Binomial Heaps
Outline for this Week

- **Binomial Heaps (Today)**
  - A simple, flexible, and versatile priority queue.

- **Lazy Binomial Heaps (Today)**
  - A powerful building block for designing advanced data structures.

- **Fibonacci Heaps (Thursday)**
  - A heavyweight and theoretically excellent priority queue.
Review: Priority Queues
Priority Queues

- A **priority queue** is a data structure that supports these operations:
  - \( pq\text{.enqueue}(v, k) \), which enqueues element \( v \) with key \( k \);
  - \( pq\text{.find-min}() \), which returns the element with the least key; and
  - \( pq\text{.extract-min}() \), which removes and returns the element with the least key.

- They’re useful as building blocks in a *bunch* of algorithms.
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Mt. Giluwe

4,368
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<table>
<thead>
<tr>
<th>Mountain</th>
<th>Elevation</th>
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<tbody>
<tr>
<td>Mt. Giluwe</td>
<td>4,368</td>
</tr>
<tr>
<td>Mt. Sidley</td>
<td>4,285</td>
</tr>
<tr>
<td>Pico de Orizaba</td>
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Binary Heaps

- Priority queues are frequently implemented as *binary heaps*.
  - *enqueue* and *extract-min* run in time $O(\log n)$; *find-min* runs in time $O(1)$.
- These heaps are surprisingly fast in practice. It’s tough to beat their performance!
  - $d$-ary heaps can outperform binary heaps for a well-tuned value of $d$, and otherwise only the *sequence heap* is known to specifically outperform this family.
  - (Is this information incorrect as of 2020? Let me know and I’ll update it.)
- In that case, why do we need other heaps?
Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - `meld(pq_1, pq_2)`: Destroy `pq_1` and `pq_2` and combine their elements into a single priority queue. *(MSTs via Cheriton-Tarjan)*
  - `pq.decrease-key(v, k')`: Given a pointer to element `v` already in the queue, lower its key to have new value `k'`. *(Shortest paths via Dijkstra, global min-cut via Stoer-Wagner)*
  - `pq.add-to-all(Δk)`: Add `Δk` to the keys of each element in the priority queue, typically used with `meld`. *(Optimum branchings via Chu-Edmonds-Liu)*
- In lecture, we'll cover binomial heaps to efficiently support `meld` and Fibonacci heaps to efficiently support `meld` and `decrease-key`.
- Assuming the TAs sign off on it, you’ll design a priority queue supporting `meld` and `add-to-all` on the next problem set.
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Meldable Priority Queues
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• A priority queue supporting the *meld* operation is called a *meldable priority queue*.

• **meld**($pq_1$, $pq_2$) destructively modifies $pq_1$ and $pq_2$ and produces a new priority queue containing all elements of $pq_1$ and $pq_2$. 
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Efficiently Meldable Queues

- Standard binary heaps do not efficiently support *meld*.
- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.
Adding Binary Numbers

• Given the binary representations of two numbers $n$ and $m$, we can add those numbers in time $O(\log m + \log n)$.

\textbf{Intuition:}
Writing out $n$ in any “reasonable” base requires $\Theta(\log n)$ digits.
Adding Binary Numbers

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\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]
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```
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+  1 1 1 1 1 1
```
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\begin{array}{c}
  1 \quad 1 \\
  1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \\
+ \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
\hline
  1 \quad 0 \quad 1 \\
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\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & \downarrow \\
\\
1 & 0 & 1 & 1 & 0 \\
\downarrow & + & & & & \\
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
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```
  1 1 1 1 1 1
+ 1 0 1 1 0 0
  1 1 1 1 1
```

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```
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- Represent \( n \) and \( m \) as a collection of “packets” whose sizes are powers of two.
- Adding together \( n \) and \( m \) can then be thought of as combining the packets together, eliminating duplicates

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\[ + \]
\[
\begin{array}{c}
16 \\
+ \\
8 \\
+ \\
8 \\
\hline
4 \\
\hline
1
\end{array}
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Building a Priority Queue

- **Idea:** Store elements in “packets” whose sizes are powers of two and *meld* by “adding” groups of packets.
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```
 26 53
31 58
 64 97 84 23
41 93 62 59
```

+ ____________________________________________
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- What properties must our packets have?
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As long as the packets provide $O(1)$ access to the minimum, we can execute $\textit{find-min}$ in time $O(\log n)$. 
Building a Priority Queue

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  • Can efficiently fuse packets of the same size.
  • Can efficiently find the minimum element of each packet.
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.

- **Idea**: Meld together the queue and a new queue with a single packet.
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![Queue elements](image)
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```
27  18  84  23  53  58
28  45  62  59  14  26
```

Time required: $O(\log n)$ fuses.
Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.
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- If we have a packet with $2^k$ elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.
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Deleting the Minimum

- If we have a packet with $2^k$ elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.
- **Fun fact**: $2^k - 1 = 2^0 + 2^1 + 2^2 + ... + 2^{k-1}$.
- **Idea**: “Fracture” the packet into $k$ smaller packets, then add them back in.
Fracturing Packets

- We can\textit{extract-min} by fracturing the packet containing the minimum and adding the fragments back in.
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- Runtime is $O(\log n)$ fuses in meld, plus fracture cost.
Building a Priority Queue

• What properties must our packets have?
  • Size is a power of two.
  • Can efficiently fuse packets of the same size.
  • Can efficiently find the minimum element of each packet.
  • Can efficiently "fracture" a packet of $2^k$ nodes into packets of $2^0, 2^1, 2^2, 2^3, \ldots, 2^{k-1}$ nodes.

• Question: How can we represent our packets to support the above operations efficiently?
Thanks to former CS166er Anna Zeng for this explanation!
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Idea: Have the minimum value as the root of a tree whose children are the smaller packets. Then recursively expand out the packets.
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Binomial Trees

- A *binomial tree of order* $k$ is a tree structure with $2^k$ nodes.
- We can *mechanically* describe binomial trees as follows:
  - Place the minimum of the $2^k$ keys at the root.
  - Regroup the remaining elements into $k$ groups of sizes $2^0$, $2^1$, $2^2$, ..., and $2^{k-1}$.
  - Recursively build binomial trees from each group.
  - Make those new trees children of the root.
- Can we operationally describe binomial trees?
Binomial Trees

- Here’s an operational definition of a binomial tree of order $k$:

  A binomial tree of order $k$ is a tree obeying the min-heap property consisting of a root node whose children are binomial trees of order $0, 1, 2, ..., k - 1$.

Why are these called binomial trees? Look across the layers of these trees and see if you notice anything!
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Make the binomial tree with the larger root the first child of the tree with the smaller root.
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[Diagram of a binomial tree]
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The Binomial Heap

- A **binomial heap** is a collection of binomial trees stored in ascending order of size.

- Operations defined as follows:
  - **meld**($pq_1$, $pq_2$): Use addition to combine all the trees.
    - Fuses $O(\log n + \log m)$ trees. Cost: $O(\log n + \log m)$. Here, assume one binomial heap has $n$ nodes, the other $m$.
  - **$pq$.enqueue**(v, k): Meld $pq$ and a singleton heap of (v, k).
    - Total time: $O(\log n)$.
  - **$pq$.find-min**(): Find the minimum of all tree roots.
    - Total time: $O(\log n)$.
  - **$pq$.extract-min**(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: $O(\log n)$. 
Draw what happens if we enqueue the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.
Draw what happens if we *enqueue* the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.
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Where We Stand

• Here’s the current scorecard for the binomial heap.

• This is a fast, elegant, and clever data structure.

• **Question:** Can we do better?

Binomial Heap

• **enqueue**: O(log n)

• **find-min**: O(log n)

• **extract-min**: O(log n)

• **meld**: O(log m + log n).
Where We Stand

- **Theorem:** No comparison-based priority queue structure can have *enqueue* and *extract-min* each take time $o(\log n)$.

- **Proof:** Suppose these operations each take time $o(\log n)$. Then we could sort $n$ elements by perform $n$ *enqueue*es and then $n$ *extract-mins* in time $o(n \log n)$. This is impossible with comparison-based algorithms. ■

### Binomial Heap

- *enqueue*: $O(\log n)$
- *find-min*: $O(\log n)$
- *extract-min*: $O(\log n)$
- *meld*: $O(\log m + \log n)$. 
Where We Stand

- We can’t make both *enqueue* and *extract-min* run in time $o(\log n)$.
- However, we could conceivably make one of them faster.

*Question:* Which one should we prioritize?

- Probably *enqueue*, since we aren’t guaranteed to have to remove all added items.

*Goal:* Make *enqueue* take time $O(1)$.

### Binomial Heap

- **enqueue**: $O(\log n)$
- **find-min**: $O(\log n)$
- **extract-min**: $O(\log n)$
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Where We Stand

• The \textit{enqueue} operation is implemented in terms of \textit{meld}.

• If we want \textit{enqueue} to run in time $O(1)$, we’ll need \textit{meld} to take time $O(1)$.

• How could we accomplish this?

Binomial Heap

• \textit{enqueue}: $O(\log n)$
• \textit{find-min}: $O(\log n)$
• \textit{extract-min}: $O(\log n)$
• \textit{meld}: $O(\log m + \log n)$. 
Thinking With Amortization
Refresher: Amortization

• In an amortized efficient data structure, some operations can take much longer than others, provided that previous operations didn’t take too long to finish.

• Think dishwashers: you may have to do a big cleanup at some point, but that’s because you did basically no work to wash all the dishes you placed in the dishwasher.
Lazy Melding

- Consider the following lazy *meld*ing approach:

  *To meld together two binomial heaps, just combine the two sets of trees together.*

- **Intuition:** Why do any work to organize keys if we’re not going to do an *extract-min*? We’ll worry about cleanup then.

![Diagram of binomial heaps](image-url)
Lazy Melding

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Lazy Melding

• If we store our list of trees as circularly, doubly-linked lists, we can concatenate tree lists in time $O(1)$.
  • Cost of a *meld*: $O(1)$.
  • Cost of an *enqueue*: $O(1)$.
• If it sounds too good to be true, it probably is.
Lazy Melding

- Imagine that we implement *extract-min* the same way as before:
  - Find the packet with the minimum.
  - “Fracture” that packet to expose smaller packets.
  - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?
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Each pass of finding the minimum value takes time $\Theta(n)$ in the worst case. We’ve lost our nice runtime guarantees!
Washing the Dishes

• Every *meld* (and *enqueue*) creates some “dirty dishes” (small trees) that we need to clean up later.

• If we never clean them up, then our *extract-min* will be too slow to be usable.

• *Idea:* Change *extract-min* to “wash the dishes” and make things look nice and pretty again.

• *Question:* What does “wash the dishes” mean here?
Washing the Dishes

- With our eager **meld** (and **enqueue**) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- **Idea:** After doing an **extract-min**, do a **coalesce step** to ensure there’s at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.
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6  4  7
   v  v
  8  5  8
   v  v
  8
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At this point, the mess is cleaned up, and we’re left with what we would have had if we had been cleaning up as we go.
Where We’re Going

- A *lazy binomial heap* is a binomial heap, modified as follows:
  - The *meld* operation is lazy. It just combines the two groups of trees together.
  - After doing an *extract-min*, we do a *coalesce* to combine together trees until there’s at most one tree of each order.
  - Intuitively, we’d expect this to amortize away nicely, since the “mess” left by *meld* gets cleaned up later on by a future *extract-min*.

- Questions left to answer:
  - How do we efficiently implement the *coalesce* operation?
  - How efficient is this approach, in an amortized sense?
Coalescing Trees

• The *coalesce* step repeatedly combines trees together until there’s at most one tree of each order.

• How do we implement this so that it runs quickly?
Coalescing Trees

- **Observation:** This would be a *lot* easier to do if all the trees were sorted by size.
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- We can sort our group of \( t \) trees by size in time \( O(t \log t) \) using a standard sorting algorithm.

- **Better idea**: All the sizes are small integers. Use counting sort!
Coalescing Trees

• Here is a fast implementation of *coalesce*:
  • Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, ..., \([\log_2 (n + 1)]\).
  • Start at bucket 0. While there’s two or more trees in the bucket, fuse them and place the result one bucket higher.
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<tbody>
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Here is a fast implementation of *coalesce*:

- Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, ..., \([\log_2 (n + 1)]\).
- Start at bucket 0. While there’s two or more trees in the bucket, fuse them and place the result one bucket higher.
Analyzing Coalesce

**Claim:** Coalescing a group of $t$ trees takes time $O(t + \log n)$.

- Time to create the array of buckets: $O(\log n)$.
- Time to distribute trees into buckets: $O(t)$.
- Time to fuse trees: $O(t)$
  - Number of fuses is $O(t)$, since each fuse decreases the number of trees by one.
  - Cost per fuse is $O(1)$.
- Total work done: $O(t + \log n)$.
- In the worst case, this is $O(n)$. 
The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - enqueue: $O(1)$
  - meld: $O(1)$
  - find-min: $O(1)$
  - extract-min: $O(n)$.
- But these are worst-case time bounds. Intuitively, things should nicely amortize away.
  - The number of trees grows slowly (one per enqueue).
  - The number of trees drops quickly (at most one tree per order) after an extract-min.
An Amortized Analysis

• We’ve seen two methods for performing amortized analysis, the *banker’s method* and the *potential method*.

• In each case, the idea is to clearly mark what “messes” we need to clean up.

• In our case, each tree is a “mess,” since our future *coalesce* operation has to clean it up.
An Amortized Analysis

- We’ll use the potential method and set $\Phi$ to be the number of trees in the lazy binomial heap.
- Recall: The amortized cost of each operation is the actual wall-clock time, plus $O(1) \cdot \Delta \Phi$.
- To perform the analysis, let’s work out how much time each operation takes, plus how it changes the potential.
Analyzing an Insertion

- To *enqueue* a key, we add a new binomial tree to the forest.
Analyzing an Insertion

• To enqueue a key, we add a new binomial tree to the forest.
• Actual time: $O(1)$. $\Delta \Phi: +1$
• Amortized cost: $O(1)$.
Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$. 
Analyzing a Meld

• Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
- We have the same number of trees before and after we do this, so $\Delta \Phi = 0$.
- Amortized cost: $O(1)$. 
Analyzing *extract-min*
Find tree with minimum key.

Work: $O(t)$

$\Phi = t$

Remove min. Add children to list of trees.

Work: $O(\log n)$

Run the coalesce algorithm.

Work: $O(t + \log n)$

$\Phi = O(\log n)$

Work: $O(t + \log n)$

$\Delta \Phi: O(-t + \log n)$
Find tree with minimum key.

Work: $O(t)$

$\Phi = t$

Remove min.
Add children to list of trees.

Work: $O(\log n)$

Run the coalesce algorithm.

Work: $O(t + \log n)$

$\Phi = O(\log n)$

Amortized cost: $O(\log n)$. 
Analyzing Extract-Min

- Suppose we perform an *extract-min* on a binomial heap with \( t \) trees in it.
- Initially, we expose the children of the minimum element. This increases the number of trees to \( t + O(\log n) \).
- The runtime for coalescing these trees is \( O(t + \log n) \).
- When we're done merging, there will be \( O(\log n) \) trees remaining, so \( \Delta \Phi = -t + O(\log n) \).
- Amortized cost is
  
  \[
  O(t + \log n) + O(1) \cdot (-t + O(\log n)) \\
  = O(t) - O(1) \cdot t + O(1) \cdot O(\log n) \\
  = O(\log n). 
  \]
Here’s the final scorecard for our lazy binomial heap.

These are great runtimes! We can’t improve upon this except by making \textbf{extract-min} worst-case efficient.

This is possible! Check out \textit{bootstrapped skew binomial heaps} or \textit{strict Fibonacci heaps} for details!
Major Ideas from Today

• Isometries are a great way to design data structures.
  • Here, binomial heaps come from binary arithmetic.

• Designing for amortized efficiency is about building up messes slowly and rapidly cleaning them up.
  • Each individual enqueue isn’t too bad, and a single extract-min fixes all the prior problems.
Next Time

- **The Need for decrease-key**
  - A powerful and versatile operation on priority queues.

- **Fibonacci Heaps**
  - A variation on lazy binomial heaps with efficient decrease-key.

- **Implementing Fibonacci Heaps**
  - ... is harder than it looks!