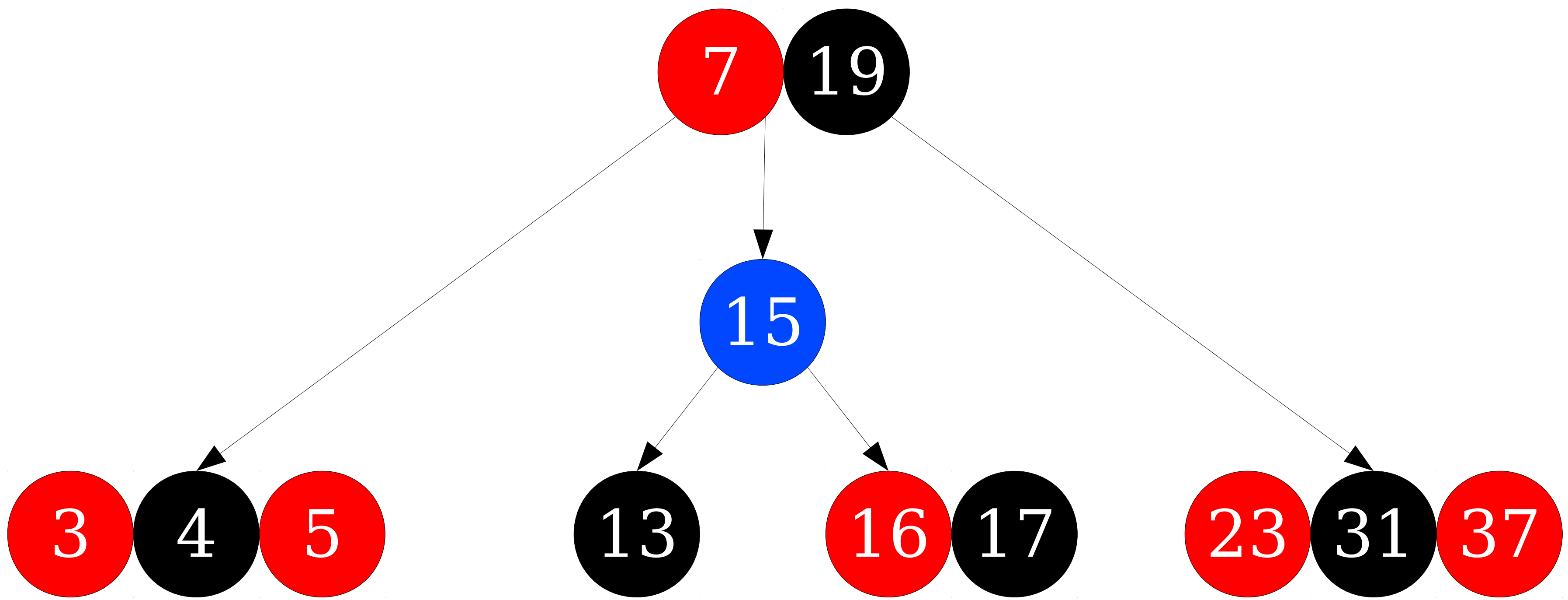
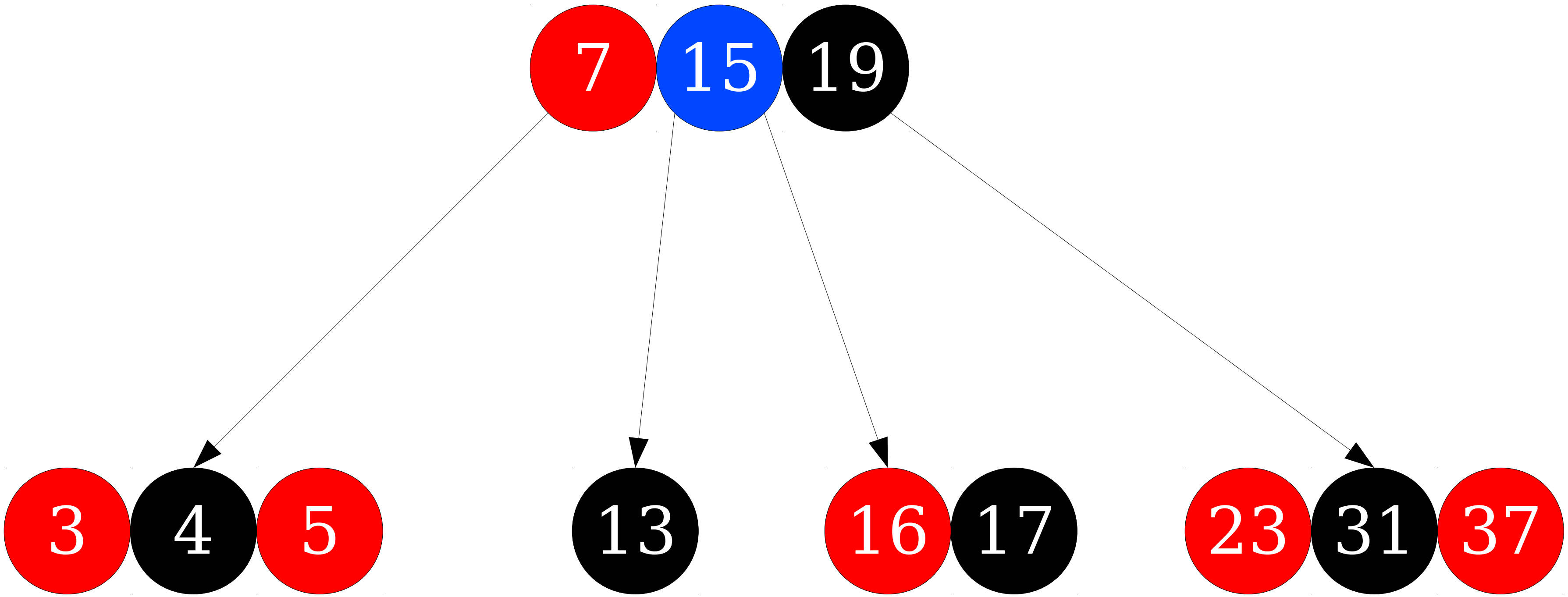
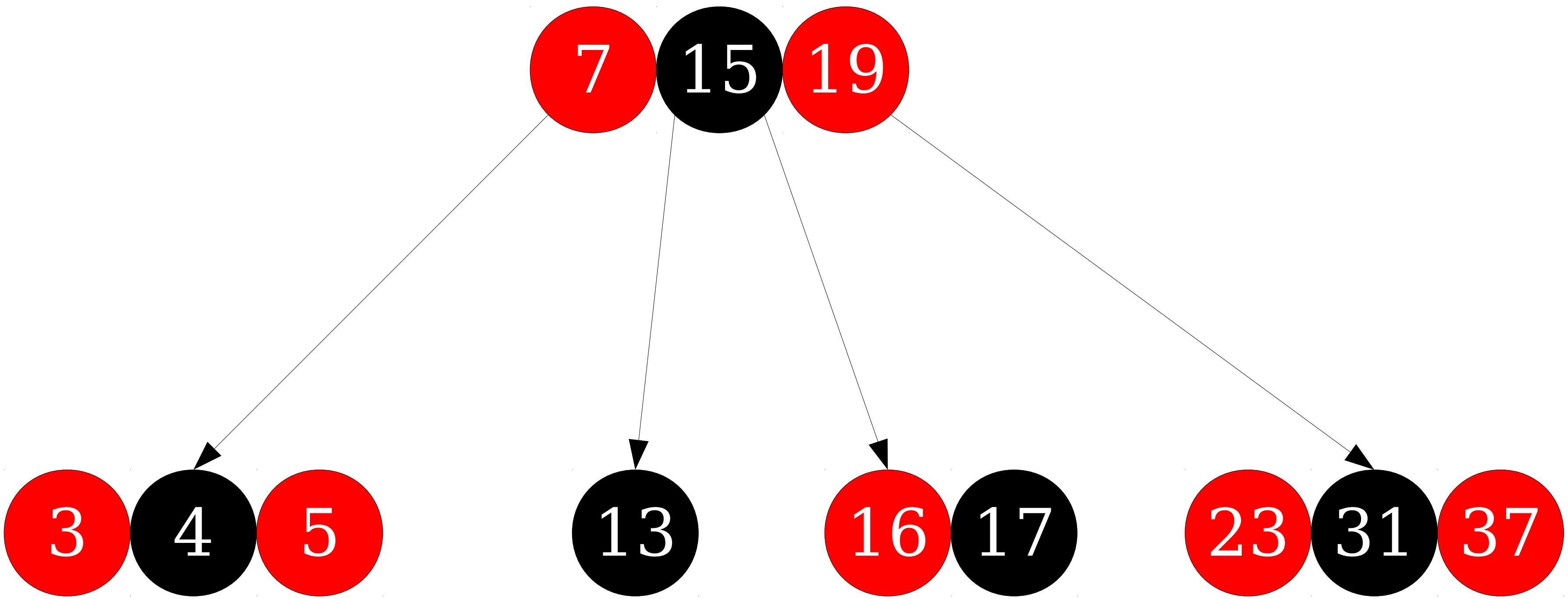


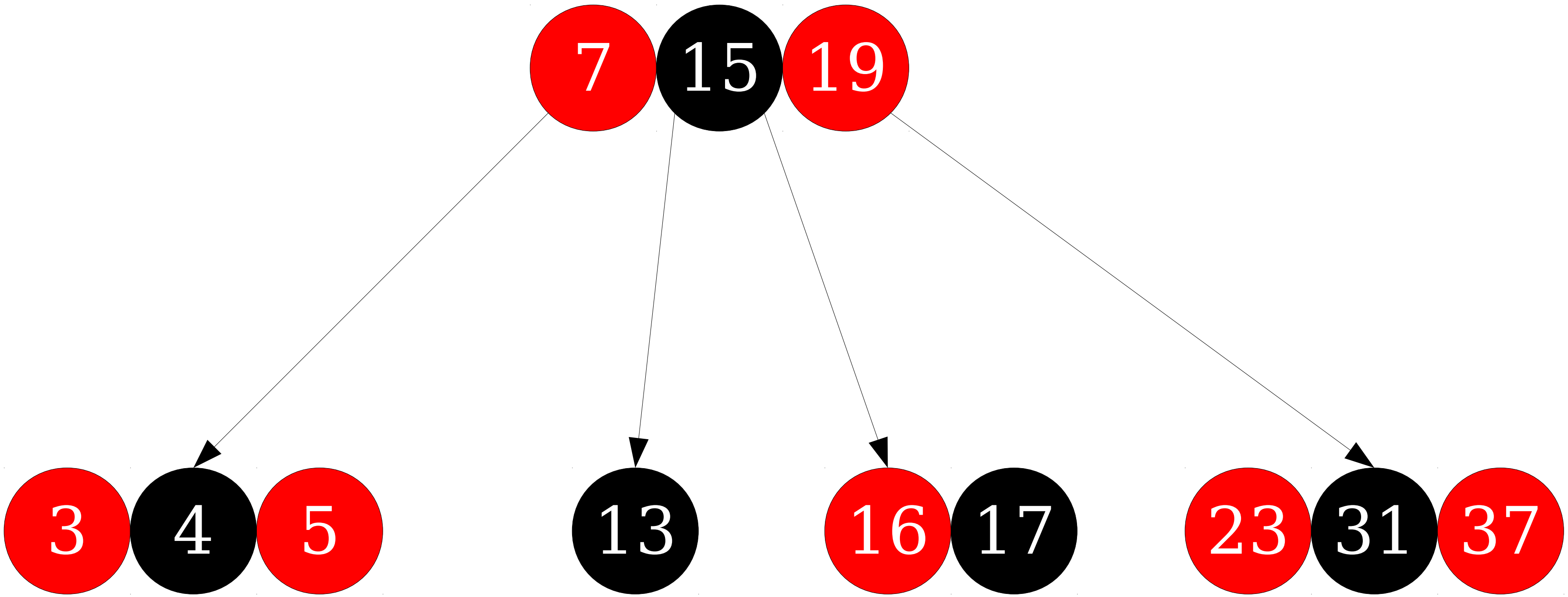
Using the Isometry



Using the Isometry

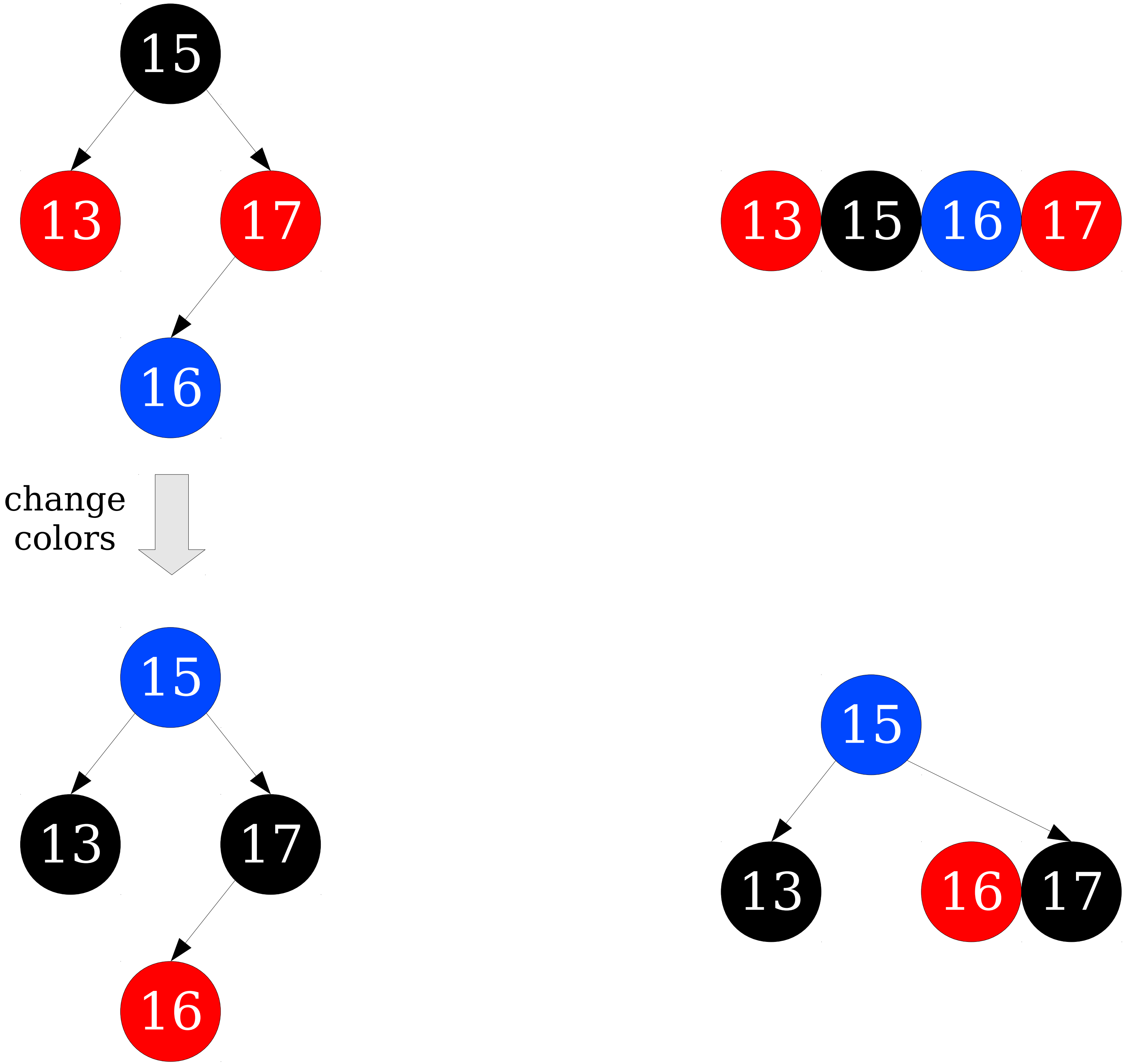


Using the Isometry



Two steps:

1. Split the “5-node” into a “2-node” and a “3-node.”
2. Insert the new parent of the two nodes into the parent node.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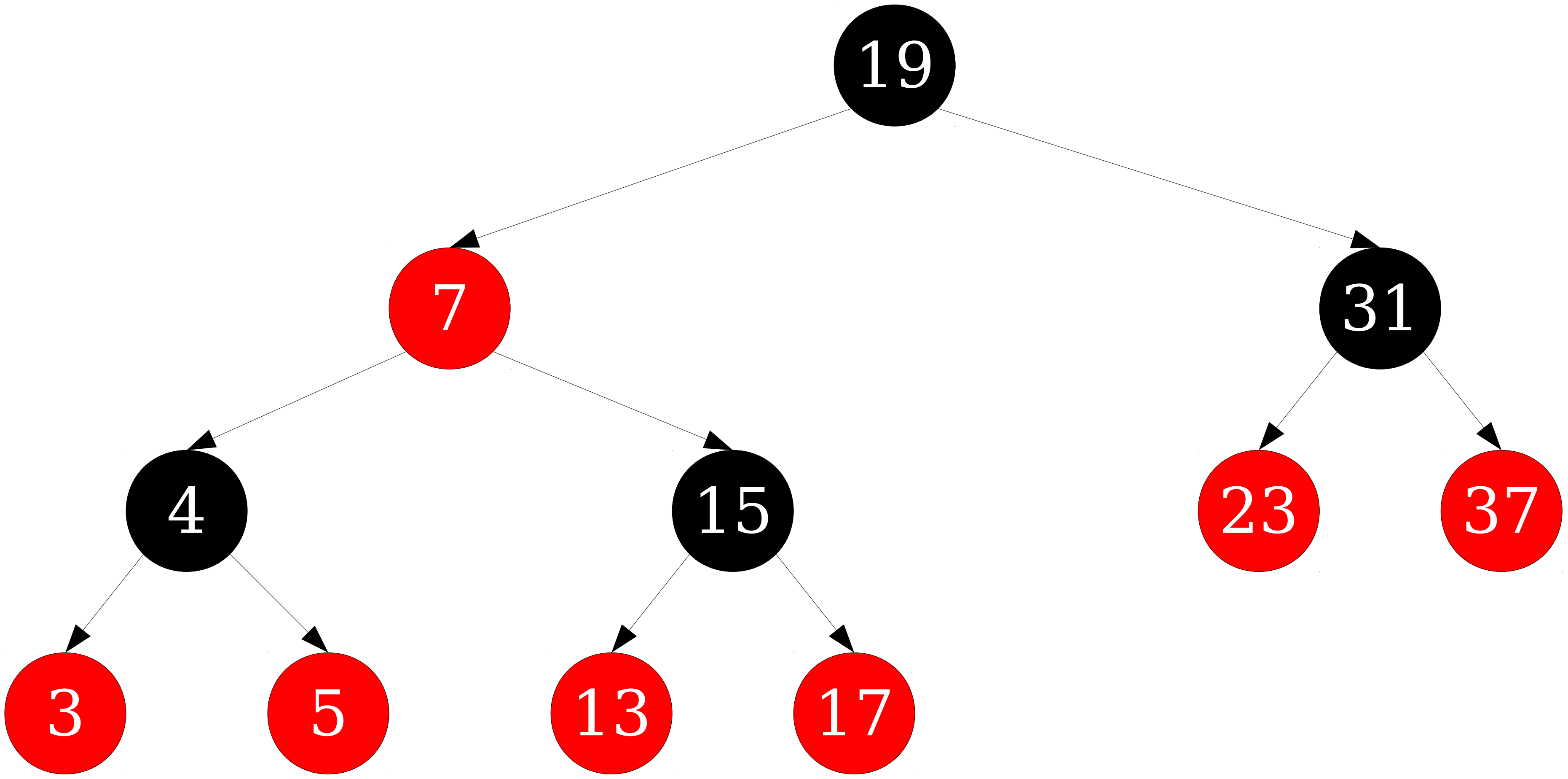
Using the Isometry



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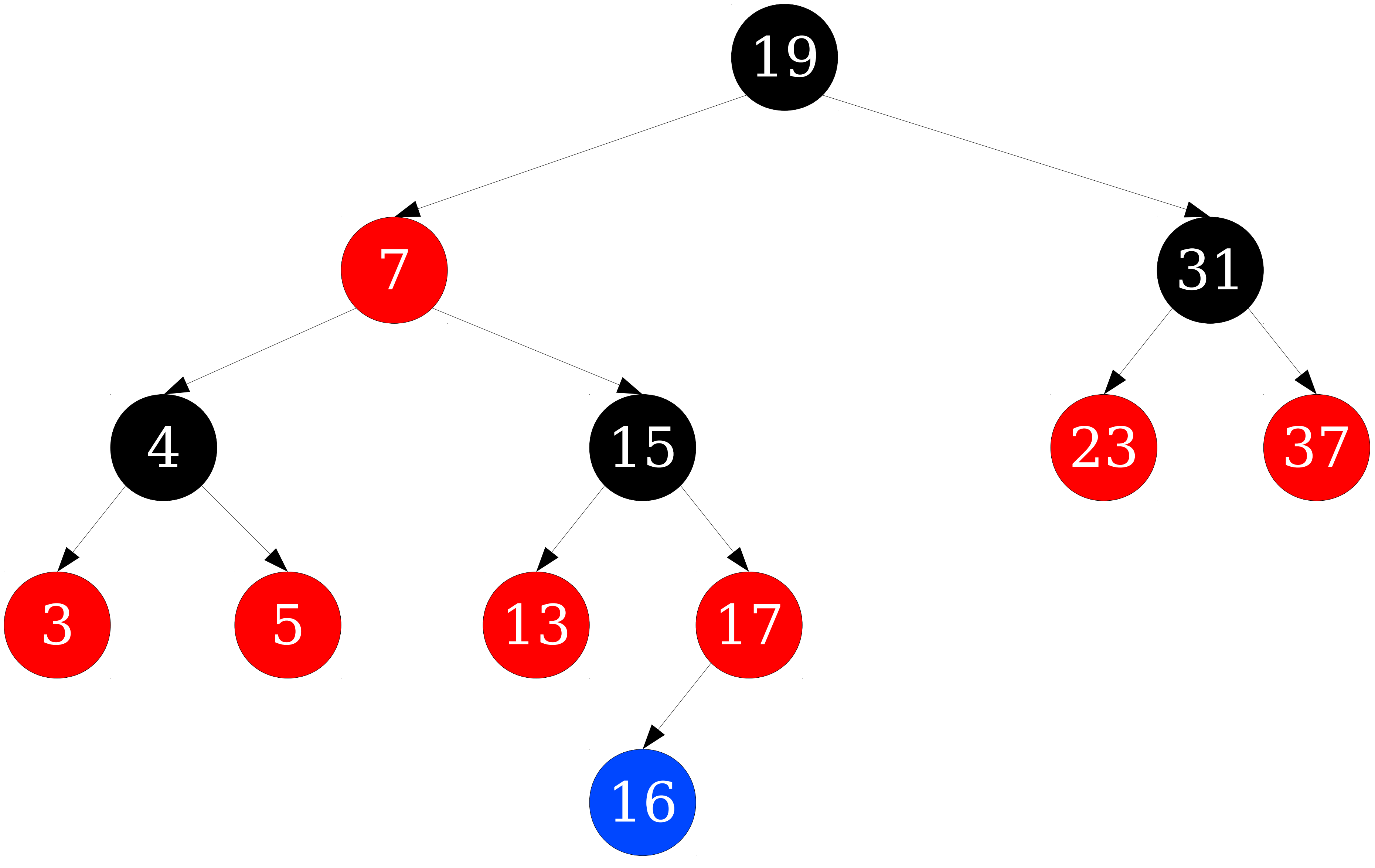

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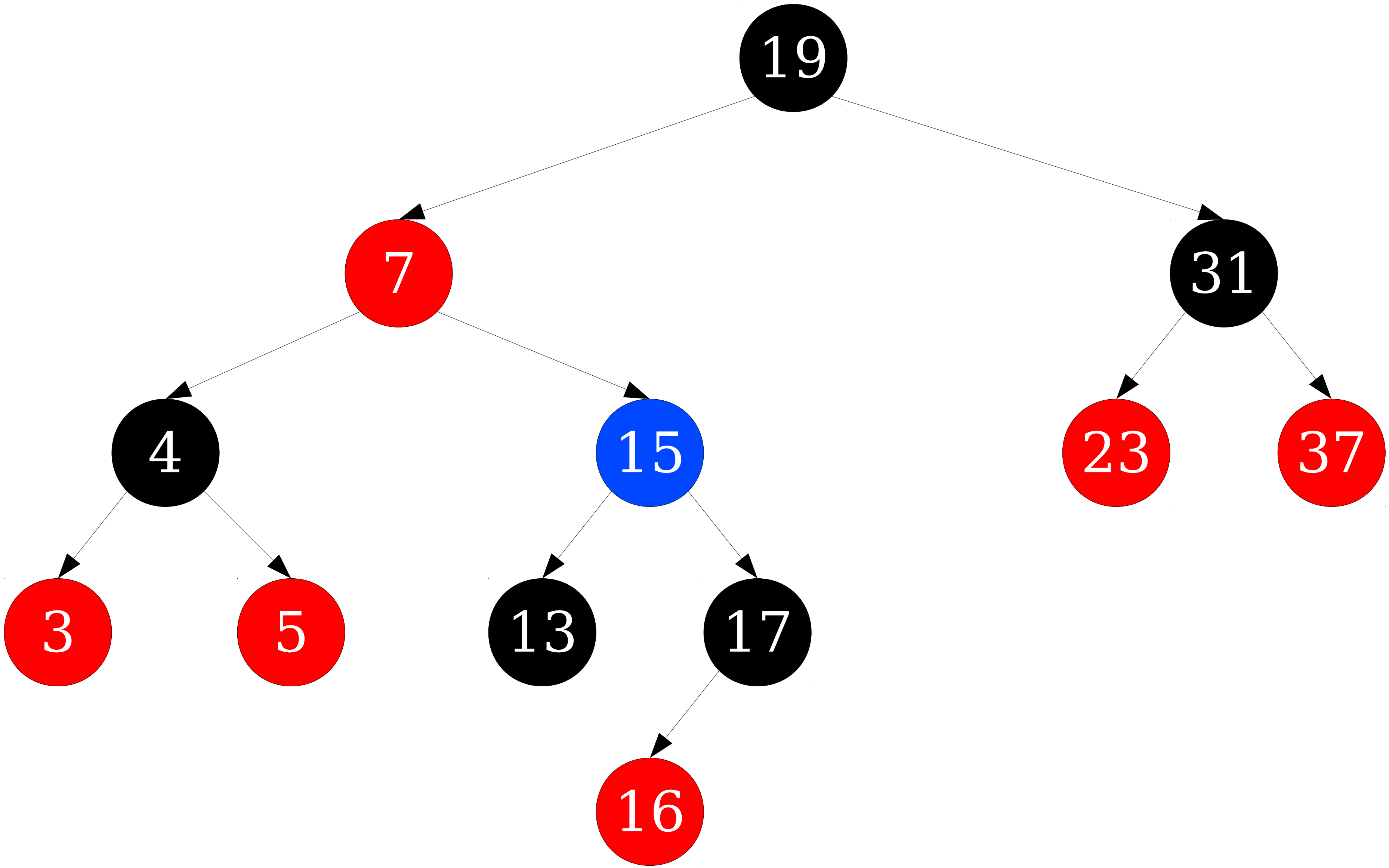
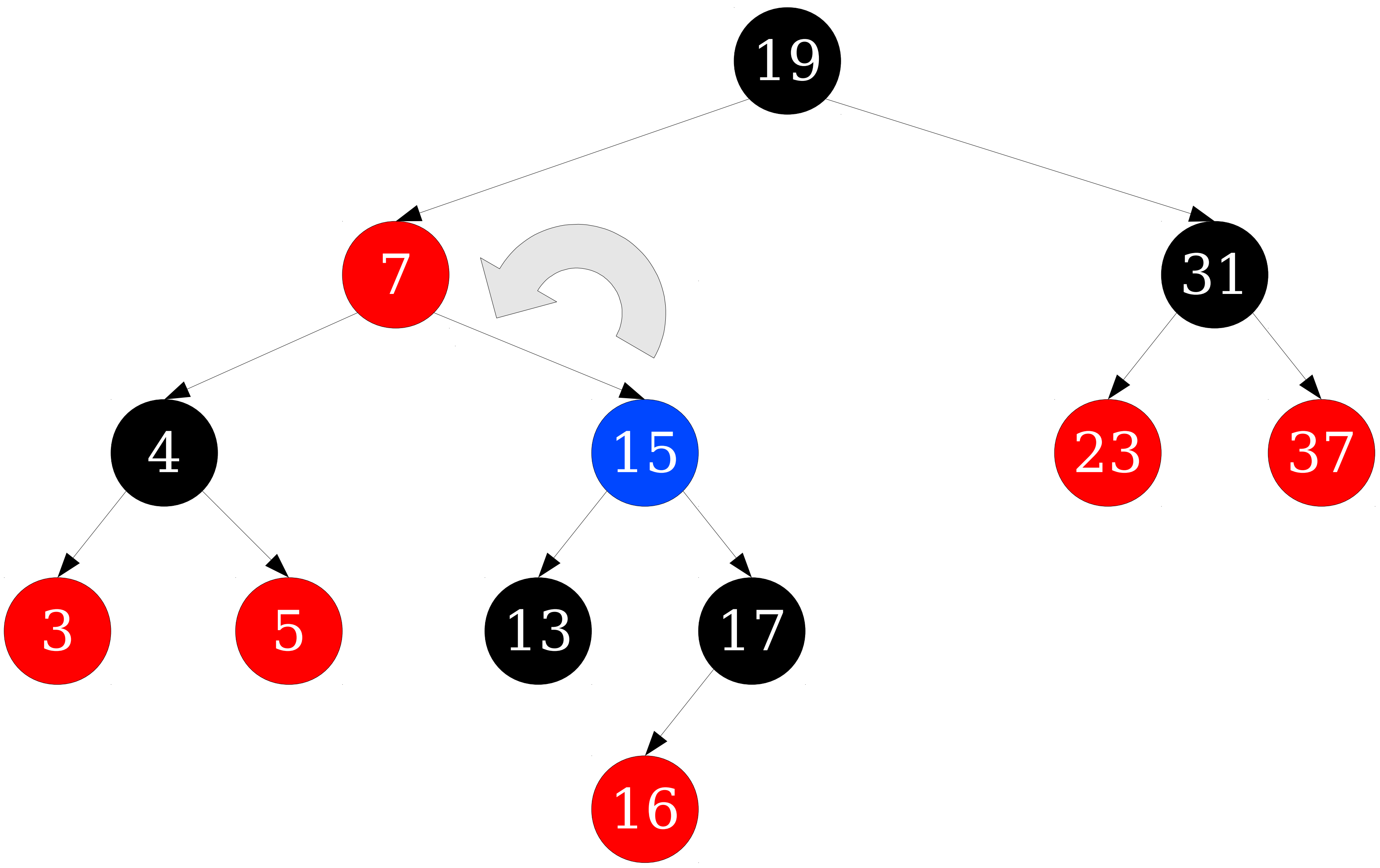
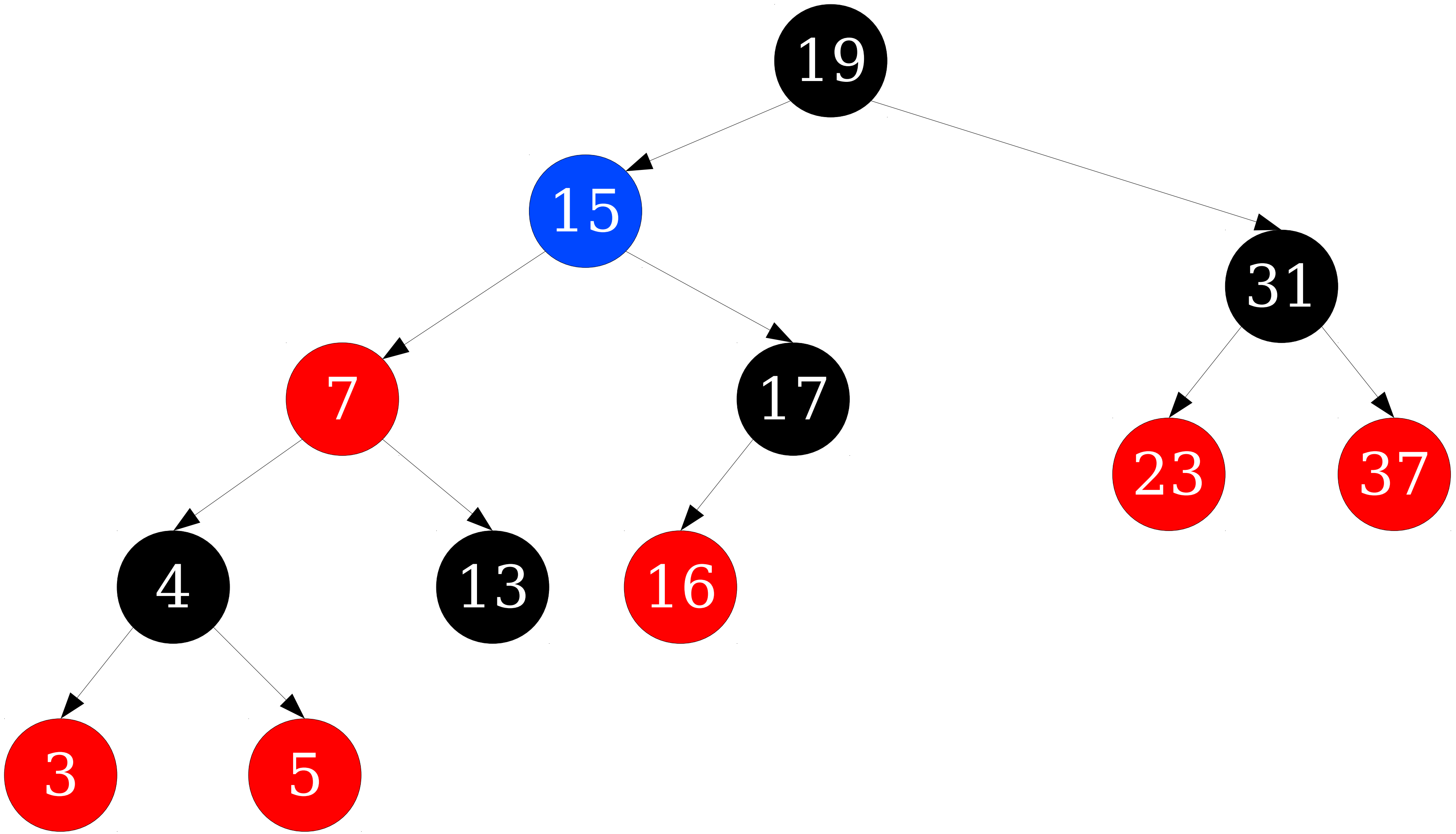


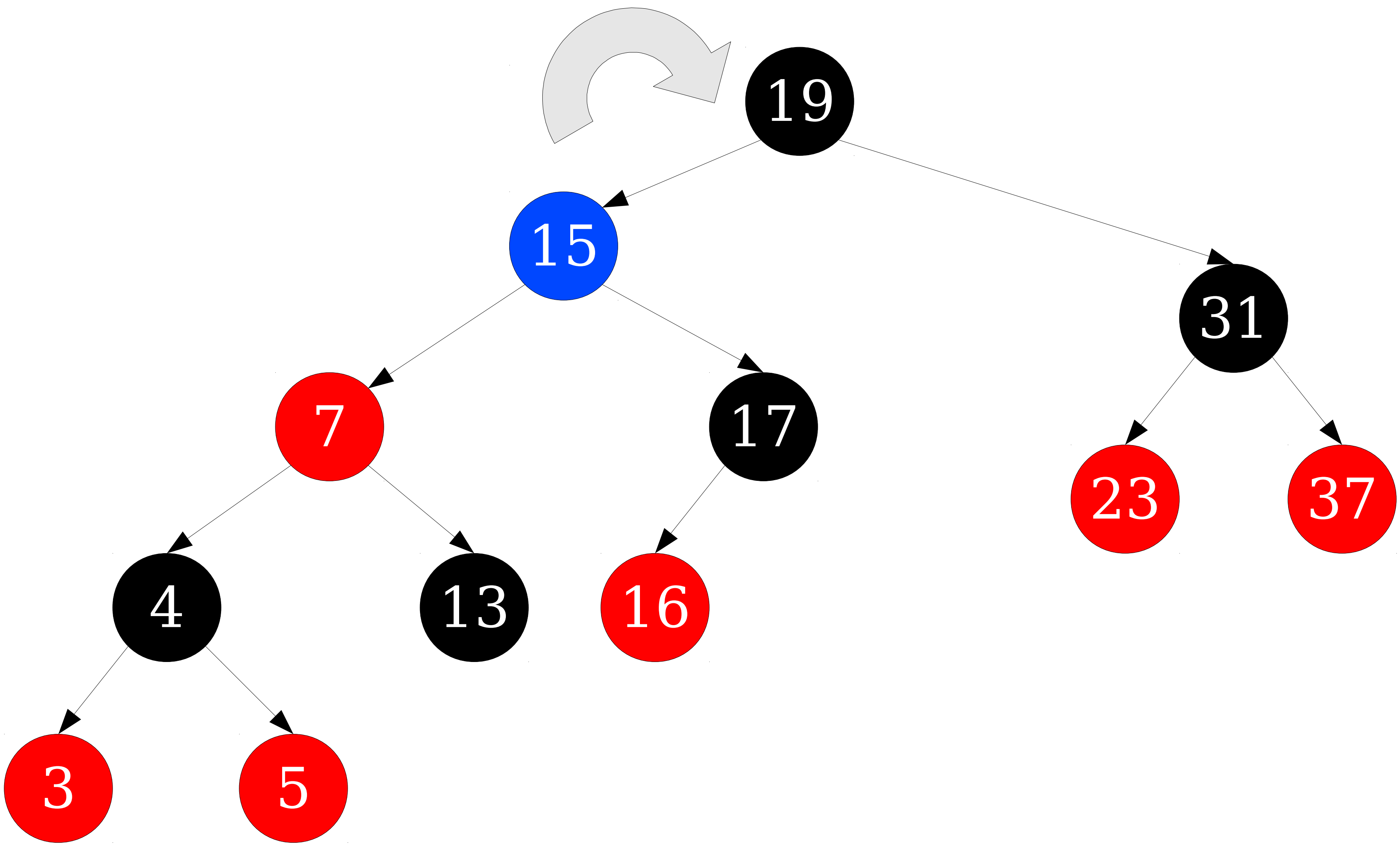
Using the Isometry



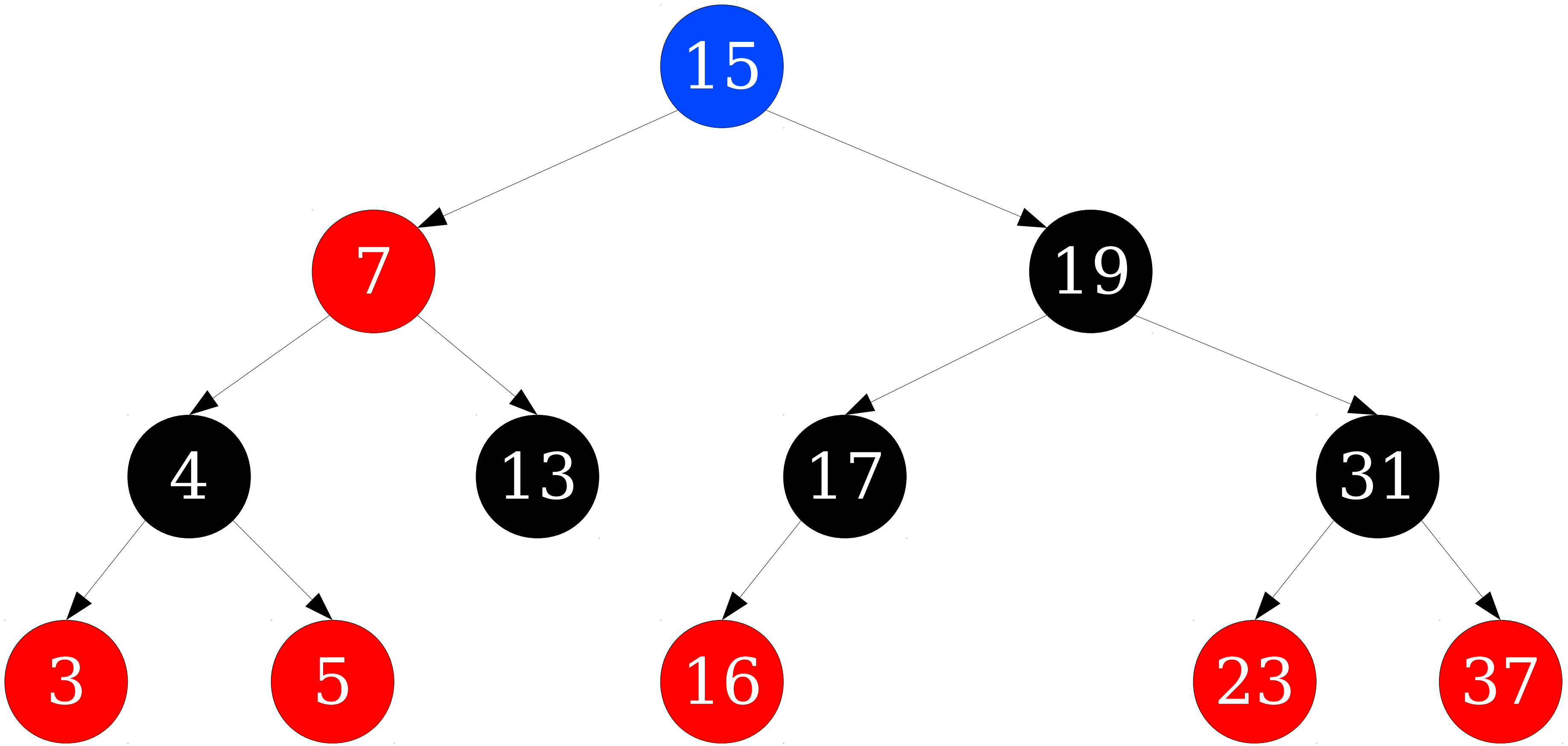
Using the Isometry



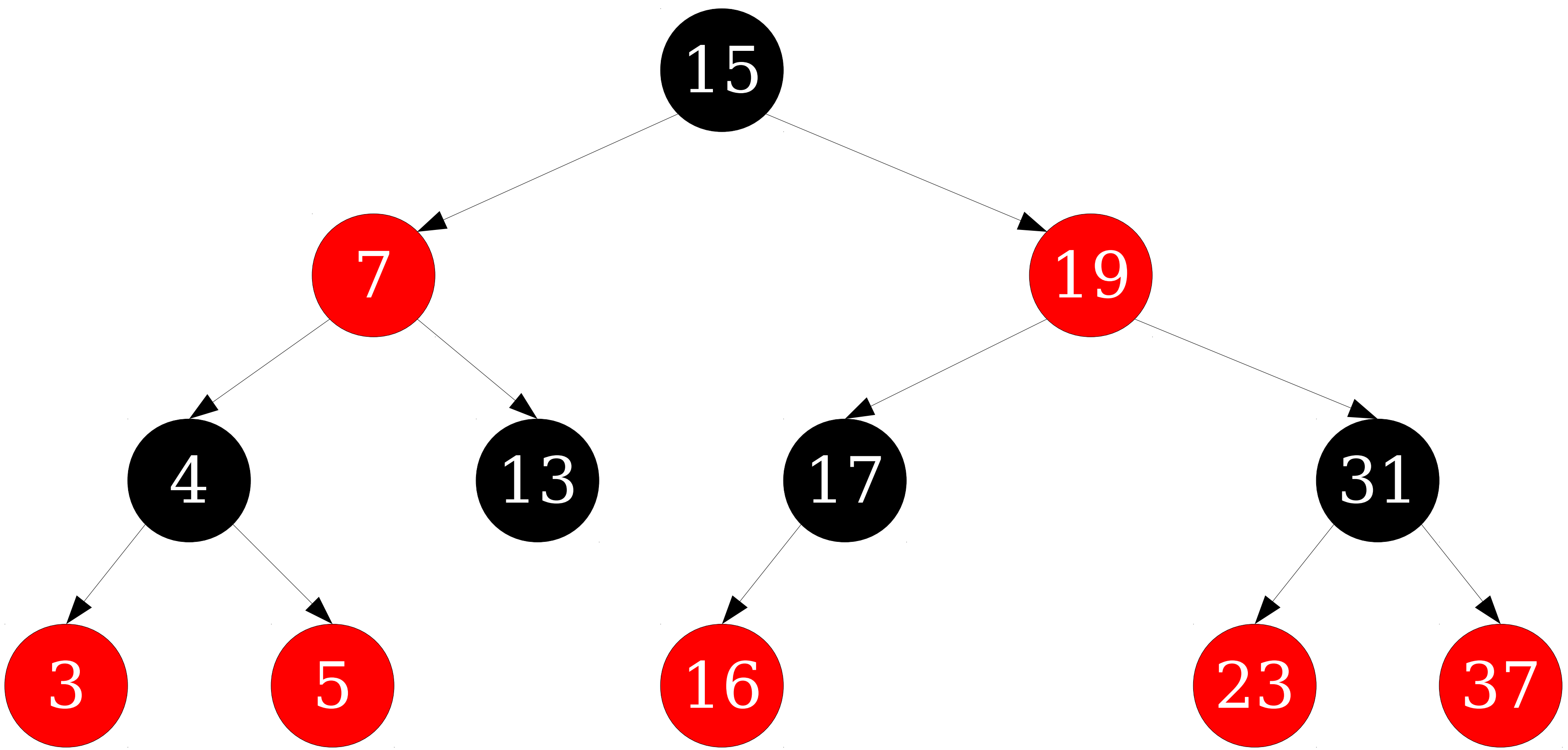
Using the Isometry



Using the Isometry



Using the Isometry



Building Up Rules

- All of the crazy insertion rules on red/black trees make perfect sense if you connect it back to 2-3-4 trees.
- There are lots of cases to consider because there are many different ways you can insert into a red/black tree.
- ***Main point:*** Simulating the insertion of a key into a node takes time O(1) in all cases. Therefore, since 2-3-4 trees support O(log *n*) insertions, red/black trees support O(log *n*) insertions.
- The same is true of deletions.

My Advice

- ***Do*** know how to do B-tree insertions and deletions.
  - You can derive these easily if you remember to split and join nodes.
- ***Do*** remember the rules for red/black trees and B-trees.
  - These are useful for proving bounds and deriving results.
- ***Do*** remember the isometry between red/black trees and 2-3-4 trees.
  - Gives immediate intuition for all the red/black tree operations.
- ***Don't*** memorize the red/black rotations and color flips.
  - This is rarely useful. If you're coding up a red/black tree, just flip open CLRS and translate the pseudocode. ☺

Next Time

- ***Augmented Trees***
  - Building data structures on top of balanced BSTs.
- ***Splitting and Joining Trees***
  - Two powerful operations on balanced trees.