Binomial Heaps
Outline for this Week

- **Binomial Heaps (Today)**
  - A simple, flexible, and versatile priority queue.

- **Lazy Binomial Heaps (Today)**
  - A powerful building block for designing advanced data structures.

- **Fibonacci Heaps (Thursday)**
  - A heavyweight and theoretically excellent priority queue.
Review: Priority Queues
Priority Queues

- A **priority queue** is a data structure that stores a set of elements annotated with totally-ordered *keys* and allows efficient extraction of the element with the least key.

- More concretely, supports these operations:
  - `pq.enqueue(v, k)`, which enqueues element `v` with key `k`;
  - `pq.find-min()`, which returns the element with the least key; and
  - `pq.extract-min()`, which removes and returns the element with the least key,
Priority queues are frequently implemented as **binary heaps**.

- *enqueue* and *extract-min* run in time $O\left(\log n\right)$; *find-min* runs in time $O\left(1\right)$.

- We're not going to cover binary heaps this quarter; I assume you've seen them before.
Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - $\text{meld}(pq_1, pq_2)$: Destroy $pq_1$ and $pq_2$ and combine their elements into a single priority queue.
  - $pq.\text{decrease-key}(v, k')$: Given a pointer to element $v$ already in the queue, lower its key to have new value $k'$.
  - $pq.\text{add-to-all}(\Delta k)$: Add $\Delta k$ to the keys of each element in the priority queue (typically used with $\text{meld}$).

- In lecture, we'll cover binomial heaps to efficiently support $\text{meld}$ and Fibonacci heaps to efficiently support $\text{meld}$ and $\text{decrease-key}$.

- You'll design a priority queue supporting efficient $\text{meld}$ and $\text{add-to-all}$ on the problem set.
Meldable Priority Queues

- A priority queue supporting the *meld* operation is called a *meldable priority queue*.
- *meld*($pq_1, pq_2$) destructively modifies $pq_1$ and $pq_2$ and produces a new priority queue containing all elements of $pq_1$ and $pq_2$. 

![Diagram of priority queues](image)
Meldable Priority Queues

- A priority queue supporting the *meld* operation is called a *meldable priority queue*.

- *meld*(pq₁, pq₂) destructively modifies pq₁ and pq₂ and produces a new priority queue containing all elements of pq₁ and pq₂.
Efficiently Meldable Queues

- Standard binary heaps do not efficiently support *meld*.
- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.
Binomial Heaps

- The **binomial heap** is an priority queue data structure that supports efficient melding.
- We'll study binomial heaps for several reasons:
  - Implementation and intuition is totally different than binary heaps.
  - Used as a building block in other data structures (Fibonacci heaps, soft heaps, etc.)
  - Has a beautiful intuition; similar ideas can be used to produce other data structures.
The Intuition: *Binary Arithmetic*
Adding Binary Numbers

- Given the binary representations of two numbers $n$ and $m$, we can add those numbers in time $\Theta(\max\{\log m, \log n\})$. 

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1
\end{array}
\]
A Different Intuition

- Represent $n$ and $m$ as a collection of “packets” whose sizes are powers of two.
- Adding together $n$ and $m$ can then be thought of as combining the packets together, eliminating duplicates.

\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]
A Different Intuition

- Represent $n$ and $m$ as a collection of “packets” whose sizes are powers of two.
- Adding together $n$ and $m$ can then be thought of as combining the packets together, eliminating duplicates.
Why This Works

• In order for this arithmetic procedure to work efficiently, the packets must obey the following properties:
  • The packets must be stored in ascending/descending order of size.
  • The packets must be stored such that there are no two packets of the same size.
  • Two packets of the same size must be efficiently “fusible” into a single packet.
Building a Priority Queue

- **Idea:** Adapt this approach to build a priority queue.
- Store elements in the priority queue in “packets” whose sizes are powers of two.
- Store packets in ascending size order.
- We'll choose a representation of a packet so that two packets of the same size can easily be fused together.
\[
\begin{array}{cccc}
64 & 97 & 84 & 23 \\
41 & 93 & 62 & 59 \\
53 & 58 & 26 & 31 \\
\end{array}
\]
Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.

- **Idea:** Meld together the queue and a new queue with a single packet.

  Time required: $O(\log n)$ fuses.
Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.
Fracturing Packets

- If we have a packet with $2^k$ elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.

- **Fun fact**: $2^k - 1 = 1 + 2 + 4 + \ldots + 2^{k-1}$.

- **Idea**: “Fracture” the packet into $k - 1$ smaller packets, then add them back in.
Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.
Fracturing Packets

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Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.

- Runtime is $O(\log n)$ fuses in *meld*, plus fragment cost.
Building a Priority Queue

• What properties must our packets have?
  • Size must be a power of two.
  • Can efficiently fuse packets of the same size.
  • Can efficiently find the minimum element of each packet.
  • Can efficiently “fracture” a packet of $2^k$ nodes into packets of $1, 2, 4, 8, \ldots, 2^{k-1}$ nodes.

• What representation of packets will give us these properties?
Binomial Trees

- A **binomial tree of order** $k$ is a type of tree recursively defined as follows:

  A binomial tree of order $k$ is a single node whose children are binomial trees of order 0, 1, 2, ..., $k - 1$.

- Here are the first few binomial trees:
Binomial Trees

- **Theorem:** A binomial tree of order $k$ has exactly $2^k$ nodes.

- **Proof:** Induction on $k$. Assuming that binomial trees of orders 0, 1, 2, ..., $k - 1$ have $2^0$, $2^1$, $2^2$, ..., $2^{k-1}$ nodes, then the number of nodes in an order-$k$ binomial tree is
  
  $$2^0 + 2^1 + ... + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k$$

  So the claim holds for $k$ as well. ■
Binomial Trees

● A *heap-ordered binomial tree* is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.

● We will use heap-ordered binomial trees to implement our “packets.”
Binomial Trees

What properties must our packets have?
- Size must be a power of two. ✓
- Can efficiently fuse packets of the same size. ✓
- Can efficiently find the minimum element of each packet.
- Can efficiently “fracture” a packet of $2^k$ nodes into packets of $1, 2, 4, 8, \ldots, 2^{k-1}$ nodes.

Make the binomial tree with the larger root the first child of the tree with the smaller root.
Binomial Trees

What properties must our packets have?

- Size must be a power of two. ✓
- Can efficiently fuse packets of the same size. ✓
- Can efficiently find the minimum element of each packet. ✓
- Can efficiently “fracture” a packet of $2^k$ nodes into packets of 1, 2, 4, 8, ..., $2^{k-1}$ nodes. ✓
The Binomial Heap

- A **binomial heap** is a collection of heap-ordered binomial trees stored in ascending order of size.

- Operations defined as follows:
  - **meld**($pq_1$, $pq_2$): Use addition to combine all the trees.
    - Fuses $O(\log n)$ trees. Total time: $O(\log n)$.
  - $pq$.enqueue($v$, $k$): Meld $pq$ and a singleton heap of $(v, k)$.
    - Total time: $O(\log n)$.
  - $pq$.find-min(): Find the minimum of all tree roots.
    - Total time: $O(\log n)$.
  - $pq$.extract-min(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: $O(\log n)$.
Time-Out for Announcements!
Problem Sets

- Problem Set Two has been graded. Check GradeScope for details!
- Problem Set Three is due on Thursday of this week.
  - Have questions? Stop by office hours or ask on Piazza!
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Open change
Back to CS166!
Analyzing Insertions

• Each enqueue into a binomial heap takes time $O(\log n)$, since we have to meld the new node into the rest of the trees.

• However, it turns out that the amortized cost of an insertion is lower in the case where we do a series of $n$ insertions.
Adding One

- Suppose we want to execute $n++$ on the binary representation of $n$.

- Do the following:
  - Find the longest span of 1's at the right side of $n$.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.

- Runtime: $\Theta(b)$, where $b$ is the number of bits flipped.
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea:** Use as a potential function the number of 1's in the number.

$\Phi = 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0$
Properties of Binomial Heaps

- Starting with an empty binomial heap, the amortized cost of each insertion into the heap is $O(1)$, assuming there are no deletions.

- **Rationale:** Binomial heap operations are isomorphic to integer arithmetic.

- Since the amortized cost of incrementing a binary counter starting at zero is $O(1)$, the amortized cost of enqueuing into an initially empty binomial heap is $O(1)$. 
Binomial vs Binary Heaps

• Interesting comparison:
  • The cost of inserting $n$ elements into a binary heap, one after the other, is $\Theta(n \log n)$ in the worst-case.
  • If $n$ is known in advance, a binary heap can be constructed out of $n$ elements in time $\Theta(n)$.
  • The cost of inserting $n$ elements into a binomial heap, one after the other, is $\Theta(n)$, even if $n$ is not known in advance!
A Catch

- This amortized time bound does not hold if `enqueue` and `extract-min` are intermixed.
- **Intuition**: Can force expensive insertions to happen repeatedly.
**Question**: Can we make insertions amortized O(1), regardless of whether we do deletions?
Where's the Cost?

- Why does enqueue take time $O(\log n)$?
- **Answer**: May have to combine together $O(\log n)$ different binomial trees together into a single tree.

- **New Question**: What happens if we don't combine trees together?
- That is, what if we just add a new singleton tree to the list?
Lazy Melding

- More generally, consider the following lazy melding approach:

  To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$. 
The Catch: Part One

- When we use eager melding, the number of trees is $O(\log n)$.
- Therefore, $\textit{find-min}$ runs in time $O(\log n)$.
- **Problem:** $\textit{find-min}$ no longer runs in time $O(\log n)$ because there can be $\Theta(n)$ trees.
A Solution

- Have the binomial heap store a pointer to the minimum element.
- Can be updated in time $O(1)$ after doing a meld by comparing the minima of the two heaps.
A Solution

- Have the binomial heap store a pointer to the minimum element.

- Can be updated in time $O(1)$ after doing a meld by comparing the minima of the two heaps.
The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.
- **Rationale**: Need to update the pointer to the minimum.
The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.

- **Rationale:** Need to update the pointer to the minimum.
Resolving the Issue

- **Idea:** When doing an `extract-min`, coalesce all of the trees so that there's at most one tree of each order.

- Intuitively:
  - The number of trees in a heap grows slowly (only during an insert or meld).
  - The number of trees in a heap drops rapidly after coalescing (down to $O(\log n)$).
  - Can backcharge the work done during an `extract-min` to `enqueue` or `meld`. 
Coalescing Trees

- Our eager melding algorithm assumes that
  - there is either zero or one tree of each order, and that
  - the trees are stored in ascending order.

- **Challenge:** When coalescing trees in this case, neither of these properties necessarily hold.
Wonky Arithmetic

• Compute the number of bits necessary to hold the sum.
  • Only $O(\log n)$ bits are needed.
• Create an array of that size, initially empty.
• For each packet:
  • If there is no packet of that size, place the packet in the array at that spot.
  • If there is a packet of that size:
    – Fuse the two packets together.
    – Recursively add the new packet back into the array.
Now With Trees!

• Compute the number of *trees* necessary to hold the *nodes*.
  • Only $O(\log n)$ *trees* are needed.
• Create an array of that size, initially empty.
• For each *tree*:
  • If there is no *tree* of that size, place the *tree* in the array at that spot.
  • If there is a *tree* of that size:
    – Fuse the two *trees* together.
    – Recursively add the new *tree* back into the array.
Analyzing Coalesce

- Suppose there are $T$ trees.
- We spend $\Theta(T)$ work iterating across the main list of trees twice:
  - Pass one: Count up number of nodes (if each tree stores its order, this takes time $\Theta(T)$).
  - Pass two: Place each node into the array.
- Each merge takes time $O(1)$.
- The number of merges is $O(T)$.
- Total work done: $\Theta(T)$.
- In the worst case, this is $O(n)$. 
The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - \textit{enqueue}: O(1)
  - \textit{meld}: O(1)
  - \textit{find-min}: O(1)
  - \textit{extract-min}: O(n).

- These are \textit{worst-case} time bounds. What about an \textit{amortized} time bounds?
An Observation

• The expensive step here is \textit{extract-min}, which runs in time proportional to the number of trees.

• Each tree can be traced back to one of three sources:
  • An \textit{enqueue}.
  • A \textit{meld} with another heap.
  • A tree exposed by an \textit{extract-min}.

• Let's use an amortized analysis to shift the blame for the \textit{extract-min} performance to other operations.
The Potential Method

• We will use the potential method in this analysis.

• When analyzing insertions with eager merges, we set $\Phi(D)$ to be the number of trees in $D$.

• Let's see what happens if we use this $\Phi$ here.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.

Actual time: $O(1)$. $\Delta \Phi: +1$

Amortized time: $O(1)$. 

Diagram:

```
  min
  /   \
 3     7
  |     |
 6     5
  |     |
 4     8
  |     |
 8
```
Analyzing a Meld

- Suppose that we **meld** two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
- Let $\Phi_{B_1}$ and $\Phi_{B_2}$ be the initial potentials of $B_1$ and $B_2$.
- The new heap $B$ has potential $\Phi_{B_1} + \Phi_{B_2}$ and $B_1$ and $B_2$ have potential 0.
- $\Delta\Phi$ is zero.
- Amortized cost: $O(1)$. 

[Diagram of min heap with numbers 3, 6, 7, 5, 1, 2, 3, 4, 8, 4, 8, 9]
Analyzing a Find-Min

- Each \textit{find-min} does $O(1)$ work and does not add or remove trees.
- Amortized cost: $O(1)$. 
Analyzing Extract-Min

- Initially, we expose the children of the minimum element. This takes time $O(\log n)$.
- Suppose that at this point there are $T$ trees. As we saw earlier, the runtime for the coalesce is $\Theta(T)$.
- When we're done merging, there will be $O(\log n)$ trees remaining, so $\Delta \Phi = -T + O(\log n)$.
- Amortized cost is

$$O(\log n) + \Theta(T) + O(1) \cdot (-T + O(\log n))$$

$$= O(\log n) + \Theta(T) - O(1) \cdot T + O(1) \cdot O(\log n)$$

$$= O(\log n).$$
The Overall Analysis

- The *amortized* costs of the operations on a lazy binomial heap are as follows:
  - *enqueue*: $O(1)$
  - *meld*: $O(1)$
  - *find-min*: $O(1)$
  - *extract-min*: $O(\log n)$
- Any series of $e$ *enqueue*es mixed with $d$ *extract-min*s will take time $O(e + d \log e)$. 
Why This Matters

- Lazy binomial heaps are a powerful building block used in many other data structures.
- We'll see one of them, the *Fibonacci heap*, when we come back on Thursday.
- You'll see another (supporting *add-to-all*) on the problem set.
Next Time

• **The Need for decrease-key**
  • A powerful and versatile operation on priority queues.

• **Fibonacci Heaps**
  • A variation on lazy binomial heaps with efficient decrease-key.

• **Implementing Fibonacci Heaps**
  • ... is harder than it looks!