Balanced Trees
Part One
Balanced Trees

- Balanced search trees are among the most useful and versatile data structures.
- Many programming languages ship with a balanced tree library.
  - C++: `std::map / std::set`
  - Java: `TreeMap / TreeSet`
  - Haskell: `Data.Map`
- Many advanced data structures are layered on top of balanced trees.
  - We'll see them used to build y-Fast Tries and (one of) Euler tour trees and link/cut trees.
Where We're Going

- **B-Trees**
  - A simple type of balanced tree developed for block storage.

- **Red/Black Trees**
  - The canonical balanced binary search tree.

- **Augmented Search Trees**
  - Adding extra information to balanced trees to supercharge the data structure.

- **Two Advanced Operations**
  - The split and join operations.
Outline for Today

- **BST Review**
  - Refresher on basic BST concepts and runtimes.

- **Overview of Red/Black Trees**
  - What we're building toward.

- **B-Trees and 2-3-4- Trees**
  - A simple balanced tree in depth.

- **Intuiting Red/Black Trees**
  - A much better feel for red/black trees.
A Quick BST Review
A **binary search tree** is a binary tree with the following properties:

- Each node in the BST stores a **key**, and optionally, some auxiliary information.
- The key of every node in a BST is strictly greater than all keys to its left and strictly smaller than all keys to its right.
- The **height** of a binary search tree is the length of the longest path from the root to a leaf, measured in the number of **edges**.
  - A tree with one node has height 0.
  - A tree with no nodes has height -1, by convention.
Case 1: If the node has just no children, just remove it.
Case 2: If the node has just one child, remove it and replace it with its child.
Deleting from a BST

**Case 3:** If the node has two children, find its inorder successor (which has zero or one child), replace the node's key with its successor's key, then delete its successor.
Runtime Analysis

- The time complexity of all these operations is $O(h)$, where $h$ is the height of the tree.
  - Represents the longest path we can take.
- In the best case, $h = O(\log n)$ and all operations take time $O(\log n)$.
- In the worst case, $h = \Theta(n)$ and some operations will take time $\Theta(n)$.
- **Challenge:** How do you efficiently keep the height of a tree low?
A Glimpse of Red/Black Trees
Red/Black Trees

- A **red/black tree** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.
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Red/Black Trees

- **Theorem:** Any red/black tree with $n$ nodes has height $O(\log n)$.
  - We could prove this now, but there's a much simpler proof of this we'll see later on.
- Given a fixed red/black tree, lookups can be done in time $O(\log n)$. 
Mutating Red/Black Trees
What are we supposed to do with this new node?
Fixing Up Red/Black Trees

- **The Good News:** After doing an insertion or deletion, can locally modify a red/black tree in time $O(\log n)$ to fix up the red/black properties.

- **The Bad News:** There are a *lot* of cases to consider and they're not trivial.

Some questions:

- How do you memorize / remember all the different types of rotations?
- How on earth did anyone come up with red/black trees in the first place?
B-Trees
Generalizing BSTs

- In a binary search tree, each node stores a single key.
- That key splits the “key space” into two pieces, and each subtree stores the keys in those halves.

![Binary Search Tree Diagram]

- Values less than two
- Values greater than two
Generalizing BSTs

- In a multiway search tree, each node stores an arbitrary number of keys in sorted order.

- In a node with \( k \) keys splits the “key space” into \( k + 1 \) pieces, and each subtree stores the keys in those pieces.
One Solution: **B-Trees**

- A **B-tree of order** $b$ is a multiway search tree with the following properties:
  - All leaf nodes are stored at the same depth.
  - All non-root nodes have between $b - 1$ and $2b - 1$ keys.
  - The root node has been $1$ and $2b - 1$ keys.
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The Tradeoff

- Because B-tree nodes can have multiple keys, when performing a search, insertion, or deletion, we have to spend more work inside each node.

- Insertion and deletion can be expensive – for large $b$, we might have to shuffle thousands or millions of keys over!

- Why would you use a B-tree?
Memory Tradeoffs

• There is an enormous tradeoff between *speed* and *size* in memory.

• SRAM (the stuff registers are made of) is fast but very expensive:
  • Can keep up with processor speeds in the GHz.
  • As of 2010, cost is $5/MB. *(Anyone know a good source for a more recent price?)*
  • Good luck buying 1TB of the stuff!

• Hard disks are cheap but very slow:
  • As of 2016, you can buy a 2TB hard drive for about $70.
  • As of 2016, good disk seek times are measured in ms (about two to four million times slower than a processor cycle!)
The Memory Hierarchy

- **Idea:** Try to get the best of all worlds by using multiple types of memory.
Why B-Trees?

- Because B-trees have a huge branching factor, they're great for on-disk storage.
  - Disk block reads/writes are glacially slow.
  - The high branching factor minimizes the number of blocks to read during a lookup.
  - Extra work scanning inside a block offset by these savings.
- Used extensively in databases, file systems, etc.
  - Typically, use a B+-tree rather than a B-tree, but idea is similar.
- Recently, have been gaining traction for main-memory data structures.
  - Memory cache effects offset extra searching costs.
The Height of a B-Tree

- What is the maximum possible height of a B-tree of order $b$?

$$\begin{align*}
&b - 1 \\
&b - 1 \quad \ldots \quad b - 1 \\
&\quad \vdots \\
&b - 1 \quad b - 1
\end{align*}$$
The Height of a B-Tree

- **Theorem:** The maximum height of a B-tree of order $b$ containing $n$ nodes is $\log_b \left( \frac{n + 1}{2} \right)$.

- **Proof:** Number of nodes $n$ in a B-tree of height $h$ is guaranteed to be at least

  $1 + 2(b - 1) + 2b(b - 1) + 2b^2(b - 1) + ... + 2b^{h-1}(b - 1)$

  $= 1 + 2(b - 1)(1 + b + b^2 + ... + b^{h-1})$

  $= 1 + 2(b - 1)((b^h - 1) / (b - 1))$

  $= 1 + 2(b^h - 1) = 2b^h - 1$

  Solving $n = 2b^h - 1$ yields $h = \log_b \left( \frac{n + 1}{2} \right)$. □

- **Corollary:** B-trees of order $b$ have height $O(\log_b n)$. 
Searching in a B-Tree

• Doing a search in a B-tree involves
  • searching the root node for the key, and
  • if it's not found, recursively exploring the correct child.

\[ \text{Time complexity} = O(\log \text{number-of-keys} \cdot \text{tree-height}) \]

Requires reading \( O(\log b \cdot \log n) \) blocks; this more directly accounts for the total runtime.
Searching in a B-Tree

- Doing a search in a B-tree involves
  - searching the root node for the key, and
  - if it's not found, recursively exploring the correct child.
- Using binary search within a given node, can find the key or the correct child in time $O(\log \text{number-of-keys})$.
- Repeat this process $O(\text{tree-height})$ times.
- Time complexity is
  
  $O(\log \text{number-of-keys} \cdot \text{tree-height})$
  
  $= O(\log b \cdot \log_b n)$
  
  $= O(\log b \cdot (\log n / \log b))$
  
  $= O(\log n)$

- Requires reading $O(\log_b n)$ blocks; this more directly accounts for the total runtime.
B-Trees are Simple

- Because nodes in a B-tree can store multiple keys, most insertions or deletions are straightforward.
- Here's a B-tree with $b = 3$ (nodes have between 2 and 5 keys):
The Trickier Cases

- What happens if you insert a key into a node that's too full?
- **Idea:** Split the node in two and propagate upward.
- Here's a 2-3-4 tree (each node has 1 to 3 keys).
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Note: B-trees grow upward, not downward.
Inserting into a B-Tree

To insert a key into a B-tree:

- Search for the key, insert at the last-visited leaf node.
- If the leaf is too big (contains $2b$ keys):
  - Split the node into two nodes of size $b$ each.
  - Remove the largest key of the first block and make it the parent of both blocks.
  - Recursively add that node to the parent, possibly triggering more upward splitting.

Time complexity:

- $O(b)$ work per level and $O(\log_b n)$ levels.
- Total work: $O(b \log_b n)$
- In terms of blocks read: $O(\log_b n)$
The Trickier Cases

- How do you delete from a leaf that has only $b - 1$ keys?
- **Idea:** Steal keys from an adjacent nodes, or merge the nodes if both are empty.
- Again, a 2-3-4 tree:
Deleting from a B-Tree

- If not in a leaf, replace the key with its successor from a leaf and delete out of a leaf.

- To delete a key from a node:
  - If the node has more than \( b - 1 \) keys, or if the node is the root, just remove the key.
  - Otherwise, find a sibling node whose shared parent is \( p \).
  - If that sibling has more than \( b - 1 \) keys, move the max/min key from that sibling into \( p \)'s place and \( p \) down into the current node, then remove the key.
  - Otherwise, fuse the node and its sibling into a single node by adding \( p \) into the block, then recursively remove \( p \) from the parent node.

- Work done is \( O(b \log_b n) \): \( O(b) \) work per level times \( O(\log_b n) \) total levels. Requires \( O(\log_b n) \) block reads/writes.
Time-Out for Announcements!
Amazingly, there aren't any announcements!

Let's just take a short break.
Back to CS166!
So... red/black trees?
Red/Black Trees

- A **red/black tree** is a BST with the following properties:
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Red/Black Trees ≡ 2-3-4 Trees

- Red/black trees are an *isometry* of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- Many data structures can be designed and analyzed in the same way.
- **Huge advantage:** Rather than memorizing a complex list of red/black tree rules, just think about what the equivalent operation on the corresponding 2-3-4 tree would be and simulate it with color flips and rotations.
The Height of a Red/Black Tree

**Theorem:** Any red/black tree with $n$ nodes has height $O(\log n)$.

**Proof:** Contract all red nodes into their parent nodes to convert the red/black tree into a 2-3-4 tree. This decreases the height of the tree by at most a factor of two. The resulting 2-3-4 tree has height $O(\log n)$, so the original red/black tree has height $2 \cdot O(\log n) = O(\log n)$. ■
Exploring the Isometry

- Nodes in a 2-3-4 tree are classified into types based on the number of children they can have.
  - **2-nodes** have one key (two children).
  - **3-nodes** have two keys (three children).
  - **4-nodes** have three keys (four children).
- How might these nodes be represented?
Exploring the Isometry
Using the Isometry

3 → 13 → 17 → 23 → 31

7 ← 19

5
Using the Isometry

3 → 5

13 → 17

23 → 31
Using the Isometry
Red/Black Tree Insertion

- **Rule #1:** When inserting a node, if its parent is black, make the node red and stop.

- **Justification:** This simulates inserting a key into an existing 2-node or 3-node.
Using the Isometry
Using the Isometry
Tree Rotations

Rotate Right

Rotate Left
Tree Rotations

- Tree rotations are a fundamental primitive on binary search trees.
- Rotations locally reorder nodes while preserving the binary search property.
- Most balanced trees use tree rotations at some point.
This applies any time we're inserting a new node into the middle of a “3-node.”

By making observations like these, we can determine how to update a red/black tree after an insertion.
Using the Isometry
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Two steps:

1. Split the “5-node” into a “2-node” and a “3-node.”
2. Insert the new parent of the two nodes into the parent node.
change colors
All of the crazy insertion rules on red/black trees make perfect sense if you connect it back to 2-3-4 trees.

There are lots of cases to consider because there are many different ways you can insert into a red/black tree.

**Main point:** Simulating the insertion of a key into a node takes time $O(1)$ in all cases. Therefore, since 2-3-4 trees support $O(\log n)$ insertions, red/black trees support $O(\log n)$ insertions.

The same is true of deletions.
My Advice

- **Do** know how to do B-tree insertions and deletions.
  - You can derive these easily if you remember to split and join nodes.
- **Do** remember the rules for red/black trees and B-trees.
  - These are useful for proving bounds and deriving results.
- **Do** remember the isometry between red/black trees and 2-3-4 trees.
  - Gives immediate intuition for all the red/black tree operations.
- **Don't** memorize the red/black rotations and color flips.
  - This is rarely useful. If you're coding up a red/black tree, just flip open CLRS and translate the pseudocode. 😊
Next Time

- **Augmented Trees**
  - Building data structures on top of balanced BSTs.

- **Splitting and Joining Trees**
  - Two powerful operations on balanced trees.