Binomial Heaps
Where We’re Going

- **Binomial Heaps (Today)**
  - A simple, flexible, and versatile priority queue.

- **Lazy Binomial Heaps (Today)**
  - A powerful building block for designing more advanced data structures.

- **Fibonacci Heaps (Tuesday)**
  - A famous and theoretically excellent priority queue.
Review: Priority Queues
Priority Queues

- A priority queue is a data structure that supports these operations:
  - `pq.enqueue(v, k)`, which enqueues element v with key k;
  - `pq.find-min()`, which returns the element with the least key; and
  - `pq.extract-min()`, which removes and returns the element with the least key.
- They’re useful as building blocks in a bunch of algorithms.
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Binary Heaps

• Priority queues are frequently implemented as *binary heaps*.
  • *enqueue* and *extract-min* run in time $O(\log n)$; *find-min* runs in time $O(1)$.
• These heaps are surprisingly fast in practice. It’s tough to beat their performance!
  • *d*-ary heaps can outperform binary heaps for a well-tuned value of $d$, and otherwise only the *sequence heap* is known to specifically outperform this family.
  • (Is this information incorrect as of 2023? Let me know and I’ll update it.)
• In that case, why do we need other heaps?
Many graph algorithms directly rely on priority queues supporting extra operations:

- **meld**($pq_1$, $pq_2$): Destroy $pq_1$ and $pq_2$ and combine their elements into a single priority queue. *(MSTs via Cheriton-Tarjan)*

- **$pq.decrease-key(v, k')$**: Given a pointer to element $v$ already in the queue, lower its key to have new value $k'$. *(Shortest paths via Dijkstra, global min-cut via Stoer-Wagner)*

- **$pq.add-to-all(\Delta k)$**: Add $\Delta k$ to the keys of each element in the priority queue, typically used with meld. *(Optimum branchings via Chu-Edmonds-Liu)*

In lecture, we'll cover binomial heaps to efficiently support meld and Fibonacci heaps to efficiently support meld and decrease-key.

You’ll design a priority queue supporting meld and add-to-all on the next problem set.
Priority Queues in Practice

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In lecture, we'll cover binomial heaps to efficiently support **meld** and Fibonacci heaps to efficiently support **meld** and **decrease-key**.

You’ll design a priority queue supporting **meld** and **add-to-all** on the next problem set.
Meldable Priority Queues
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• A priority queue supporting the *meld* operation is called a *meldable priority queue*.

• \textit{meld}(pq_1, pq_2) destructively modifies \(pq_1\) and \(pq_2\) and produces a new priority queue containing all elements of \(pq_1\) and \(pq_2\).
Meldable Priority Queues

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Meldable Priority Queues

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Meldable Priority Queues

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- \textit{meld}(pq_1, pq_2) destructively modifies \textit{pq}_1 and \textit{pq}_2 and produces a new priority queue containing all elements of \textit{pq}_1 and \textit{pq}_2.
Efficiently Meldable Queues

- Standard binary heaps do not efficiently support *meld*.
- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.
What things *can* be combined together efficiently?
Adding Binary Numbers

- Given the binary representations of two numbers \( n \) and \( m \), we can add those numbers in time \( O(\log m + \log n) \).

**Intuition:**
Writing out \( n \) in any “reasonable” base requires \( \Theta(\log n) \) digits.
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\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1
\end{array}
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- Represent $n$ and $m$ as a collection of “packets” whose sizes are powers of two.
- Adding together $n$ and $m$ can then be thought of as combining the packets together, eliminating duplicates

\[
\begin{array}{ccccccc}
1 & 0 & 1 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 \\
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\[
\begin{array}{cccc}
16 & 4 & 2 \\
+ & 8 & 4 & 2 & 1
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+ \\
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$$
\begin{array}{c}
32 \\
+ \\
\hline \\
4 \\
1
\end{array}
$$
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Building a Priority Queue

- **Idea:** Store elements in “packets” whose sizes are powers of two and *meld* by “adding” groups of packets.
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As long as the packets provide $O(1)$ access to the minimum, we can execute `find-min` in time $O(\log n)$. 
Building a Priority Queue

- What properties must our packets have?
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  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.

- **Idea:** Meld together the queue and a new queue with a single packet.
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Time required: $O(\log n)$ fuses.
Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.
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Deleting the Minimum

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- **Fun fact**: $2^k - 1 = 2^0 + 2^1 + 2^2 + \ldots + 2^{k-1}$.

- **Idea**: “Fracture” the packet into $k$ smaller packets, then add them back in.
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Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.
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Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.
- Runtime is $O(\log n)$ fuses in *meld*, plus fracture cost.
Building a Priority Queue

What properties must our packets have?
- Size is a power of two.
- Can efficiently fuse packets of the same size.
- Can efficiently find the minimum element of each packet.
- Can efficiently “fracture” a packet of $2^k$ nodes into packets of $2^0$, $2^1$, $2^2$, $2^3$, ..., $2^{k-1}$ nodes.

**Question:** How can we represent our packets to support the above operations efficiently?
Binomial Trees

- A **binomial tree of order k** is a type of tree recursively defined as follows:

  A binomial tree of order $k$ is a single node whose children are binomial trees of order 0, 1, 2, ..., $k - 1$.

- Here are the first few binomial trees:

![Binomial Trees Diagram]

Why are these called binomial heaps? Look across the layers of these trees and see if you notice anything!
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```
  2
 / \  /
6  4 3  7
 / \   / \\
8  5  1
```
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Make the binomial tree with the larger root the first child of the tree with the smaller root.
Binomial Trees

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The Binomial Heap

• A **binomial heap** is a collection of binomial trees stored in ascending order of size.

• Operations defined as follows:
  
  • **meld**(pq₁, pq₂): Use addition to combine all the trees.
    - Fuses O(log \( n \) + log \( m \)) trees. Cost: O(log \( n \) + log \( m \)). Here, assume one binomial heap has \( n \) nodes, the other \( m \).
  
  • **pq.enqueue**(v, k): Meld \( pq \) and a singleton heap of (v, k).
    - Total time: O(log \( n \)).
  
  • **pq.find-min**(): Find the minimum of all tree roots.
    - Total time: O(log \( n \)).
  
  • **pq.extract-min**(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: O(log \( n \)).
Draw what happens if we enqueue the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.
Draw what happens if we *enqueue* the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.
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Draw what happens after performing an *extract-min* in this binomial heap.
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Where We Stand

- Here’s the current scorecard for the binomial heap.
- This is a fast, elegant, and clever data structure.
- **Question:** Can we do better?

### Binomial Heap

- **enqueue:** $O(\log n)$
- **find-min:** $O(\log n)$
- **extract-min:** $O(\log n)$
- **meld:** $O(\log m + \log n)$. 
Where We Stand

• **Theorem:** No comparison-based priority queue structure can have **enqueue** and **extract-min** each take time $o(\log n)$.

• **Proof:** Suppose these operations each take time $o(\log n)$. Then we could sort $n$ elements by perform $n$ **enqueue**s and then $n$ **extract-mins** in time $o(n \log n)$. This is impossible with comparison-based algorithms. ■

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**Binomial Heap**

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• **find-min:** $O(\log n)$
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• **meld:** $O(\log m + \log n)$. 
Where We Stand

• We can’t make both *enqueue* and *extract-min* run in time $o(\log n)$.

• However, we could conceivably make one of them faster.

• **Question:** Which one should we prioritize?

• Probably *enqueue*, since we aren’t guaranteed to have to remove all added items.

• **Goal:** Make *enqueue* take time $O(1)$.

### Binomial Heap

- **enqueue**: $O(\log n)$
- **find-min**: $O(\log n)$
- **extract-min**: $O(\log n)$
- **meld**: $O(\log m + \log n)$. 
Where We Stand

- The *enqueue* operation is implemented in terms of *meld*.
- If we want *enqueue* to run in time $O(1)$, we’ll need *meld* to take time $O(1)$.
- How could we accomplish this?

Binomial Heap

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- *find-min*: $O(\log n)$
- *extract-min*: $O(\log n)$
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Thinking With Amortization
Refresher: Amortization

- In an amortized efficient data structure, some operations can take much longer than others, provided that previous operations didn’t take too long to finish.
- Think dishwashers: you may have to do a big cleanup at some point, but that’s because you did basically no work to wash all the dishes you placed in the dishwasher.

![Graph showing work over time](image-url)
Lazy Melding

- Consider the following lazy *meld*ing approach:

  *To meld together two binomial heaps, just combine the two sets of trees together.*

- **Intuition:** Why do any work to organize keys if we’re not going to do an *extract-min*? We’ll worry about cleanup then.
Lazy Melding

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Lazy Melding

- If we store our list of trees as circularly, doubly-linked lists, we can concatenate tree lists in time $O(1)$.
  - Cost of a *meld*: $O(1)$.
  - Cost of an *enqueue*: $O(1)$.
- If it sounds too good to be true, it probably is.
Lazy Melding

• Imagine that we implement *extract-min* the same way as before:
  • Find the packet with the minimum.
  • “Fracture” that packet to expose smaller packets.
  • Meld those packets back in with the master list.
• What happens if we do this with lazy melding?
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Each pass of finding the minimum value takes time $\Theta(n)$ in the worst case. We’ve lost our nice runtime guarantees!
Washing the Dishes

• Every *meld* (and *enqueue*) creates some “dirty dishes” (small trees) that we need to clean up later.

• If we never clean them up, then our *extract-min* will be too slow to be usable.

• **Idea:** Change *extract-min* to “wash the dishes” and make things look nice and pretty again.

• **Question:** What does “wash the dishes” mean here?
Washing the Dishes

- With our eager **meld** (and **enqueue**) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- **Idea:** After doing an **extract-min**, do a **coalesce step** to ensure there’s at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.
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At this point, the mess is cleaned up, and we’re left with what we would have had if we had been cleaning up as we go.
Where We’re Going

• A lazy binomial heap is a binomial heap, modified as follows:
  • The meld operation is lazy. It just combines the two groups of trees together.
  • After doing an extract-min, we do a coalesce to combine together trees until there’s at most one tree of each order.
  • Intuitively, we’d expect this to amortize away nicely, since the “mess” left by meld gets cleaned up later on by a future extract-min.

• Questions left to answer:
  • How do we efficiently implement the coalesce operation?
  • How efficient is this approach, in an amortized sense?
Coalescing Trees

- The *coalesce* step repeatedly combines trees together until there’s at most one tree of each order.
- How do we implement this so that it runs quickly?
Coalescing Trees

- **Observation:** This would be a *lot* easier to do if all the trees were sorted by size.
Coalescing Trees

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```
  3  7  2  1  3
 /   /   /   /
6   4   8   9
```

```
  8
```

```
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```
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- **Observation:** This would be a lot easier to do if all the trees were sorted by size.
- We can sort our group of $t$ trees by size in time $O(t \log t)$ using a standard sorting algorithm.
- **Better idea:** All the sizes are small integers. Use counting sort!
Coalescing Trees

• Here is a fast implementation of *coalesce*:
  • Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, ..., $\lceil \log_2 (n + 1) \rceil$.
  • Start at bucket 0. While there’s two or more trees in the bucket, fuse them and place the result one bucket higher.
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![Diagram](image)
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```
Order 3 | Order 2 | Order 1 | Order 0
---|---|---|---

5 7

3 5

3 4

1

4

5

6
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Analyzing Coalesce

**Claim:** Coalescing a group of \( t \) trees takes time \( O(t + \log n) \).

- Time to create the array of buckets: \( O(\log n) \).
- Time to distribute trees into buckets: \( O(t) \).
- Time to fuse trees: \( O(t + \log n) \)
  - Number of fuses is \( O(t) \), since each fuse decreases the number of trees by one. Cost per fuse is \( O(1) \).
  - Need to iterate across \( O(\log n) \) buckets.

**Total work done:** \( O(t + \log n) \).

- In the worst case, this is \( O(n) \).
The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - *enqueue*: $O(1)$
  - *meld*: $O(1)$
  - *find-min*: $O(1)$
  - *extract-min*: $O(n)$.

- But these are *worst-case* time bounds. Intuitively, things should nicely amortize away.
  - The number of trees grows slowly (one per *enqueue*).
  - The number of trees drops quickly (at most one tree per order) after an *extract-min*. 
An Amortized Analysis

- This is a great spot to use an amortized analysis by defining a potential function $\Phi$.
- In each case, the idea is to clearly mark what "messes" we need to clean up.
- In our case, each tree is a "mess," since our future coalesce operation has to clean it up.

Set $\Phi$ to the number of trees in the lazy binomial heap.
An Amortized Analysis

- **Recall:** We assign amortized costs as
  \[
  \text{amortized-cost} = \text{real-cost} + k \cdot \Delta \Phi,
  \]
  where \( \Delta \Phi = \Phi_{\text{after}} - \Phi_{\text{before}} \).
  - Increasing \( \Phi \) (adding more trees) artificially boosts costs.
  - Decreasing \( \Phi \) (removing trees) artificially lowers costs.
- Let’s work out the amortized costs of each operation on a lazy binomial heap.

Set \( \Phi \) to the number of trees in the lazy binomial heap.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest.

Set $\Phi$ to the number of trees in the lazy binomial heap.
Analyzing an Insertion

- To *enqueue* a key, we add a new binomial tree to the forest.
- Real cost: $O(1)$. $\Delta \Phi: +1$
- Amortized cost: $O(1)$.

Set $\Phi$ to the number of trees in the lazy binomial heap.
Analyzing a Meld

- What is the amortized cost of *meld*?
- The real cost is $O(1)$.
- What’s $\Delta \Phi$?
- That’s trickier – there are two separate collections of trees here.

Set $\Phi$ to the number of trees in the lazy binomial heap.
Analyzing a Meld

- What is the amortized cost of *meld*?

- **Common trick:** When working with mergeable data structures, define $\Phi$ globally across all instances of the data structure.

Set $\Phi$ to the number of trees in the lazy binomial heap.
Analyzing a Meld

• What is the amortized cost of *meld*?

• *Common trick*: When working with mergeable data structures, define $\Phi$ globally across all instances of the data structure.

Set $\Phi$ to the number of trees in *all* lazy binomial heaps.
Analyzing a Meld

- What is the amortized cost of *meld*?
- **Common trick:** When working with mergeable data structures, define $\Phi$ globally across all instances of the data structure.
- Now $\Delta \Phi = 0$ and the amortized cost is $O(1)$.

Set $\Phi$ to the number of trees in *all* lazy binomial heaps.
Analyzing *extract-min*
Find tree with minimum key.

Work: $O(t)$

$\Phi = t$

Remove min.
Add children to list of trees.

Work: $O(\log n)$

Run the coalesce algorithm.

Work: $O(t + \log n)$

$\Phi = O(\log n)$

Work: $O(t + \log n)$

$\Delta \Phi: O(-t + \log n)$
Find tree with minimum key.

Work: $O(t)$

$\Phi = t$

Remove min. Add children to list of trees.

Work: $O(\log n)$

Run the coalesce algorithm.

Work: $O(t + \log n)$

$\Phi = O(\log n)$

Amortized cost: $O(\log n)$. 
Analyzing Extract-Min

• Suppose we perform an *extract-min* on a lazy binomial heap with \( t \) trees in it.

• Initially, we fracture the tree containing the minimum. This increases the number of trees to \( t + O(\log n) \).

• The runtime for coalescing these trees is \( O(t + \log n) \).

• When we're done merging, there will be \( O(\log n) \) trees remaining, so \( \Delta \Phi = -t + O(\log n) \).

• Amortized cost is

\[
O(t + \log n) + k \cdot (-t + O(\log n)) \\
= O(t) - k \cdot t + k \cdot O(\log n) \\
= O(\log n).
\]
The Final Scorecard

- Here’s the final scorecard for our lazy binomial heap.
- These are great runtimes! We can’t improve upon this except by making extract-min worst-case efficient.
  - This is possible! Check out bootstrapped skew binomial heaps for details!

<table>
<thead>
<tr>
<th>Lazy Binomial Heap</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Insert</strong>: O(1)</td>
</tr>
<tr>
<td><strong>Find-Min</strong>: O(1)</td>
</tr>
<tr>
<td><strong>Extract-Min</strong>: O(log n)*</td>
</tr>
<tr>
<td><strong>Meld</strong>: O(1)</td>
</tr>
</tbody>
</table>

* amortized
Major Ideas from Today

• Isometries are a *great* way to design data structures.
  • Here, binomial heaps come from binary arithmetic.

• Designing for amortized efficiency is about building up messes slowly and rapidly cleaning them up.
  • Each individual *enqueue* isn’t too bad, and a single *extract-min* fixes all the prior problems.
Next Time

• **The Need for decrease-key**
  • A powerful and versatile operation on priority queues.

• **Fibonacci Heaps**
  • A variation on lazy binomial heaps with efficient *decrease-key*.

• **Analyzing Fibonacci Heaps**
  • A clever analysis.