Binomial Heaps
Outline for this Week

- **Binomial Heaps (Today)**
  - A simple, flexible, and versatile priority queue.

- **Lazy Binomial Heaps (Today)**
  - A powerful building block for designing advanced data structures.

- **Fibonacci Heaps (Thursday)**
  - A heavyweight and theoretically excellent priority queue.
Review: Priority Queues
Priority Queues

A priority queue is a data structure that stores a set of elements annotated with totally-ordered keys and allows efficient extraction of the element with the least key.

More concretely, supports these operations:

- `pq.enqueue(v, k)`, which enqueues element v with key k;
- `pq.find-min()`, which returns the element with the least key; and
- `pq.extract-min()`, which removes and returns the element with the least key,
Binary Heaps

- Priority queues are frequently implemented as **binary heaps**.
- **enqueue** and **extract-min** run in time $O(\log n)$; **find-min** runs in time $O(1)$.
- We're not going to cover binary heaps this quarter; I assume you've seen them before.
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Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - **meld**($pq_1$, $pq_2$): Destroy $pq_1$ and $pq_2$ and combine their elements into a single priority queue.
  - $pq$.decrease-key($v$, $k'$): Given a pointer to element $v$ already in the queue, lower its key to have new value $k'$.
  - $pq$.add-to-all($\Delta k$): Add $\Delta k$ to the keys of each element in the priority queue (typically used with **meld**).

- In lecture, we'll cover binomial heaps to efficiently support **meld** and Fibonacci heaps to efficiently support **meld** and **decrease-key**.

- You'll design a priority queue supporting efficient **meld** and **add-to-all** on the problem set.
Meldable Priority Queues

- A priority queue supporting the *meld* operation is called a *meldable priority queue*.

- \( \text{meld}(pq_1, pq_2) \) destructively modifies \( pq_1 \) and \( pq_2 \) and produces a new priority queue containing all elements of \( pq_1 \) and \( pq_2 \).
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\begin{itemize}
  \item \begin{tikzpicture}
    \node (A) at (0,0) [cloud, draw, aspect=2, minimum width=1.5cm, fill=cyan!30] {
      \begin{tabular}{c}
        24 \\
        137 \\
        13 \\
        16 \\
      \end{tabular}\ 
    };
    \node (B) at (3,0) [cloud, draw, aspect=2, minimum width=1.5cm, fill=yellow!30] {
      \begin{tabular}{c}
        19 \\
        6 \\
        18 \\
        72 \\
      \end{tabular}\ 
    };
  \end{tikzpicture}
\end{itemize}
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Efficiently Meldable Queues

- Standard binary heaps do not efficiently support *meld*.

- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.
Binomial Heaps

- The **binomial heap** is an priority queue data structure that supports efficient melding.
- We'll study binomial heaps for several reasons:
  - Implementation and intuition is totally different than binary heaps.
  - Used as a building block in other data structures (Fibonacci heaps, soft heaps, etc.)
  - Has a beautiful intuition; similar ideas can be used to produce other data structures.
The Intuition: *Binary Arithmetic*
Adding Binary Numbers

• Given the binary representations of two numbers $n$ and $m$, we can add those numbers in time $\Theta(\max\{\log m, \log n\})$. 

\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]
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    1
```
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\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 \\
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\end{array}
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A Different Intuition

- Represent $n$ and $m$ as a collection of “packets” whose sizes are powers of two.
- Adding together $n$ and $m$ can then be thought of as combining the packets together, eliminating duplicates.
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\[
\begin{array}{c}
\text{16} \\
\text{16} \\
\hline
\text{4} \quad \text{1}
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Why This Works

- In order for this arithmetic procedure to work efficiently, the packets must obey the following properties:
  - The packets must be stored in ascending/descending order of size.
  - The packets must be stored such that there are no two packets of the same size.
  - Two packets of the same size must be efficiently “fusable” into a single packet.
Building a Priority Queue

- **Idea:** Adapt this approach to build a priority queue.
- Store elements in the priority queue in “packets” whose sizes are powers of two.
- Store packets in ascending size order.
- We'll choose a representation of a packet so that two packets of the same size can easily be fused together.
Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.
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Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.

- **Idea:** Meld together the queue and a new queue with a single packet.
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Time required: \( O(\log n) \) fuses.
Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.
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Fracturing Packets

• If we have a packet with $2^k$ elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.

• **Fun fact**: $2^k - 1 = 1 + 2 + 4 + ... + 2^{k-1}$.

• **Idea**: “Fracture” the packet into $k - 1$ smaller packets, then add them back in.
Fracturing Packets

• We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.
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- Runtime is $O(\log n)$ fuses in *meld*, plus fragment cost.
Building a Priority Queue

- What properties must our packets have?
  - Size must be a power of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently “fracture” a packet of $2^k$ nodes into packets of 1, 2, 4, 8, ..., $2^{k-1}$ nodes.

- What representation of packets will give us these properties?
Binomial Trees

- A **binomial tree of order** $k$ is a type of tree recursively defined as follows:

  A binomial tree of order $k$ is a single node whose children are binomial trees of order 0, 1, 2, ..., $k - 1$.

- Here are the first few binomial trees:
Binomial Trees

- **Theorem:** A binomial tree of order $k$ has exactly $2^k$ nodes.

- **Proof:** Induction on $k$. Assuming that binomial trees of orders 0, 1, 2, ..., $k - 1$ have $2^0, 2^1, 2^2, ..., 2^{k-1}$ nodes, then the number of nodes in an order-$k$ binomial tree is

  $$2^0 + 2^1 + ... + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k$$

  So the claim holds for $k$ as well. ■
Binomial Trees

- A *heap-ordered binomial tree* is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.

- We will use heap-ordered binomial trees to implement our “packets.”
Binomial Trees

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```
     1
   / | \
  2 3 7
 / \
6   4
 /   \
5
 /    \
8
```
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```
  1
 /\   \
2  3   7
 / \  /  \
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   \
8
```

Make the binomial tree with the larger root the first child of the tree with the smaller root.
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The Binomial Heap

A **binomial heap** is a collection of heap-ordered binomial trees stored in ascending order of size.

Operations defined as follows:

- **meld**(\(pq_1, pq_2\)): Use addition to combine all the trees.
  - Fuses O(log \(n\)) trees. Total time: O(log \(n\)).
- **\(pq\).enqueue**(\(v, k\)): Meld \(pq\) and a singleton heap of (\(v, k\)).
  - Total time: O(log \(n\)).
- **\(pq\).find-min\()**: Find the minimum of all tree roots.
  - Total time: O(log \(n\)).
- **\(pq\).extract-min\()**: Find the min, delete the tree root, then meld together the queue and the exposed children.
  - Total time: O(log \(n\)).
Time-Out for Announcements!
Problem Sets

- Problem Set Two has been graded. Check GradeScope for details!
- Problem Set Three is due on Thursday of this week.
  - Have questions? Stop by office hours or ask on Piazza!
COMBATTING INEQUITY IN EDUCATION
A CRITICAL CONVERSATION ABOUT THE PATH TOWARD EDUCATION EQUITY IN AMERICA

APRIL 27, 7PM
PAUL BREST HALL
BIT.LY/INEQUITYEDU


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STANFORD UNIVERSITY

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LINDA DARLING-HAMMOND
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STANFORD CENTER
FOR OPPORTUNITY AND POLICY IN EDUCATION

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OFFICE OF THE VICE PROVOST FOR GRADUATE EDUCATION (VPGE)
OFFICE OF THE VICE PROVOST FOR UNDERGRADUATE EDUCATION (VPUE)

Open change
Back to CS166!
Analyzing Insertions

- Each enqueue into a binomial heap takes time $O(\log n)$, since we have to meld the new node into the rest of the trees.

- However, it turns out that the amortized cost of an insertion is lower in the case where we do a series of $n$ insertions.
Adding One

- Suppose we want to execute $n++$ on the binary representation of $n$.

- Do the following:
  - Find the longest span of 1's at the right side of $n$.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.
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\[
1 \ 1 \ 0 \ 0 \ 0 \ 1
\]
Adding One

• Suppose we want to execute $n++$ on the binary representation of $n$.

• Do the following:
  • Find the longest span of 1's at the right side of $n$.
  • Flip those 1's to 0's.
  • Set the preceding bit to 1.
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\[ 1 \ 1 \ 0 \ 1 \ 0 \]
Adding One

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- Do the following:
  - Find the longest span of 1's at the right side of $n$.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.
- Runtime: $\Theta(b)$, where $b$ is the number of bits flipped.
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea:** Use as a potential function the number of 1's in the number.

$\Phi = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$
An Amortized Analysis

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$\Phi = 1$  \hspace{1cm} 0 0 0 0 0 0 1
An Amortized Analysis

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\[
\Phi = 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1
\]

Actual cost: 1
ΔΦ: +1
Amortized cost: 2
An Amortized Analysis

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\[ \Phi = 1 \]

\[ \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ \end{array} \]
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• **Idea:** Use as a potential function the number of 1's in the number.

$$\Phi = 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0$$

Actual cost: 2
$\Delta \Phi$: 0
Amortized cost: 2
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea:** Use as a potential function the number of 1's in the number.

\[
\Phi = 2 \\
0 \ 0 \ 0 \ 1 \ 1 \ 1
\]
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.

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<table>
<thead>
<tr>
<th>Φ = 2</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual cost: 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ΔΦ: 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Amortized cost: 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
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0 0 0 1 1
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\[ \Phi = 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \]

- Actual cost: 3
- $\Delta \Phi$: -1
- Amortized cost: 2
Properties of Binomial Heaps

• Starting with an empty binomial heap, the amortized cost of each insertion into the heap is $O(1)$, assuming there are no deletions.

• **Rationale:** Binomial heap operations are isomorphic to integer arithmetic.

• Since the amortized cost of incrementing a binary counter starting at zero is $O(1)$, the amortized cost of enqueuing into an initially empty binomial heap is $O(1)$. 
Binomial vs Binary Heaps

• Interesting comparison:
  
  • The cost of inserting \( n \) elements into a binary heap, one after the other, is \( \Theta(n \log n) \) in the worst-case.
  
  • If \( n \) is known in advance, a binary heap can be constructed out of \( n \) elements in time \( \Theta(n) \).
  
  • The cost of inserting \( n \) elements into a binomial heap, one after the other, is \( \Theta(n) \), even if \( n \) is not known in advance!
A Catch

- This amortized time bound does not hold if \textit{enqueue} and \textit{extract-min} are intermixed.

\textbf{Intuition:} Can force expensive insertions to happen repeatedly.
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```
3
  ↓
  6
    ↓
    8

7
  ↓
  4
  ↓
  8

2
  ↓
  5
```


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\begin{tikzpicture}
  \node[circle, draw, fill=blue!20] (1) at (0,0) {3};
  \node[circle, draw, fill=blue!20] (2) at (0,-1) {6};
  \node[circle, draw, fill=blue!20] (3) at (0,-2) {8};
  \node[circle, draw, fill=blue!20] (4) at (1,-1) {4};
  \node[circle, draw, fill=blue!20] (5) at (1,0) {7};
  \node[circle, draw, fill=blue!20] (6) at (1,-2) {8};
  \node[circle, draw, fill=blue!20] (7) at (2,0) {5};
  \node[circle, draw, fill=orange!50] (8) at (2,-2) {2};
  \draw[->] (1) -- (2);
  \draw[->] (2) -- (3);
  \draw[->] (5) -- (4);
  \draw[->] (4) -- (6);
  \draw[->] (5) -- (7);
  \draw[->] (7) -- (6);
  \draw[->] (7) -- (8);
\end{tikzpicture}
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- This amortized time bound does not hold if *enqueue* and *extract-min* are intermixed.
- **Intuition:** Can force expensive insertions to happen repeatedly.
Question: Can we make insertions amortized $O(1)$, regardless of whether we do deletions?
Where's the Cost?

• Why does \texttt{enqueue} take time $O(\log n)$?

• \textbf{Answer}: May have to combine together $O(\log n)$ different binomial trees together into a single tree.

• \textbf{New Question}: What happens if we don't combine trees together?

• That is, what if we just add a new singleton tree to the list?
Lazy Melding

• More generally, consider the following lazy melding approach:

  To meld together two binomial heaps, just combine the two sets of trees together.

• If we assume the trees are stored in doubly-linked lists, this can be done in time O(1).
Lazy Melding

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Lazy Melding

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  To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$. 
The Catch: Part One

- When we use eager melding, the number of trees is $O(\log n)$.
- Therefore, $\text{find-min}$ runs in time $O(\log n)$.
- **Problem: $\text{find-min}$ no longer runs in time $O(\log n)$ because there can be $\Theta(n)$ trees.**
A Solution

- Have the binomial heap store a pointer to the minimum element.
- Can be updated in time $O(1)$ after doing a meld by comparing the minima of the two heaps.
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The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.
- **Rationale:** Need to update the pointer to the minimum.
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- **Rationale:** Need to update the pointer to the minimum.
Resolving the Issue

- **Idea:** When doing an *extract-min*, coalesce all of the trees so that there's at most one tree of each order.

- Intuitively:
  - The number of trees in a heap grows slowly (only during an insert or meld).
  - The number of trees in a heap drops rapidly after coalescing (down to $O(\log n)$).
  - Can backcharge the work done during an *extract-min* to *enqueue* or *meld*. 

Coalescing Trees

- Our eager melding algorithm assumes that
  - there is either zero or one tree of each order, and that
  - the trees are stored in ascending order.

- **Challenge:** When coalescing trees in this case, neither of these properties necessarily hold.
 Wonky Arithmetic

• Let's turn back to arithmetic to get an intuition for how to solve this problem.

4 2 8 2 1 1 1
Wonky Arithmetic

• Let's turn back to arithmetic to get an intuition for how to solve this problem.

Sum: 19
Bits Needed: 5
Wonky Arithmetic

• Let's turn back to arithmetic to get an intuition for how to solve this problem.
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Let's turn back to arithmetic to get an intuition for how to solve this problem.
Wonky Arithmetic

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![Image of blocks with numbers]
Wonky Arithmetic

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Wonky Arithmetic

- Compute the number of bits necessary to hold the sum.
  - Only $O(\log n)$ bits are needed.
- Create an array of that size, initially empty.
- For each packet:
  - If there is no packet of that size, place the packet in the array at that spot.
  - If there is a packet of that size:
    - Fuse the two packets together.
    - Recursively add the new packet back into the array.
Now With Trees!

- Compute the number of trees necessary to hold the nodes.
  - Only $O(\log n)$ trees are needed.
- Create an array of that size, initially empty.
- For each tree:
  - If there is no tree of that size, place the tree in the array at that spot.
  - If there is a tree of that size:
    - Fuse the two trees together.
    - Recursively add the new tree back into the array.
Coalescing Trees
Coalescing Trees

Total number of nodes: \textbf{15}

(Can compute in time $\Theta(T)$, where $T$ is the number of trees, if each tree is tagged with its order)

Bits needed: \textbf{4}
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Analyzing Coalesce

- Suppose there are $T$ trees.
- We spend $\Theta(T)$ work iterating across the main list of trees twice:
  - Pass one: Count up number of nodes (if each tree stores its order, this takes time $\Theta(T)$).
  - Pass two: Place each node into the array.
- Each merge takes time $O(1)$.
- The number of merges is $O(T)$.
- Total work done: $\Theta(T)$.
- In the worst case, this is $O(n)$. 
The Story So Far

• A binomial heap with lazy melding has these worst-case time bounds:
  • enqueue: O(1)
  • meld: O(1)
  • find-min: O(1)
  • extract-min: O(n).

• These are worst-case time bounds. What about an amortized time bounds?
An Observation

• The expensive step here is *extract-min*, which runs in time proportional to the number of trees.

• Each tree can be traced back to one of three sources:
  • An *enqueue*.
  • A *meld* with another heap.
  • A tree exposed by an *extract-min*.

• Let's use an amortized analysis to shift the blame for the *extract-min* performance to other operations.
The Potential Method

- We will use the potential method in this analysis.
- When analyzing insertions with eager merges, we set $\Phi(D)$ to be the number of trees in $D$.
- Let's see what happens if we use this $\Phi$ here.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.

```
min
3 -> 6 -> 8
   |     |
   4    8
7 -> 5
```
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.

![Binary tree diagram](image-url)
Analyzing an Insertion

- To **enqueue** a key, we add a new binomial tree to the forest and possibly update the min pointer.

  Actual time: \(O(1)\). \(\Delta \Phi: +1\)

  Amortized time: \(O(1)\).
Analyzing a Meld

- Suppose that we **meld** two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
Analyzing a Meld

- Suppose that we meld two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$. 

![Diagram of melding two binomial heaps.](attachment:meld_diagram.png)
Analyzing a Meld

- Suppose that we meld two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
- Let $\Phi_{B_1}$ and $\Phi_{B_2}$ be the initial potentials of $B_1$ and $B_2$.
- The new heap $B$ has potential $\Phi_{B_1} + \Phi_{B_2}$ and $B_1$ and $B_2$ have potential 0.
- $\Delta\Phi$ is zero.
- Amortized cost: $O(1)$. 

\[
\begin{array}{ccccccc}
3 & 7 & 5 & 1 & 2 & 3 & 4 \\
6 & 4 & 8 & 1 & 9 & & \\
8 & & & & & & \\
\end{array}
\]
Analyzing a Find-Min

- Each \texttt{find-min} does $O(1)$ work and does not add or remove trees.
- Amortized cost: $O(1)$. 
Analyzing Extract-Min

• Initially, we expose the children of the minimum element. This takes time $O(\log n)$.

• Suppose that at this point there are $T$ trees. As we saw earlier, the runtime for the coalesce is $\Theta(T)$.

• When we're done merging, there will be $O(\log n)$ trees remaining, so $\Delta \Phi = -T + O(\log n)$.

• Amortized cost is

  $$O(\log n) + \Theta(T) + O(1) \cdot (-T + O(\log n))$$

  $$= O(\log n) + \Theta(T) - O(1) \cdot T + O(1) \cdot O(\log n)$$

  $$= O(\log n).$$
The Overall Analysis

- The *amortized* costs of the operations on a lazy binomial heap are as follows:
  - `enqueue`: $O(1)$
  - `meld`: $O(1)$
  - `find-min`: $O(1)$
  - `extract-min`: $O(\log n)$

- Any series of $e$ *enqueue*es mixed with $d$ *extract-min*es will take time $O(e + d \log e)$. 
Why This Matters

- Lazy binomial heaps are a powerful building block used in many other data structures.
- We'll see one of them, the *Fibonacci heap*, when we come back on Thursday.
- You'll see another (supporting *add-to-all*) on the problem set.
Next Time

- **The Need for decrease-key**
  - A powerful and versatile operation on priority queues.

- **Fibonacci Heaps**
  - A variation on lazy binomial heaps with efficient decrease-key.

- **Implementing Fibonacci Heaps**
  - ... is harder than it looks!