Balanced Trees
Part Two
Outline for Today

• **Red/Black Trees**
  • Using our isometry!

• **Tree Rotations**
  • A key primitive in restructuring trees.

• **Augmented Binary Search Trees**
  • Leveraging red/black trees.
Recap from Last Time
2-3-4 Trees

- A **2-3-4 tree** is a multiway search tree where
  - every node has 1, 2, or 3 keys,
  - any non-leaf node with \( k \) keys has exactly \( k+1 \) children, and
  - all leaves are at the same depth.

- To insert a key, place it in a leaf. If out of space, split the leaf and kick the median key one level higher, repeating this process.
A red/black tree is a BST with the following properties:

- Every node is either red or black.
- The root is black.
- No red node has a red child.
- Every root-null path in the tree passes through the same number of black nodes.
A **red/black tree** is a BST with the following properties:

- Every node is either red or black.
- The root is black.
- No red node has a red child.
- Every root-null path in the tree passes through the same number of black nodes.

After we hoist red nodes into their parents:

- Each “meta node” has 1, 2, or 3 keys in it. (No red node has a red child.)
- Each “meta node” is either a leaf or has one more child than key. (Root-null path property.)
- Each “meta leaf” is at the same depth. (Root-null path property.)
New Stuff!
Data Structure Isometries

- Red/black trees are an **isometry** of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- That gives us some really easy theorems basically for free.
- **Theorem:** The maximum height of a red/black tree with $n$ nodes is $O(\log n)$.
- **Proof idea:** Pulling red nodes into their parents forms a 2-3-4 tree with $n$ keys, which has height $O(\log n)$. Undoing this at most doubles the height of the tree. ■-ish
Exploring the Isometry

- Nodes in a 2-3-4 tree are classified into types based on the number of children they can have.
  - **2-nodes** have one key (two children).
  - **3-nodes** have two keys (three children).
  - **4-nodes** have three keys (four children).
- How might these nodes be represented?
Exploring the Isometry
Goal
Goal
Goal
add 21

add 21
Goal
Red/Black Tree Insertion

- **Rule #1**: When inserting a node, if its parent is black, make the node red and stop.

- **Justification**: This simulates inserting a key into an existing 2-node or 3-node.
Goal

We need to move nodes around in a binary search tree. How do we do this?
Tree Rotations
Tree Rotations
This applies any time we're inserting a new node into the middle of a “3-node” in this pattern.

By making observations like these, we can determine how to update a red/black tree after an insertion.
Goal
Goal
Goal
Goal
Building Up Rules

• The complex rules on red/black trees make perfect sense if you connect it back to 2-3-4 trees.
• There are lots of cases to consider because there are many different ways you can insert into a red/black tree.

**Main point:** Simulating the insertion of a key into a node takes time $O(1)$ in all cases. Therefore, since 2-3-4 trees support $O(\log n)$ insertions, red/black trees support $O(\log n)$ insertions.
• The same is true of deletions.
My Advice

• **Do** know how to do B-tree insertions and searches.
  • You can derive these easily if you remember to split nodes.

• **Do** remember the rules for red/black trees and B-trees.
  • These are useful for proving bounds and deriving results.

• **Do** remember the isometry between red/black trees and 2-3-4 trees.
  • Gives immediate intuition for all the red/black tree operations.

• **Don't** memorize the red/black rotations and color flips.
  • This is rarely useful. If you're coding up a red/black tree, just flip open CLRS and translate the pseudocode. ☺
Dynamic Problems
Classical Algorithms

• The “classical” algorithms model goes something like this:

   Given some input $X$, compute some interesting function $f(X)$.

• The input $X$ is provided up front, and only a single answer is produced.
Dynamic Problems

- **Dynamic versions** of problems are framed like this:

  > Given an input $X$ that can change in fixed ways, maintain $X$ while being able to compute $f(X)$ efficiently at any point in time.

- These problems are typically harder to solve efficiently than the “classical” static versions.
Dynamic Selection

• The *selection* problem is the following:

  *Given a list of distinct values and a number k, return the kth-smallest value.*

• In the static case, where the data set is fixed in advance and k is known, we can solve this in time $O(n)$ using quickselect or the median-of-medians algorithm.

• **Goal:** Solve this problem efficiently when the data set is changing – that is, the underlying set of elements can have insertions and deletions intermixed with queries.
Dynamic Selection

```
14

7

4 13

3 5 17

```

```
19

17

15

23

31
```
Dynamic Selection
Dynamic Selection

Diagram of a data structure with nodes labeled from 0 to 31, illustrating dynamic selection.
Dynamic Selection
Dynamic Selection
Dynamic Selection
Dynamic Selection

A diagram of a binary search tree with nodes containing numbers from 0 to 31.
Dynamic Selection
Dynamic Selection
Dynamic Selection
**Problem:** After inserting a new value, we may have to update $\Theta(n)$ values.

This is inherent in this solution route. These numbers track *global* properties of the tree.
Dynamic Selection
Dynamic Selection
Dynamic Selection
Dynamic Selection
If new nodes are added to the left subtree, the numbers on the right don’t need to update.
Dynamic Selection
Dynamic Selection

Diagram of a selection algorithm.
Dynamic Selection

```
0 3 2 5
1 4
3
7
5 14
1 17
4 13
2 19
1
1 31
0 23
```
Dynamic Selection

The diagram represents a selection tree with values 0, 1, 3, 4, 5, 7, 13, 14, 15, 17, 19, 23, and 31.
Dynamic Selection

Diagram of a dynamic selection algorithm with nodes labeled with values and arrows indicating the selection process.
Dynamic Selection

**Mechanically:** Number each key so that it only stores its order statistic in the subtree rooted at itself.

**Operationally:** Annotate each key with the number of keys in its left subtree.
Dynamic Selection
Dynamic Selection
Dynamic Selection

1? 5
3 7
1 4 0 13
0 5
1 17
0 15
1 31
0 23
Dynamic Selection
Dynamic Selection
Dynamic Selection
Dynamic Selection

9?
Dynamic Selection
Dynamic Selection

3

14

19

3?
Dynamic Selection
Dynamic Selection
Dynamic Selection
Dynamic Selection

Diagram of a binary search tree with node values.
Dynamic Selection
Dynamic Selection

Graph representation of a dynamic selection algorithm.
Dynamic Selection
We only update values on nodes that gained a new key in their left subtree. And there are only $O(\log n)$ of these!
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We only update values on nodes that gained a new key in their left subtree. And there are only $O(\log n)$ of these!
Dynamic Selection

![Binary Search Tree Diagram]

The diagram illustrates a binary search tree with node values ranging from 0 to 31, used to demonstrate dynamic selection.
Dynamic Selection

![Binary Tree](image)
Dynamic Selection

Dynamic Selection is a data selection algorithm that efficiently selects a specific rank of elements from a collection of elements. It uses a binary search tree as its underlying data structure to perform operations such as insertion, deletion, and selection in logarithmic time. The diagram illustrates a binary search tree with nodes containing values and pointers to their left and right subtrees.
Dynamic Selection
Dynamic Selection

![Dynamic Selection Diagram]
How do we update the numbers after the rotation?
\[ n_a \quad n_b \quad n_{b - n_a - 1} \]

- **Rotate Right**
  - \[ <A \quad >A \quad n_a \quad n_b \quad >B \]
  - \[ <A \quad >A \quad n_a \quad n_b \quad >B \]

- **Rotate Left**
  - \[ <A \quad >A \quad n_a \quad n_b \quad >B \]
  - \[ <A \quad >A \quad n_a \quad n_b \quad >B \]
\[ n_a + n_a + 1 \]

\[ n_b + n_a + 1 \]

\[ n_b - n_a - 1 \]
How do we update the numbers after the rotation?
Dynamic Selection
Dynamic Selection
Dynamic Selection
Dynamic Selection
This modified red/black tree is called an \textit{order statistics tree}.

- Start with a red/black tree.
- Tag each node with the number of nodes in its left subtree.
- Use the preceding update rules to preserve values during rotations.
- Propagate other changes up to the root of the tree.
- Only $O(\log n)$ values must be updated on an insertion or deletion and each can be updated in time $O(1)$.
- Supports all BST operations plus \textit{select} (find $k$th order statistic) and \textit{rank} (given a key, report its order statistic) in time $O(\log n)$. 
Generalizing our Idea
Edits to values are localized along the access path.
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Imagine we cache some value in each node that can be computed just from (1) the node itself and (2) its children’s values.
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Theorem: Suppose we want to cache some computed value in each node of a red/black tree. Provided that the value can be recomputed purely from the node’s value and from it’s children’s values, and provided that each value can be computed in time $O(1)$, then these values can be cached in each node with insertions, lookups, and deletions still taking time $O(\log n)$. 
Example: Hierarchical Clustering
1D Hierarchical Clustering
1D Hierarchical Clustering

| 20 | 42 | 44 | 60 | 66 | 71 | 86 | 92 | 100 |
1D Hierarchical Clustering

[Table with values: 20, 42, 44, 60, 66, 71, 86, 92, 100]
1D Hierarchical Clustering

\[
\begin{array}{cccccccc}
20 & 43 & 60 & 66 & 71 & 86 & 92 & 100 \\
42 & 44 \\
\end{array}
\]
1D Hierarchical Clustering

Diagram showing a 1D hierarchical clustering with numbers 20, 43, 60, 66, 71, 86, 92, 100, 42, and 44.
1D Hierarchical Clustering

Diagram showing a dendrogram with data points 20, 42, 44, 60, 66, 71, 86, 92, and 100 clustered in a hierarchical manner.
1D Hierarchical Clustering
1D Hierarchical Clustering
1D Hierarchical Clustering

```
20  43  60 | 68.5 | 86  92  100
  42  44
  66  71
```

```
| 20 | 42 | 44 | 60 | 66 | 71 | 86 | 92 | 100 |
```
1D Hierarchical Clustering

<table>
<thead>
<tr>
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20 42 44 60 66 71 86 92 100
1D Hierarchical Clustering
1D Hierarchical Clustering
1D Hierarchical Clustering

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![Cluster Diagram](image)
1D Hierarchical Clustering

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20  42  44  60  66  71  86  92  100
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1D Hierarchical Clustering

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| 86 | 92 |

20 42 44 60 66 71 86 92 100
1D Hierarchical Clustering

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20  43  65.67  89  100
  42  44  60  66  71  86  92
```

![Hierarchical Clustering Diagram]
1D Hierarchical Clustering

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Diagram of hierarchical clustering with numbers representing data points.
## 1D Hierarchical Clustering

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Diagram of a hierarchical clustering with numbers 20, 42, 44, 60, 66, 71, 86, 92, and 100.
1D Hierarchical Clustering

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Diagram of hierarchical clustering with numbers 20, 42, 44, 60, 66, 71, 86, 92, and 100.
1D Hierarchical Clustering

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Diagram showing the hierarchical clustering with values 20 to 100.
1D Hierarchical Clustering
This tree is called a *dendrogram*.
Analyzing the Runtime

• How efficient is this algorithm?
  • Number of rounds: $\Theta(n)$.
  • Work to find closest pair: $O(n)$.
  • Total runtime: $\Theta(n^2)$.

• Can we do better?
Analyzing the Runtime

How efficient is this algorithm?

Number of rounds: $\Theta(n)$.

- Work to find closest pair: $O(n)$.

Total runtime: $\Theta(n^2)$.

Can we do better?
Dynamic 1D Closest Points

• The **dynamic 1D closest points problem** is the following:

  Maintain a set of real numbers undergoing insertion and deletion while efficiently supporting queries of the form “what is the closest pair of points?”

• Can we build a better data structure for this?
Dynamic 1D Closest Points

\[ k \]

\[ \text{max} \]

\[ \text{min} \]
A Tree Augmentation

- Augment each node to store the following:
  - The maximum value in the tree.
  - The minimum value in the tree.
  - The closest pair of points in the tree.

- **Claim:** Each of these properties can be computed in time $O(1)$ from the left and right subtrees.

- These properties can be augmented into a red/black tree so that insertions and deletions take time $O(\log n)$ and “what is the closest pair of points?” can be answered in time $O(1)$. 
Dynamic 1D Closest Points

137

Min: ?
Max: ?
Closest: ?, ?

42

Min: -17
Max: 67
Closest: 15, 21

271

Min: 142
Max: 415
Closest: 300, 310
Dynamic 1D Closest Points

**42**
- Min: -17
- Max: 67
- Closest: 15, 21

**137**
- Min: -17
- Max: ?
- Closest: ?, ?

**271**
- Min: 142
- Max: 415
- Closest: 300, 310
Dynamic 1D Closest Points

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Dynamic 1D Closest Points

- **137**
  - Min: -17
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  - Closest: 137, 142

- **42**
  - Min: -17
  - Max: 67
  - Closest: 15, 21

- **271**
  - Min: 142
  - Max: 415
  - Closest: 300, 310
Some Other Questions

• How would you augment this tree so that you can efficiently (in time $O(1)$) compute the appropriate weighted averages?

• **Trickier:** Is this the fastest possible algorithm for this problem?
  
  • What if you’re guaranteed that the keys are all integers in some nice range?
A Helpful Intuition
Divide-and-Conquer

• Initially, it can be tricky to come up with the right tree augmentations.

• *Useful intuition:* Imagine you're writing a divide-and-conquer algorithm over the elements and have $O(1)$ time per “conquer” step.

\[
\begin{array}{c|c|c}
< k & k & > k \\
\end{array}
\]
Divide-and-Conquer

- Initially, it can be tricky to come up with the right tree augmentations.

- **Useful intuition:** Imagine you're writing a divide-and-conquer algorithm over the elements and have $O(1)$ time per "conquer" step.
Next Time

• **Amortized Analysis**
  • Lying about runtime costs in an honest manner.

• **Frameworks for Amortization**
  • How can we think about assigning costs?

• **Some Applications**
  • Building queues and handling red/black tree deletions