Suffix and LCP Arrays
Recap from Last Time
Theorem: $w$ is a substring of $x$ if and only if $w$ is a prefix of a suffix of $x$. 

012345678
New Stuff!
Representing Suffix Trees
We know that a suffix tree has $O(m)$ nodes, where $m$ is the number of characters in the input string.

This means that there are $O(m)$ edges.

**Question:** Why can’t we immediately claim that the space usage of the suffix tree is $O(m)$?
Representing a Suffix Tree

- **Claim:** Writing out all suffixes of a string of length $m$ requires $\Theta(m^2)$ characters.

- **Proof idea:** Those suffixes have length $1 + 2 + ... + (m+1)$, factoring in the special $\$ character.

- **Problem:** It is indeed possible to build a suffix tree with $\Theta(m^2)$ total letters on the edges.
Representing a Suffix Tree

- By being clever with our representation, we can guarantee that a suffix tree uses only $\Theta(m)$ space, regardless of the input string.
- **Observation:** Each edge is labeled with a substring of the original input string.
- **Idea:** Don’t actually write out the labels on the edges. Just write down the start and end index!
Representing a Suffix Tree

nonsense$

012345678

start
8 4 0 1 3
end
8 4 0 8 4
child

nonsense$
012345678
Representing a Suffix Tree

• Space usage required for a suffix tree:
  • $O(m)$ space for all the nodes.
  • $O(m)$ space for a copy of the original string.
  • $O(m)$ space for the edges.
• Total space: $O(m)$. 

Suffix Tree Space Usage

- Suffix tree edges take up a *lot* of space.
  - Two machine words per edge to denote the range of characters visited.
  - One machine word per edge for the pointer itself.
  - Number of edges ranges from $m$ to $2m - 1$, so this is between $3m$ and $6m$ machine words for the whole string!
- Example: a human genome is about three billion characters long.
  - With clever techniques, that can be packed into about 800MB.
  - On a 32-bit machine, the suffix tree needs about 48GB – too big to fit into memory!
  - On a 64-bit machine, the suffix tree needs about 96GB – way more than a typical machine can hold!
Key Question: Can we get the benefits of a suffix tree without the space penalty?
What is it about suffix trees that make them so useful algorithmically?
**Theorem:** There is a node labeled $\omega$ in a suffix tree for $T$ if and only if $\omega$ is a suffix of $T\$ or $\omega$ is a branching word in $T\$. 
A string $\omega$ is a **branching word** in $T\$ if there are distinct characters $a$ and $b$ where $\omega a$ and $\omega b$ are substrings of $T\$.

**Theorem:** There is a node labeled $\omega$ in a suffix tree for $T$ if and only if

- $\omega$ is a suffix of $T\$  
- $\omega$ is a branching word in $T\$.
Key Intuition: The efficiency in a suffix tree is largely due to 1. keeping the suffixes in sorted order, and 2. exposing branching words.
Where We’re Going

• Today, we’ll see two data structures that encode much of the same information as suffix trees, but in much less space.
  • The *suffix array* stores information about the ordering of the suffixes of a string.
  • The *LCP array* stores information about the branching words of a string.
• Together, they’ll provide algorithms that match or are comparable to the time bounds from last time.
Suffix Arrays
Theorem: There is a node labeled $\omega$ in a suffix tree for $T$ if and only if
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**Diagram:**

- Nodes labeled with $\$e$, $\ensense$, $\nonsense$, $\nse$, $\nsense$, $\onsense$, $\se$, and $\sense$.
- Edges connecting these nodes form a suffix tree.
- The root node is indicated by the label $\text{nonsense}\$.
- The leaves of the tree are marked with the sequence $012345678$. 

The diagram illustrates the structure of the suffix tree and how it relates to the theorem's conditions.
A *suffix array* for a string $T$ is a sorted array of the suffixes of the string $T$.

Suffix arrays distill out just the first component of suffix trees: they store suffixes in sorted order.

**Non-obvious fact:** Suffix arrays can be built in time $O(m)$. We can cover this later in the quarter if you’re interested.
Suffix Arrays

• The way we’ve drawn suffix arrays is terribly space-inefficient.
  • It always uses space $\Theta(m^2)$, since that’s how many total characters occur in all suffixes.

• Can we do better?
Suffix Arrays

• We reduced the space usage of suffix trees by representing substrings, implicitly, as ranges within the original string.

• **Idea:** Don’t store the suffixes themselves. Just store the starting positions of the suffixes.

• Space: $\Theta(m)$, and with only one machine word used per character of input.
Suffix Arrays

• Although the picture to the right is how we’d represent the suffix array in memory, for this lecture we’ll draw things out the longer way.

• This is just to build intuition; we wouldn’t actually do that in practice.
Using Suffix Arrays

- Last time, we saw how to find all instances of a pattern $P$ in a text $T$ using suffix trees.

- How could we do that with suffix arrays?
Using Suffix Arrays

- **Reminder:** Our text string \( T \) has length \( m \). Our pattern string \( P \) has length \( n \).

- **Claim:** With a suffix array, we can determine whether \( P \) appears in \( T \) in time \( O(n \log m) \).

Binay search has \( O(\log m) \) rounds. Each probe takes time \( O(n) \). This bound can be made tight.

Figure that \( m \) is often much bigger than \( n \), so this is a huge win over a raw scan.

How?

Answer at [https://pollev.com/cs166spr23](https://pollev.com/cs166spr23)
Using Suffix Arrays

- **Reminder:** Our text string $T$ has length $m$. Our pattern string $P$ has length $n$.

- **Claim:** With a suffix array, we can determine whether $P$ appears in $T$ in time $O(n \log m)$.
  - Binary search has $O(\log m)$ rounds.
  - Each probe takes time $O(n)$.
- This bound can be made tight. (*How?*)
- Figure that $m$ is often much bigger than $n$, so this is a huge win over a raw scan.
Using Suffix Arrays

• **Claim:** With a suffix array, we can find all matches of a pattern $P$ in $T$ in time $O(n \log m + z)$, where $z$ is the number of matches.

• **Idea:** Binary search can be used to find a range of values equal to some key. Adapt that idea to find all suffixes beginning with the same prefix.
The Story So Far

• Suffix arrays store all the suffixes of a string in sorted order.

• They provide an \(\langle O(m), O(n \log m + z)\rangle\) solution to the substring search problem.

• **Intuition:** Suffix trees are valuable in large part because they just keep the suffixes sorted.

• What else are suffix trees doing?
Theorem: There is a node labeled $\omega$ in a suffix tree for $T$ if and only if $\omega$ is a suffix of $T$ or $\omega$ is a branching word in $T$. 

$nonsense$ 
$012345678$
**Theorem:** There is a node labeled $\omega$ in a suffix tree for $T$ if and only if $\omega$ is a suffix of $T$ or $\omega$ is a branching word in $T$.
Branching Words

- **Recall:** If $T$ is a string, then $\omega$ is a branching word in $T$ if there are characters $a \neq b$ such that $\omega a$ and $\omega b$ are substrings of $T$.
Branching Words

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Although ABA is a repeated substring, it is not a branching word because all appearances are followed by N.
Branching Words

- **Recall:** If $T$ is a string, then $\omega$ is a branching word in $T$ if there are characters $a \neq b$ such that $\omega a$ and $\omega b$ are substrings of $T$.

The substring ANANA only appears once, so it’s not a branching word.
Branching Words

- Notice that, by sorting suffixes, we’ve made it easier to spot branching words.

- Specifically, all suffixes starting with a branching word will be adjacent in the suffix array.

- The branching word will be the *longest common prefix* (or *LCP*) of those adjacent suffixes.
Branching Words

- Notice that, by sorting suffixes, we’ve made it easier to spot branching words.
- Specifically, all suffixes starting with a branching word will be adjacent in the suffix array.
- The branching word will be the *longest common prefix* (or *LCP*) of those adjacent suffixes.
Theorem: A string $\omega$ is a branching word in string $T\$ if and only if it’s the longest common prefix of two adjacent suffixes in $T$’s suffix array.

Proof idea: If $\omega$ is the longest common prefix of two adjacent suffixes, let $a$ and $b$ be the characters immediately following $\omega$ in those two suffixes. Then $\omega a$ and $\omega b$ are substrings of $T\$. If $\omega$ is branching, choose the lexicographically smallest $a$ and $b$ making the definition work. Then the last suffix starting with $\omega a$ and the first suffix starting with $\omega b$ are adjacent in the suffix array. ■
ω is an internal node in the suffix tree for T if and only if
ω is a branching word in T$
if and only if
ω is the LCP of two adjacent suffixes in the suffix array for T
Key Intuition: Adjacent suffixes with long shared prefixes correspond to subtrees of the suffix tree.
Harnessing this Connection
Longest Repeated Substring

- Last time, we saw how to solve the longest repeated substring problem by using suffix trees.

- **Algorithm:** Find the internal node in the suffix tree with the longest label.

- **Question:** Can we do this with just a suffix array?
Longest Repeated Substring

- We can list all branching words from a suffix array in time $O(m^2)$.
  - $O(m)$ pairs; each pair takes time $O(m)$ to process.
- This worst-case bound can be realized.
- Contrast this with $O(m)$ for a suffix tree.
- Can we do better?

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ANABANDANA$ \\
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NDANA$
\end{array}$
**Longest Repeated Substring**

- **Observation:** We don’t actually need to know what all the branching words are to find the longest repeated substring.
- We just need to know how long they are.
- That way, we can figure out which is longest.
- Is there some nice way to do this?

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ABANANABANDANA$  
```
LCP Arrays
LCP Arrays

- The **LCP array**, often denoted $H$, is an array where $H[i]$ is the length of the LCP of the $i$th and $(i+1)$st suffixes in the suffix array.
- (The letter $H$ comes from “height.”)
Key intuition: The suffix array gives the leaves of the suffix tree. The LCP array gives the internal nodes of the suffix tree.
Using LCP Arrays

- If you already have a suffix array and LCP array, you can solve longest repeated substring in time $O(m)$:
  - Find the largest element in the LCP array.
  - Return the string it corresponds to.
- **Question**: How fast can we construct an LCP array?

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Building LCP Arrays
Building LCP Arrays

- It never hurts to start with the naive algorithm and see what happens!
- **Algorithm:** For each consecutive pair of strings in the suffix array, compute the length of their longest common prefix.
- We can upper-bound the runtime at $O(m^2)$.
- **Question:** Can we realize this upper bound?
Building LCP Arrays

- Why is our naive algorithm slow?
- **Intuition:** We aren’t able to carry work from one suffix over to the next.

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Building LCP Arrays

- **Key intuition:** Suffixes overlap one another! It should be possible to share LCP information across suffixes.
- For example, suppose we compute the LCP entry shown here.
- Look at the suffixes formed by dropping the first letter of these two suffixes.
- What do we know about their LCP?
Building LCP Arrays

- Let’s do another example. Suppose we know the LCP of these suffixes.
- As before, drop the first letter from each suffix.
- What can we say about the LCP of the resulting suffixes?
Building LCP Arrays

- Sometimes, in dropping the first letter, two adjacent suffixes get spread out.

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Building LCP Arrays

- Sometimes, in dropping the first letter, two adjacent suffixes get spread out.

- **Claim:** Look at the second suffix in the pair. Its LCP with the suffix before it is at least the previous LCP minus one.

- Think about the suffix tree. The two shorter suffixes are in the same subtree, so everything between them is also in that subtree.
Building LCP Arrays

- We know that these two new suffixes must have an LCP of at least 1, because the two old suffixes have an LCP of 2.
- However, the LCP may be longer than 1, since we’ve never seen one of these two suffixes.
- We still need to some scanning, but we won’t necessarily have to rescan the entire suffix.
Kasai’s Algorithm

- For each suffix of the original string, except the last:
  - Find that suffix in the suffix array.
  - Look at the suffix that comes before it.
  - (★) Find the length of the longest common prefix of those suffixes.
  - Write that down in the $H$ array.
  - Use the insight from the previous slides to speed up step (★).
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With $O(m)$ preprocessing time, can be done in time $O(1)$.

*Question to Ponder:* How would you do this?
Kasai’s Algorithm

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With $O(m)$ preprocessing time, can be done in time $O(1)$.

**Question to Ponder:** How would you do this?

The runtime of this step is proportional to how much the LCP increases on that step.

*Had to scan these characters*

*Already known to match*
Kasai’s Algorithm

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The runtime of this step is proportional to how much the LCP increases on that step.

The LCP value decreases by at most one per suffix. *(We saw this earlier.)*

The LCP value maxes out at $m$. *(Can’t match more than all the characters.)*

Therefore, the LCP value can grow at most $2m$ times. *(Prove this!)*

**Claim:** Across all iterations, this step takes a total of $O(m)$ time.
Kasai’s Algorithm

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Total runtime: $O(m)$. 
More to Explore

- We could easily spend a whole quarter talking about suffix arrays. Here’s what we didn’t cover:
  - **Bottom-up tree simulations:** Using LCP arrays, you can simulate any $O(m)$-time suffix tree algorithm that works with a bottom-up DFS in time $O(m)$.
  - **Faster substring searching:** Using LCP arrays, plus RMQ, you can improve the cost of a substring search to $O(n + z + \log m)$.
  - **Burrows-Wheeler transforms:** Suffix arrays, plus LCP arrays, can be used to significantly improve the performance of text compressors.
- Check these out – they’re super interesting!
Next Time

• *Amortized Analysis*
  • Lying in a runtime analysis.

• *The Potential Method*
  • Physics meets data structure design.