Randomized Data Structures

• Randomization is a powerful tool for improving efficiency and solving problems under seemingly impossible constraints.

• Over the next three lectures, we’ll explore a sampler of data structures that give a feel for the breadth of what’s out there.

• You can easily spend an entire academic career just exploring this space; take CS265 for more on randomized algorithms!
Where We’re Going

• **Hashing and Sketching (Thursday / Tuesday)**
  • Using hash functions to count without counting.

• **Cuckoo Hashing (Next Thursday)**
  • Hashing with *worst-case* $O(1)$ lookups, along with a splash of random hypergraph theory.
Outline for Today

• **Hash Functions**
  • Understanding our basic building blocks.

• **Frequency Estimation**
  • Estimating how many times we’ve seen something.

• **Probabilistic Techniques**
  • Standard but powerful tools for reasoning about randomized data structures.
Preliminaries: *Hash Functions*
Hashing in Practice

• Hash functions are used extensively in programming and software engineering:
  • They make hash tables possible: think C++ `std::hash`, Python’s `__hash__`, or Java’s `Object.hashCode()`.
  • They’re used in cryptography: SHA-256, HMAC, etc.

• **Question:** When we’re in Theoryland, what do we mean when we say “hash function?”
Hashing in Theoryland

• In Theoryland, a hash function is a function from some domain called the universe (typically denoted $\mathcal{U}$) to some codomain.

• The codomain is usually a set of the form $[m] = \{0, 1, 2, 3, ..., m - 1\}$

$$h : \mathcal{U} \rightarrow [m]$$
Hashing in Theoryland

- **Intuition:** No matter how clever you are with designing a specific hash function, that hash function isn’t random, and so there will be pathological inputs.
  - You can formalize this with the pigeonhole principle.
- **Idea:** Rather than finding the One True Hash Function, we’ll assume we have a collection of hash functions to pick from, and we’ll choose which one to use randomly.
Families of Hash Functions

- A **family** of hash functions is a set $\mathcal{H}$ of hash functions with the same domain and codomain.
- We can then introduce randomness into our data structures by sampling a random hash function from $\mathcal{H}$.
- **Key Point:** The randomness in our data structures almost always derives from the random choice of hash functions, not from the data.
  
  *Data is adversarial.*  
  *Hash function selection is random.*

- **Question:** What makes a family of hash functions $\mathcal{H}$ a “good” family of hash functions?
Goal: If we pick \( h \in \mathcal{H} \) uniformly at random, then \( h \) should distribute elements uniformly randomly.
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**Problem:** A hash function that distributes \( n \) elements uniformly at random over \([m]\) requires \( \Omega(n \log m) \) space in the worst case.

**Question:** Do we actually need true randomness? Or can we get away with something weaker?
**Distribution Property:**
Each element should have an equal probability of being placed in each slot.

For any \( x \in \mathcal{U} \) and random \( h \in \mathcal{H} \), the value of \( h(x) \) is uniform over its codomain.
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Find an “obviously bad” family of hash functions that satisfies the distribution property.

Answer at [https://pollev.com/cs166spr23](https://pollev.com/cs166spr23)
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**Problem:**
This rule doesn’t guarantee that elements are spread out.
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Where one element is placed shouldn’t impact where a second goes.

For any $x \in U$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over its codomain.

For any distinct $x, y \in U$ and random $h \in \mathcal{H}$, $h(x)$ and $h(y)$ are independent random variables.
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A family of hash functions $\mathcal{H}$ is called **2-independent** (or **pairwise independent**) if it satisfies the distribution and independence properties.

```
0 1 2 3 4 5 6 7 ... m-1
```
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2-independence means any pair of elements is unlikely to collide.
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**Question:** Where did these elements collide with one another?

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This is the same as if $h$ were a truly random function.

For any distinct $x, y \in \mathcal{U}$ and random $h \in \mathcal{H}$, $h(x)$ and $h(y)$ are independent random variables.
For more on hashing outside of Theoryland, check out this Stack Exchange post.
Frequency Estimation
Frequency Estimators

- A frequency estimator is a data structure supporting the following operations:
  - \texttt{increment}(x), which increments the number of times that \(x\) has been seen, and
  - \texttt{estimate}(x), which returns an estimate of the frequency of \(x\).

- Using BSTs, we can solve this in space \(\Theta(n)\) with worst-case \(O(\log n)\) costs on the operations.

- Using hash tables, we can solve this in space \(\Theta(n)\) with expected \(O(1)\) costs on the operations.
Frequency Estimators

• Frequency estimation has many applications:
  • Search engines: Finding frequent search queries.
  • Network routing: Finding common source and destination addresses.

• In these applications, \( \Theta(n) \) memory can be impractical.

• **Goal:** Get *approximate* answers to these queries in sublinear space.
The Count-Min Sketch
# How to Build an Estimator

## Step One: Build a Simple Estimator

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<th>Count-Min Sketch</th>
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Revisiting the Exact Solution

In the exact solution to the frequency estimation problem, we maintained a single counter for each distinct element. This is too space-inefficient.

**Idea:** Store a fixed number of counters and assign a counter to each $x \in \mathcal{U}$. Multiple objects might be assigned to the same counter.

- To **increment**($x$), increment the counter for $x$.
- To **estimate**($x$), read the value of the counter for $x$. 

![Diagram with counters and objects]

11, 6, 4, 7
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\begin{figure}
\begin{tabular}{|c|c|c|c|}
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12 & 6 & 4 & 7 \\
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• To *increment*($x$), increment the counter for $x$.

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![Diagram with counters and objects]

12 6 5 7
Our Initial Structure

- Create an array of counters, all initially 0, called \textit{count}. It will have \( w \) elements for some \( w \) we choose later.
- Choose, from a family of 2-independent hash functions \( \mathcal{H} \), a uniformly-random hash function \( h : \mathcal{U} \rightarrow [w] \).
- To \textit{increment}(x), increment \textit{count}[h(x)].
- To \textit{estimate}(x), return \textit{count}[h(x)].
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# How to Build an Estimator

## Count-Min Sketch

| Step One: Build a Simple Estimator | Hash items to counters; add +1 when item seen. |
| Step Two: Compute Expected Value of Estimator | |

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**Step One:**
Build a Simple Estimator

**Step Two:**
Compute Expected Value of Estimator

---

**Count-Min Sketch**

| Count-Min Sketch | |
|------------------| |
| Hash items to counters; add +1 when item seen. | |
Some Notation

- Let $x_1, x_2, x_3, \ldots$ denote the list of distinct items whose frequencies are being stored.
- Let $a_1, a_2, a_3, \ldots$ denote the frequencies of those items.
  - e.g. $a_i$ is the true number of times $x_i$ is seen.
- Let $\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots$ denote the estimate our data structure gives for the frequency of each item.
  - e.g. $\hat{a}_i$ is our estimate for how many times $x_i$ has been seen.
  - **Important detail:** the $a_i$ values are not random variables (data are chosen adversarially), while the $\hat{a}_i$ values are random variables (they depend on a randomly-sampled hash function).
- In what follows, imagine we’re querying the frequency of some specific element $x_i$. We want to analyze $\hat{a}_i$. 
Analyzing our Estimator

- We’re interested in learning more about $\hat{a}_i$. A good first step is to work out $E[\hat{a}_i]$.
- $\hat{a}_i$ will be equal to $a_i$, plus some “noise” terms from colliding elements.
- Each of those elements is very unlikely to collide with us, though. (There’s a $\frac{1}{w}$ chance of a collision for any one other element.)
- **Reasonable guess:** $E[\hat{a}_i] = a_i + \sum_{j \neq i} \frac{a_j}{w}$
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  Frequency of each other item, scaled to account for chance of a collision.
Let’s make this more rigorous.

For each element $x_j$:

- If $h(x_i) = h(x_j)$, then $x_j$ contributes $a_j$ to $\text{count}[h(x_i)]$.
- If $h(x_i) \neq h(x_j)$, then $x_j$ contributes 0 to $\text{count}[h(x_i)]$. 
Making Things Formal

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- To pin this down precisely, let’s define a set of random variables $X_1, X_2, \ldots$, as follows:

$$X_j = \begin{cases} 
1 & \text{if } h(x_i) = h(x_j) \\
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\end{cases}$$
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Each of these variables is called an \textit{indicator random variable}, since it “indicates” whether some event occurs.
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  X_j = \begin{cases} 
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  \end{cases}
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- The value of $\hat{a}_i - a_i$ is then given by
  \[
  \hat{a}_i - a_i = \sum_{j \neq i} a_j X_j
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\[ E[\hat{a}_i - a_i] = E[\sum_{j \neq i} a_j X_j] \]
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This follows from \textit{linearity of expectation}. We’ll use this property extensively over the next few days.
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The values of $a_j$ are not random. *The randomness comes from our choice of hash function.*
\[ E[\hat{a}_i - a_i] = E[\sum_{j \neq i} a_j X_j] \]

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$\mathbb{E}[X_j] = \text{...}$
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E[X_j] = \begin{cases} 
1 & \text{if } h(x_i) = h(x_j) \\
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If \( X \) is an indicator variable for some event \( \mathcal{E} \), then \( E[X] = Pr[\mathcal{E}] \). This is really useful when using linearity of expectation!
\[ E[\hat{a}_i - a_i] = E[\sum_{j \neq i} a_j X_j] \]
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**Idea:** Think of our element frequencies \( a_1, a_2, a_3, \ldots \) as a vector \( a = [a_1, a_2, a_3, \ldots] \).

The total number of objects is the sum of the vector entries.

This is called the \textbf{L\textsubscript{1} norm} of \( a \), and is denoted \( \|a\|_1 \):

\[ \|a\|_1 = \sum_i |a_i| \]
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\[ \leq \frac{\|a\|_1}{w} \]

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# How to Build an Estimator

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| **Step Three:** | Apply Concentration Inequality | }
On Expected Values

- We know that $E[\hat{a}_i - a_i] \leq ||a||_1 / w$. This means that the expected overestimate is low.

  **Claim:** This fact, in isolation, is not very useful.

- Below is a probability distribution for a random variable whose expected value is 9 that never takes values near 9.

- If this is the sort of distribution we get for $\hat{a}_i$, then our estimator is not very useful!
On Expected Values

- We’re looking for a way to say something like the following:
  
  “Not only is our estimate’s expected value pretty close to the real value, our estimate has a high probability of being close to the real value.”

- In other words, if the true frequency is 9, we want the distribution of our estimate to kinda sorta look like this:

If the true frequency is 9, why isn’t there any probability mass below 9?

Answer at [https://pollev.com/cs166spr23](https://pollev.com/cs166spr23)
On Expected Values

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How Close is Close?

- In some applications, we might be okay overshooting by a larger amount (e.g. roughly estimating which restaurants people are visiting).
- In others, it’s really bad if we overestimate by too much (e.g. polling for an election).
- **Idea:** Allow the client of the estimator to pick some value $\varepsilon$ between 0 and 1 indicating how close they want to be to the true value. The closer $\varepsilon$ is to 0, the better the approximation we want.
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How Close is Close?

- Our overestimate is related to $||a||_1$.
- We’ll formalize how $\varepsilon$ works as follows: we’ll say we’re okay with any estimate that’s within $\varepsilon||a||_1$ of the true value.
- This is okay for high-frequency elements, but not so great for low-frequency elements. (Why?)
- But that’s okay. In practice, we are most interested in finding the high-frequency items.
We know that

\[ E[\hat{a}_i - a_i] \leq \frac{\|a\|_1}{W} \]

We want to bound this quantity:

\[ \Pr[\hat{a}_i - a_i > \varepsilon \|a\|_1] \]

Let’s run the numbers!
\[ \Pr \left[ \hat{a}_i - a_i > \varepsilon \| a \|_1 \right] \]
\[
\Pr \left[ \hat{a}_i - a_i > \varepsilon \|a\|_1 \right]
\]
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We don’t know the exact distribution of this random variable.

However, we have a \textit{one-sided error}: our estimate can never be lower than the true value. This means that \( \hat{a}_i - a_i \geq 0 \).

\textit{Markov’s inequality} says that if \( X \) is a nonnegative random variable, then

\[
\Pr[ X \geq c ] \leq \frac{\mathbb{E}[X]}{c}.
\]
\[
\Pr [ \hat{a}_i - a_i > \varepsilon \| a \|_1 ] \\
\leq \frac{\mathbb{E} [ \hat{a}_i - a_i ]}{\varepsilon \| a \|_1}
\]

We don’t know the exact distribution of this random variable.

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\]

\[E[\hat{a}_i - a_i] \leq \frac{\|a\|_1}{w} \]
$$\Pr \left[ \hat{a}_i - a_i > \varepsilon \|a\|_1 \right]$$

$$\leq \frac{\mathbb{E} \left[ \hat{a}_i - a_i \right]}{\varepsilon \|a\|_1}$$

$$\leq \frac{\|a\|_1}{\omega} \cdot \frac{1}{\varepsilon \|a\|_1}$$
\[
\Pr \left[ \hat{a}_i - a_i > \varepsilon \|a\|_1 \right] \\
\leq \frac{\mathbb{E} \left[ \hat{a}_i - a_i \right]}{\varepsilon \|a\|_1} \\
\leq \frac{\|a\|_1}{w} \cdot \frac{1}{\varepsilon \|a\|_1} \\
= \frac{1}{\varepsilon w}
\]
Interpreting this Result

- Here’s what we just proved:
  \[
  \Pr \left[ \hat{a}_i - a_i > \varepsilon \|a\|_1 \right] \leq \frac{1}{\varepsilon w}
  \]

- What does this tell us?
  - Increasing \( w \) decreases the chance of an overestimate. Decreasing \( w \) increases the chance of an overestimate.
  - As the user decreases \( \varepsilon \), we have to proportionally increase \( w \) for this bound to tell us anything useful.

- **Idea:** Choose \( w = e \cdot \varepsilon^{-1} \).
  - The choice of \( e \) is “somewhat” arbitrary in that any constant will work – but I peeked ahead and there’s a good reason to choose \( e \) here.
Interpreting this Result

• Here’s what we just proved:

\[
\Pr \left[ \hat{a}_i - a_i > \varepsilon \|a\|_1 \right] \leq e^{-1}
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  • Increasing \( w \) decreases the chance of an overestimate. Decreasing \( w \) increases the chance of an overestimate.
  
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  • The choice of \( e \) is “somewhat” arbitrary in that any constant will work – but I peeked ahead and there’s a good reason to choose \( e \) here.
The Story So Far

• The user chooses a value $\varepsilon \in (0, 1)$. We pick $w = e \cdot \varepsilon^{-1}$.
• Create an array `count` of $w$ counters, each initially zero.
• Choose, from a family of 2-independent hash functions $\mathcal{H}$, a uniformly-random hash function $h : \mathcal{U} \to [w]$.
• To `increment`(x), increment `count`[h(x)].
• To `estimate`(x), return `count`[h(x)].
• With probability at least $1 - \frac{1}{e}$, the estimate for the frequency of item $x_i$ is within $\varepsilon \cdot \|a\|_1$ of the true frequency.

\[ w = O(\varepsilon^{-1}) \text{ counters} \]

\[
\begin{array}{llllll}
  h & \quad & 31 & 41 & 59 & 26 & \ldots & 58 \\
\end{array}
\]
# How to Build an Estimator

<table>
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<th>Step Four: Boost Confidence</th>
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The Story So Far

• We now have a simple estimator where
  \[ \Pr[\hat{a}_i - a_i > \varepsilon \|a\|_1] \leq e^{-1} \]

• This means we have a decent chance of getting an estimate we’re happy with.

• **Problem:** We probably want to be more confident than this.
  
  • In some applications, maybe it’s okay to have a 63% success rate.
  
  • In others (say, election polling) we’ll need to be a lot more confident than this.

• **Question:** How do you define “confident enough”?
The Parameter $\delta$

• The user already can select a parameter $\varepsilon$ tuning the accuracy of the estimator: how close we want to be to the true value.

• Let’s have them also select a parameter $\delta$ tuning the confidence of the estimator: how likely it is that we achieve this goal.

• $\delta$ ranges from 0 to 1. Lower $\delta$ means a higher chance of getting a good estimate.
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Our Goal

• Right now, we have this statement:
  \[ \Pr [ \hat{a}_i - a_i > \varepsilon \|a\|_1 ] \leq e^{-1} \]

• We want to get to this one:
  \[ \Pr [ \hat{a}_i - a_i > \varepsilon \|a\|_1 ] \leq \delta \]

• How might we achieve this?
A Key Technique
It’s *super unlikely* that you’ll miss the center of the target every single time!
Running in Parallel

- Let’s run $d$ copies of our data structure in parallel with one another.
- Each row has its hash function sampled uniformly at random from our hash family.
- Each time we **increment** an item, we perform the corresponding **increment** operation on each row.

$$w = \lceil e \cdot \varepsilon^{-1} \rceil$$

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d &= ???
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\[
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_d$</td>
<td>69</td>
<td>31</td>
<td>47</td>
<td>18</td>
<td>5</td>
<td>...</td>
<td>60</td>
</tr>
</tbody>
</table>
Running in Parallel

- Imagine we call \textit{estimate}(\chi) on each of our estimators and get back these estimates.
- We need to give back a single number.
- \textbf{Question:} How should we aggregate these numbers into a single estimate?

\begin{itemize}
  \item \textbf{Estimator 1:} 137
  \item \textbf{Estimator 2:} 271
  \item \textbf{Estimator 3:} 166
  \item \textbf{Estimator 4:} 103
  \item \textbf{Estimator 5:} 261
\end{itemize}

Answer at \url{https://pollev.com/cs166spr23}
Running in Parallel

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\end{itemize}

\textbf{Intuition:} The smallest estimate returned has the least “noise,” and that’s the best guess for the frequency.
Let $\hat{a}_{ij}$ be the estimate from the $j$th copy of the data structure. Our final estimate is $\min \{ \hat{a}_{ij} \}$.
\[ \Pr \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \| a \|_1 \right] \]

Let \( \hat{a}_{ij} \) be the estimate from the \( j \)th copy of the data structure.

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\Pr \left[ \min \left\{ \hat{a}_{ij} \right\} - a_i > \varepsilon \|a\|_1 \right]
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The only way the minimum estimate is inaccurate is if every estimate is inaccurate.

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\Pr \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \|a\|_1 \right] \\
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\[
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Our final estimate is

$$\min \{ \hat{a}_{ij} \}$$

Each copy of the data structure is independent of the others.
\[ \Pr \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \|a\|_1 \right] \]

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= \prod_{j=1}^{d} \Pr \left[ \hat{a}_{ij} - a_i > \epsilon \| a \|_1 \right]
\]

\[
\Pr[\hat{a}_i - a_i \geq \epsilon \| a \|_1] \leq e^{-1}
\]

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Our final estimate is \( \min \{ \hat{a}_{ij} \} \).
\[
\Pr \left[ \min \left\{ \hat{a}_{ij} \right\} - a_i > \varepsilon \left\| a \right\|_1 \right]
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= \Pr \left[ \bigwedge_{j=1}^{d} \left( \hat{a}_{ij} - a_i > \varepsilon \left\| a \right\|_1 \right) \right]
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\[
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\]

\[
\leq \prod_{j=1}^{d} e^{-1}
\]

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Our final estimate is \( \min \left\{ \hat{a}_{ij} \right\} \).
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Let $\hat{a}_{ij}$ be the estimate from the $j$th copy of the data structure.

Our final estimate is $\min \{ \hat{a}_{ij} \}$.
Finishing Touches

• We now see that
  \[ \Pr [ \hat{a}_i - a_i > \varepsilon \| a \|_1 ] \leq e^{-d} \]

• We want to reach this goal:
  \[ \Pr [ \hat{a}_i - a_i > \varepsilon \| a \|_1 ] \leq \delta \]

• So set \( d = \ln \delta^{-1} \).
The Count-Min Sketch

\[ w = \lceil e \cdot \varepsilon^{-1} \rceil \]

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>41</td>
<td>59</td>
<td>26</td>
</tr>
<tr>
<td>27</td>
<td>18</td>
<td>28</td>
<td>18</td>
</tr>
<tr>
<td>16</td>
<td>18</td>
<td>3</td>
<td>39</td>
</tr>
<tr>
<td>69</td>
<td>31</td>
<td>47</td>
<td>18</td>
</tr>
</tbody>
</table>

Sampled uniformly and independently from a 2-independent family of hash functions
The Count-Min Sketch

\[
\begin{array}{ccccccc}
 h_1 & 31 & 41 & 59 & 26 & 53 & \ldots & 58 \\
h_2 & 27 & 18 & 28 & 18 & 28 & \ldots & 45 \\
h_3 & 16 & 18 & 3 & 39 & 88 & \ldots & 75 \\
\ldots & \ldots & \ldots & \ldots \\
h_d & 69 & 31 & 47 & 18 & 5 & \ldots & 59 \\
\end{array}
\]

**increment(x):**
\[
\text{for } i = 1 \ldots d:
\text{count}[i][h_i(x)]++
\]
The Count-Min Sketch

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>31</th>
<th>41</th>
<th>59</th>
<th>26</th>
<th>53</th>
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<tbody>
<tr>
<td>$h_2$</td>
<td>27</td>
<td>18</td>
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<td>18</td>
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**increment**(x):

```python
for i = 1 ... d:
    count[i][h_i(x)]++
```
The Count-Min Sketch

increment(x):
    for i = 1 … d:
        count[i][h_i(x)]++
The Count-Min Sketch

increment(x):
    for i = 1 ... d:
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The Count-Min Sketch

\[ h_1 \]

\[ h_2 \]

\[ h_3 \]

\[ \ldots \]

\[ h_d \]

\[
\begin{array}{ccccccc}
32 & 41 & 59 & 26 & 53 & \ldots & 58 \\
27 & 18 & 28 & 19 & 28 & \ldots & 45 \\
16 & 19 & 3 & 39 & 88 & \ldots & 75 \\
69 & 31 & 47 & 18 & 5 & \ldots & 60 \\
\end{array}
\]

**increment**\( (x) \):

\[
\text{for } i = 1 \ldots d:
\text{count}[i][h_i(x)]++
\]

**estimate**\( (x) \):

\[
\text{result} = \infty \\
\text{for } i = 1 \ldots d:
\text{result} = \min(\text{result}, \text{count}[i][h_i(x)]) \\
\text{return } \text{result}
\]
The Count-Min Sketch

increment(x):
    for i = 1 ... d:
        count[i][h_i(x)]++

estimate(x):
    result = ∞
    for i = 1 ... d:
        result = min(result, count[i][h_i(x)])
    return result
The Count-Min Sketch

- Update and query times are $\Theta(\log \delta^{-1})$.
  - That’s the number of replicated copies, and we do $O(1)$ work at each.
- Space usage: $\Theta(\varepsilon^{-1} \cdot \log \delta^{-1})$ counters.
  - Each individual estimator has $\Theta(\varepsilon^{-1})$ counters, and we run $\Theta(\log \delta^{-1})$ copies in parallel.
- Provides an estimate to within $\varepsilon \|a\|_1$ with probability at least $1 - \delta$.
- This can be *significantly* better than just storing a raw frequency count – especially if your goal is to find items that appear very frequently.
How to Build an Estimator

<table>
<thead>
<tr>
<th>Step One: Build a Simple Estimator</th>
<th>Count-Min Sketch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hash items to counters; add +1 when item seen.</td>
<td></td>
</tr>
</tbody>
</table>

| Step Two: Compute Expected Value of Estimator | |
| Sum of indicators; 2-independent hashes have low collision rate. |

| Step Three: Apply Concentration Inequality | |
| One-sided error; use expected value and Markov’s inequality. |

| Step Four: Replicate to Boost Confidence | |
| Take min; only fails if all estimates are bad. |
Major Ideas From Today

- **2-independent hash families** are useful when we want to keep collisions low.

- A “good” approximation of some quantity should have tunable *confidence* and *accuracy* parameters.

- **Sums of indicator variables** are useful for deriving expected values of estimators.

- **Concentration inequalities** like *Markov’s inequality* are useful for showing estimators don’t stay too much from their expected values.

- Good estimators can be built from *multiple parallel copies* of weaker estimators.
Next Time

- **Count Sketches**
  - An alternative frequency estimator with different time/space bounds.

- **Cardinality Estimation**
  - Estimating how many different items you’ve seen in a data stream.