Binomial Heaps
Outline for this Week

- **Binomial Heaps (Today)**
  - A simple, flexible, and versatile priority queue.

- **Lazy Binomial Heaps (Today)**
  - A powerful building block for designing advanced data structures.

- **Fibonacci Heaps (Thursday)**
  - A heavyweight and theoretically excellent priority queue.
Review: Priority Queues
Priority Queues

• A priority queue is a data structure that stores a set of elements annotated with totally-ordered keys and allows efficient extraction of the element with the least key.

• More concretely, supports these operations:
  • \texttt{pq.enqueue}(v, k), which enqueues element \( v \) with key \( k \);
  • \texttt{pq.find-min}(), which returns the element with the least key; and
  • \texttt{pq.extract-min}(), which removes and returns the element with the least key,
Binary Heaps

- Priority queues are frequently implemented as *binary heaps*.
- *enqueue* and *extract-min* run in time $O(\log n)$; *find-min* runs in time $O(1)$.
- We're not going to cover binary heaps this quarter; I assume you've seen them before.
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Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - `meld(pq_1, pq_2)`: Destroy `pq_1` and `pq_2` and combine their elements into a single priority queue.
  - `pq.decrease-key(v, k')`: Given a pointer to element `v` already in the queue, lower its key to have new value `k'`.
  - `pq.add-to-all(Δk)`: Add `Δk` to the keys of each element in the priority queue (typically used with `meld`).

- In lecture, we'll cover binomial heaps to efficiently support `meld` and Fibonacci heaps to efficiently support `meld` and `decrease-key`.

- You'll design a priority queue supporting efficient `meld` and `add-to-all` on the problem set.
Meldable Priority Queues

- A priority queue supporting the \textit{meld} operation is called a \textit{meldable priority queue}.
- \textit{meld}(pq_1, pq_2) destructively modifies \(pq_1\) and \(pq_2\) and produces a new priority queue containing all elements of \(pq_1\) and \(pq_2\).
Meldable Priority Queues

• A priority queue supporting the *meld* operation is called a *meldable priority queue*.

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Efficiently Meldable Queues

- Standard binary heaps do not efficiently support *meld*.
- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.
Binomial Heaps

• The *binomial heap* is an priority queue data structure that supports efficient melding.

• We'll study binomial heaps for several reasons:
  
  • Implementation and intuition is totally different than binary heaps.
  
  • Used as a building block in other data structures (Fibonacci heaps, soft heaps, etc.)
  
  • Has a beautiful intuition; similar ideas can be used to produce other data structures.
Supporting Efficient Melding
The Intuition: *Binary Arithmetic*
Adding Binary Numbers

- Given the binary representations of two numbers $n$ and $m$, we can add those numbers in time $\Theta(\max\{\log m, \log n\})$. 

\[
\begin{array}{ccccccc}
1 & 0 & 1 & 1 & 1 & 0 \\
\hline
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
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+ \\
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\[
\begin{array}{c}
16 \\
8 \\
4 \\
1 \\
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\[ \underline{32} \quad \underline{4} \quad \underline{1} \]
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Building a Priority Queue

- **Idea:** Adapt this approach to build a priority queue.
- Store elements in the priority queue in “packets” whose sizes are powers of two.
- Store packets in ascending size order.
- We'll choose a representation of a packet so that two packets of the same size can easily be fused together.
+ 

\[
\begin{array}{cccc}
64 & 97 & 84 & 23 \\
41 & 93 & 62 & 59 \\
53 & 26 & 58 & 31
\end{array}
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As long as the packets provide $O(1)$ access to the minimum, we can execute $\text{find-min}$ in time $O(\log n)$. 
Building a Priority Queue

• What properties must our packets have?
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  • Can efficiently find the minimum element of each packet.
Inserting into the Queue

• If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.

• **Idea:** Meld together the queue and a new queue with a single packet.
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Time required: $O(\log n)$ fuses.
Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.
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Fracturing Packets

• If we have a packet with $2^k$ elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.

• **Fun fact**: $2^k - 1 = 1 + 2 + 4 + \ldots + 2^{k-1}$.

• **Idea**: “Fracture” the packet into $k - 1$ smaller packets, then add them back in.
Fracturing Packets

- We can \textit{extract-min} by fracturing the packet containing the minimum and adding the fragments back in.
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• Runtime is $O(\log n)$ fuses in *meld*, plus fragment cost.
Building a Priority Queue

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  • Size must be a power of two.
  • Can efficiently fuse packets of the same size.
  • Can efficiently find the minimum element of each packet.
  • Can efficiently “fracture” a packet of $2^k$ nodes into packets of $1, 2, 4, 8, \ldots, 2^{k-1}$ nodes.

• What representation of packets will give us these properties?
Binomial Trees

- A binomial tree of order $k$ is a type of tree recursively defined as follows:

  A binomial tree of order $k$ is a single node whose children are binomial trees of order 0, 1, 2, ..., $k - 1$.

- Here are the first few binomial trees:
Binomial Trees

- **Theorem:** A binomial tree of order $k$ has exactly $2^k$ nodes.

- **Proof:** Induction on $k$. Assuming that binomial trees of orders 0, 1, 2, ..., $k - 1$ have $2^0$, $2^1$, $2^2$, ..., $2^{k-1}$ nodes, then then number of nodes in an order-$k$ binomial tree is

\[
2^0 + 2^1 + ... + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k
\]

So the claim holds for $k$ as well. ■
Binomial Trees

• A *heap-ordered binomial tree* is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.

• We will use heap-ordered binomial trees to implement our “packets.”
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The Binomial Heap

- A **binomial heap** is a collection of heap-ordered binomial trees stored in ascending order of size.

- Operations defined as follows:
  - **meld**\((pq_1, pq_2)\): Use addition to combine all the trees.
    - Fuses \(O(\log n)\) trees. Total time: \(O(\log n)\).
  - **pq.enqueue\((v, k)\)**: Meld \(pq\) and a singleton heap of \((v, k)\).
    - Total time: \(O(\log n)\).
  - **pq.find-min()**: Find the minimum of all tree roots.
    - Total time: \(O(\log n)\).
  - **pq.extract-min()**: Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: \(O(\log n)\).
An Issue of Representation

- Binomial trees are *logically* multiway trees, but are typically *implemented* as binary trees.
- We use the *left-child/right-sibling* representation.
- Each node’s left pointer points to its first child.
- Each node’s right pointer points to its next sibling.
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- The LCRS representation of binomial trees improves efficiency.
- Fusion takes time $O(1)$.
- Fracturing takes time $O(\log n)$. 
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Time-Out for Announcements!
Stanford Women in Computer Science

Casual Dinner

{w}

Tuesday, May 1st from 6-7 PM at Gates 403

Come mingle with CS professors and friends with delicious Lotus Thai!
The Final Project

- We’ve just posted information online (and in hardcopy here) about the CS166 final project.

- The quick summary:
  - Work in teams of two or three.
  - Pick a data structure, algorithm, or technique of your choice.
  - Become experts on it. Put together a writeup and presentation on the topic.
  - Do something “interesting” with it. You have broad latitude how to interpret what “interesting” means – pick something you’re excited about!

- Projects and presentations are due in the last week of class. They’re usually a highlight of the quarter for everyone involved!
Project Proposals

• Before working on the project, you’ll need to submit a proposal about what you’d like to work on.

• Your proposal should consist of a ranked list of five data structures you’d be interested in exploring, along with some preliminary information about each one.

• The proposal is due next Thursday, May 10th at 2:30PM.

• We’ll do a global matchmaking to assign topics over that weekend.
Problem Sets

• Problem Set Three is due Thursday at 2:30PM.
  • There’s plenty of space to ask us questions – let us know what we can do to help out!

• Problem Set Four will go out next Tuesday. You’ll have a little gap between those problem sets.
  • We recommend using this gap to work on or think about your final project proposals.
Back to CS166!
Analyzing Insertions

- Each `enqueue` into a binomial heap takes time $O(\log n)$, since we have to meld the new node into the rest of the trees.

- However, it turns out that the amortized cost of an insertion is lower in the case where we do a series of $n$ insertions.
Adding One

- Suppose we want to execute $n++$ on the binary representation of $n$.

- Do the following:
  - Find the longest span of 1's at the right side of $n$.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.

1 0 1 1 1 0
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  - Find the longest span of 1's at the right side of \( n \).
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.
Adding One

• Suppose we want to execute $n++$ on the binary representation of $n$.

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1 1 0 1 1 0
Adding One

- Suppose we want to execute $n++$ on the binary representation of $n$.
- Do the following:
  - Find the longest span of 1's at the right side of $n$.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.
- Runtime: $\Theta(b)$, where $b$ is the number of bits flipped.
An Amortized Analysis

• **Claim**: Starting at zero, the amortized cost of adding one to the total is $O(1)$.

• **Idea**: Use as a potential function the number of 1's in the number.

\[ \Phi = 0 \]

0 0 0 0 0 0 0
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.
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<table>
<thead>
<tr>
<th>Actual cost: 1</th>
<th>$\Delta \Phi$: +1</th>
<th>Amortized cost: 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi = 1$</td>
<td>0 0 0 0 0 0 1</td>
<td></td>
</tr>
</tbody>
</table>
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\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
\]
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0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0
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An Amortized Analysis

• **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.

• **Idea:** Use as a potential function the number of 1's in the number.

\[
\Phi = 1 \\
0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0
\]

- Actual cost: 2
- $\Delta \Phi$: 0
- Amortized cost: 2
An Amortized Analysis

• **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.

• **Idea:** Use as a potential function the number of 1's in the number.

$\Phi = 2$
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.

- **Idea:** Use as a potential function the number of 1's in the number.

\[
\Phi = 2 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1
\]

- Actual cost: 1
- $\Delta \Phi$: 1
- Amortized cost: 2
An Amortized Analysis

- **Claim**: Starting at zero, the amortized cost of adding one to the total is $O(1)$.

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\[ \Phi = 2 \]

0 0 0 0 1 1
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\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
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\[
\Phi = 1 \\
0 \ 0 \ 1 \ 0 \ 0 \ 0
\]

Actual cost: 3  
\(\Delta \Phi: -1\)  
Amortized cost: 2
Properties of Binomial Heaps

- Starting with an empty binomial heap, the amortized cost of each insertion into the heap is $O(1)$, assuming there are no deletions.

- **Rationale:** Binomial heap operations are isomorphic to integer arithmetic.

- Since the amortized cost of incrementing a binary counter starting at zero is $O(1)$, the amortized cost of enqueueing into an initially empty binomial heap is $O(1)$. 
Binomial vs Binary Heaps

• Interesting comparison:
  • The cost of inserting \( n \) elements into a binary heap, one after the other, is \( \Theta(n \log n) \) in the worst-case.
  • If \( n \) is known in advance, a binary heap can be constructed out of \( n \) elements in time \( \Theta(n) \).
  • The cost of inserting \( n \) elements into a binomial heap, one after the other, is \( \Theta(n) \), even if \( n \) is not known in advance!
A Catch

- This amortized time bound does not hold if *enqueue* and *extract-min* are intermixed.
- *Intuition*: Can force expensive insertions to happen repeatedly.
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![Diagram](image)
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- \textbf{Intuition:} Can force expensive insertions to happen repeatedly.
**Question:** Can we make insertions amortized $O(1)$, regardless of whether we do deletions?
Where's the Cost?

- Why does *enqueue* take time $O(\log n)$?

  - **Answer**: May have to combine together $O(\log n)$ different binomial trees together into a single tree.

- **New Question**: What happens if we don't combine trees together?

  - That is, what if we just add a new singleton tree to the list?
Lazy Melding

- More generally, consider the following lazy melding approach:

  To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$.
Lazy Melding

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  To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$. 

![Diagram of trees](image-url)
Lazy Melding

• More generally, consider the following lazy melding approach:

To meld together two binomial heaps, just combine the two sets of trees together.

• If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$.
The Catch: Part One

- When we use eager melding, the number of trees is $O(\log n)$.
- Therefore, $\textit{find-min}$ runs in time $O(\log n)$.
- **Problem:** $\textit{find-min}$ no longer runs in time $O(\log n)$ because there can be $\Theta(n)$ trees.
A Solution

• Have the binomial heap store a pointer to the minimum element.

• Can be updated in time O(1) after doing a meld by comparing the minima of the two heaps.
A Solution

- Have the binomial heap store a pointer to the minimum element.
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- Can be updated in time $O(1)$ after doing a meld by comparing the minima of the two heaps.
The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.
- **Rationale:** Need to update the pointer to the minimum.
The Catch: Part Two

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Resolving the Issue

• **Idea:** When doing an *extract-min*, coalesce all of the trees so that there's at most one tree of each order.

• Intuitively:
  
  • The number of trees in a heap grows slowly (only during an insert or meld).
  
  • The number of trees in a heap drops rapidly after coalescing (down to $O(\log n)$).
  
  • Can backcharge the work done during an *extract-min* to *enqueue* or *meld*. 
Coalescing Trees

- Our eager melding algorithm assumes that:
  - there is either zero or one tree of each order, and that
  - the trees are stored in ascending order.
- **Challenge:** When coalescing trees in this case, neither of these properties necessarily hold.
Wonky Arithmetic

• Let's turn back to arithmetic to get an intuition for how to solve this problem.
Wonky Arithmetic

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  Sum: 19
  Bits Needed: 5
Wonky Arithmetic

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Wonky Arithmetic

• Compute the number of bits necessary to hold the sum.
  • Only $O(\log n)$ bits are needed.
• Create an array of that size, initially empty.
• For each packet:
  • If there is no packet of that size, place the packet in the array at that spot.
  • If there is a packet of that size:
    – Fuse the two packets together.
    – Recursively add the new packet back into the array.
Now With Trees!

- Compute the number of trees necessary to hold the nodes.
  - Only $O(\log n)$ trees are needed.
- Create an array of that size, initially empty.
- For each tree:
  - If there is no tree of that size, place the tree in the array at that spot.
  - If there is a tree of that size:
    - Fuse the two trees together.
    - Recursively add the new tree back into the array.
Coalescing Trees
Coalescing Trees

Total number of nodes: 15

(Can compute in time $\Theta(T)$, where $T$ is the number of trees, if each tree is tagged with its order)

Bits needed: 4
Coalescing Trees
Coalescing Trees

```
8  7
5  8
7  8
4
3
1
2
5
6
0
8
9
7
9
9
```
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees

```
   3
  / \   \
 5   4
 /     /
7  1
`-
/                      /
2                      8
 |                     /
5                      7
 |                     /
6                      8
 |                     9
5                     9
```
Coalescing Trees
Coalescing Trees
Coalescing Trees
Analyzing Coalesce

- Suppose there are $T$ trees.
- We spend $\Theta(T)$ work iterating across the main list of trees twice:
  - Pass one: Count up number of nodes (if each tree stores its order, this takes time $\Theta(T)$).
  - Pass two: Place each node into the array.
- Each merge takes time $O(1)$.
- The number of merges is $O(T)$.
- Total work done: $\Theta(T)$.
- In the worst case, this is $O(n)$. 
The Story So Far

• A binomial heap with lazy melding has these worst-case time bounds:
  • enqueue: O(1)
  • meld: O(1)
  • find-min: O(1)
  • extract-min: O(n).

• These are worst-case time bounds. What about an amortized time bounds?
An Observation

• The expensive step here is \textit{extract-min}, which runs in time proportional to the number of trees.

• Each tree can be traced back to one of three sources:
  • An \textit{enqueue}.
  • A \textit{meld} with another heap.
  • A tree exposed by an \textit{extract-min}.

• Let's use an amortized analysis to shift the blame for the \textit{extract-min} performance to other operations.
The Potential Method

• We will use the potential method in this analysis.

• When analyzing insertions with eager merges, we set $\Phi(D)$ to be the number of trees in $D$.

• Let's see what happens if we use this $\Phi$ here.
Analyzing an Insertion

• To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.

```
3
  \-----
  | 6    |
  | 4    |
  |      |
  \-----
```

```
7
  \-----
  | 5    |
  \-----
```

```
1
```

Actual time: $O(1)$. Amortized time: $O(1)$. ΔΦ: +1
Analyzing an Insertion

• To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.

```
  3
  |  
 6   4
  |  |  
8   7   5
  |  |  
8
```

Actual time: $O(1)$. ΔΦ: +1
Amortized time: $O(1)$.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.

Actual time: $O(1)$. $\Delta \Phi$: +1

Amortized cost: $O(1)$. 

---

Diagram:

```
     3
    /\  
   6   4
     \   
      8
```

```
     7
    /\  
   5   1
     \   
      8
```
Analyzing a Meld

• Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$. 
Analyzing a Meld

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Analyzing a Meld

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Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
- Let $\Phi_{B_1}$ and $\Phi_{B_2}$ be the initial potentials of $B_1$ and $B_2$.
- The new heap $B$ has potential $\Phi_{B_1} + \Phi_{B_2}$ and $B_1$ and $B_2$ have potential 0.
- $\Delta \Phi$ is zero.
- Amortized cost: $O(1)$. 

\[ \begin{array}{c}
\min
\end{array} \]

![Diagram of melding binomial heaps]
Analyzing a Find-Min

- Each \textit{find-min} does $O(1)$ work and does not add or remove trees.
- Amortized cost: $O(1)$.
Analyzing Extract-Min

- Suppose we perform an *extract-min* on a binomial heap with $T$ trees in it.
- Initially, we expose the children of the minimum element. This increases the number of trees to $T + O(\log n)$.
- The runtime for coalescing these trees is $O(T + \log n)$.
- When we're done merging, there will be $O(\log n)$ trees remaining, so $\Delta \Phi = -T + O(\log n)$.
- Amortized cost is
  \[
  \Theta(T + \log n) + O(1) \cdot (-T + O(\log n)) \\
  = \Theta(T) - O(1) \cdot T + O(1) \cdot O(\log n) \\
  = O(\log n).
  \]
The Overall Analysis

- The amortized costs of the operations on a lazy binomial heap are as follows:
  - `enqueue`: O(1)
  - `meld`: O(1)
  - `find-min`: O(1)
  - `extract-min`: O(log \( n \))

- Any series of \( e \) `enqueue`es mixed with \( d \) `extract-min`s will take time O\( (e + d \log e) \).
Why This Matters

• Lazy binomial heaps are a powerful building block used in many other data structures.

• We'll see one of them, the *Fibonacci heap*, when we come back on Thursday.

• You'll see another (supporting *add-to-all*) on the problem set.
Next Time

• **The Need for decrease-key**
  • A powerful and versatile operation on priority queues.

• **Fibonacci Heaps**
  • A variation on lazy binomial heaps with efficient decrease-key.

• **Implementing Fibonacci Heaps**
  • ... is harder than it looks!