Count-Min Sketches
Randomization

• Randomization opens up new routes for tradeoffs in data structures:
  • Trade worst-case guarantees for average-case guarantees.
  • Trade exact answers for approximate answers.
• These data structures are used \textit{extensively} in practice. Each of the next lectures is on something you’re likely to encounter IRL.
• Each of the next lectures explores powerful techniques that are useful in navigating the rivers of Theoryland.
Outline for Today

• **Hash Functions**
  • Understanding our basic building blocks.

• **Count-Min Sketches**
  • Estimating how many times we’ve seen something.

• **Concentration Inequalities**
  • “Correct on expectation” versus “correct with high probability.”

• **Probability Amplification**
  • Increasing our confidence in our answers.
Preliminaries: *Hash Functions*
Hashing in Practice

• Hash functions are used extensively in programming and software engineering:
  • They make hash tables possible: think C++ std::hash, Python’s __hash__, or Java’s Object.hashCode().
  • They’re used in cryptography: SHA-256, HMAC, etc.

• **Question:** When we’re in Theoryland, what do we mean when we say “hash function?”
Hashing in Theoryland

- In Theoryland, a hash function is a function from some domain called the **universe** (typically denoted $\mathcal{U}$) to some codomain.

- The codomain is usually a set of the form
  \[ [m] = \{0, 1, 2, 3, \ldots, m - 1\} \]

\[
h : \mathcal{U} \rightarrow [m]
\]
Hashing in Theoryland

- **Intuition:** No matter how clever you are with designing a hash function, that hash function isn’t random, and so there will be pathological inputs.
  - You can formalize this with the pigeonhole principle.
- **Idea:** Rather than finding the One True Hash Function, we’ll assume we have a collection of hash functions to pick from, and we’ll choose which one to use randomly.
Families of Hash Functions

- A family of hash functions is a set $\mathcal{H}$ of hash functions with the same domain and codomain.
- We can then introduce randomness into our data structures by sampling a random hash function from $\mathcal{H}$.
- **Key Point:** The randomness in our data structures almost always derives from the random choice of hash functions, not from the data.

  *Data is adversarial.  
  Hash function selection is random.*

- **Question:** What makes a family of hash functions $\mathcal{H}$ a “good” family of hash functions?
Goal: If we pick $h \in \mathcal{H}$ uniformly at random, then $h$ should distribute elements uniformly randomly.
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**Problem:** A hash function that distributes $n$ elements uniformly at random over $[m]$ requires $\Omega(n \log m)$ space in the worst case.

**Question:** Do we actually need true randomness? Or can we get away with something weaker?
**Distribution Property**: Each element should have an equal probability of being placed in each slot.

For any $x \in \mathcal{U}$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over $[m]$. 
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Some “obviously bad” hash functions obey this rule. How is this possible?
**Distribution Property:** Each element should have an equal probability of being placed in each slot.

**Problem:** This rule doesn’t guarantee that elements are spread out.

For any $x \in \mathcal{U}$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over $[m]$. 
**Distribution Property:** Each element should have an equal probability of being placed in each slot.

For any $x \in U$ and random $h \in H$, the value of $h(x)$ is uniform over $[m]$.

**Problem:** This rule doesn’t guarantee that elements are spread out.

Diagram: 0 1 2 3 4 5 6 7 ... m-1 with elements W, Z, y, x placed.
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**Problem:** This rule doesn’t guarantee that elements are spread out.
**Distribution Property:**
Each element should have an equal probability of being placed in each slot.

**Independence Property:**
Where one element is placed shouldn’t impact where a second goes.

For any $x \in \mathcal{U}$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over $[m]$.

For any distinct $x, y \in \mathcal{U}$ and random $h \in \mathcal{H}$, $h(x)$ and $h(y)$ are independent random variables.
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A family of hash functions $\mathcal{H}$ is called **2-independent** (or pairwise independent) if it satisfies the distribution and independence properties.
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For any distinct $x, y \in \mathcal{U}$ and random $h \in \mathcal{H}$, $h(x)$ and $h(y)$ are independent random variables.

**Intuition:**
2-independence means any pair of elements is unlikely to collide.
For any \( x \in \mathcal{U} \) and random \( h \in \mathcal{H} \), the value of \( h(x) \) is uniform over \([m]\).

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\[
\Pr[h(x) = h(y)] = \sum_{i=0}^{m-1} \Pr[h(x) = i] \cdot \Pr[h(y) = i] = \frac{1}{m}
\]

**Question:** Where did these elements collide with one another?
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```
0 1 2 ... m-1
```

\( y \)
\( x \)
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This is the same as if \( h \) were a truly random function.

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2-independence means any pair of elements is unlikely to collide.
For more on hashing outside of Theoryland, check out this Stack Exchange post.
Approximating Quantities
What makes for a good “approximate” solution?
What does it mean for an approximation to be “good”?

Let $A$ be the true answer. Let $\hat{A}$ be a random variable denoting our estimate.

This would not make for a good estimate. However, we have $E[\hat{A}] = A$.

*Observation 1:* Being correct in expectation isn’t sufficient.

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This would not make for a good estimate. However, we have $E[\hat{A}] = A$.

*Observation 1:* Being correct in expectation isn’t sufficient.
Let $A$ be the true answer. Let $\hat{A}$ be a random variable denoting our estimate. It’s unlikely that we’ll get the right answer, but we’re probably going to be close.

**Observation 2:** The difference $|\hat{A} - A|$ between our estimate and the truth should ideally be small.

What does it mean for an approximation to be “good”?
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Let $A$ be the true answer. Let $\hat{A}$ be a random variable denoting our estimate.

This estimate skews low, but it’s very close to the true value.

**Observation 3:** An estimate doesn’t have to be unbiased to be useful.
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Let $A$ be the true answer. Let $\hat{A}$ be a random variable denoting our estimate.

The more resources we allocate, the better our estimate should be.

Observation 4: A good approximation should be tunable.

Memory used: 256MB
What does it mean for an approximation to be “good”?

Suppose there are two tunable values

\[ \varepsilon \in (0, 1] \]
\[ \delta \in (0, 1] \]

where \( \varepsilon \) represents \textit{accuracy} and \( \delta \) represents \textit{confidence}.

\textbf{Goal:} Make an estimator \( \hat{A} \) for some quantity \( A \) where

With probability at least \( 1 - \delta \),

\[ |\hat{A} - A| \leq \varepsilon \cdot \text{size}(\text{input}) \]

for some measure of the size of the input.
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What does it mean for an approximation to be “good”? 

---

**Probably**
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**Goal:** Make an estimator \( \hat{A} \) for some quantity \( A \) where

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for some measure of the size of the input.

\( \delta = \frac{1}{2} \)

\( \varepsilon \) small
**Goal:** Make an estimator $\hat{A}$ for some quantity $A$ where

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\[ \delta = \frac{1}{2} \]

\[ \varepsilon \text{ medium} \]

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Goal: Make an estimator $\hat{A}$ for some quantity $A$ where

$$|A - \hat{A}| \leq \varepsilon \cdot \text{size(input)}$$

for some measure of the size of the input.

$\delta = \frac{1}{2}$
$\varepsilon$ large

Probably
Approximately
Correct

True answer
What does it mean for an approximation to be “good”?

**Goal:** Make an estimator $\hat{A}$ for some quantity $A$ where

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for some measure of the size of the input.

$\delta = \frac{1}{2}$

$\varepsilon$ small

**Correct**

**Approximately**

**Probably**
**Goal:** Make an estimator \( \hat{A} \) for some quantity \( A \) where

\[
\text{With probability at least } 1 - \delta, \quad |A - \hat{A}| \leq \varepsilon \cdot \text{size(input)}
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for some measure of the size of the input.

\( \delta = \frac{1}{4} \)

\( \varepsilon \) small

What does it mean for an approximation to be “good”?
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|A - \hat{A}| \leq \varepsilon \cdot \text{size(input)}
\]

for some measure of the size of the input.

\( \delta = \frac{1}{16} \)
\( \varepsilon \) small

What does it mean for an approximation to be “good”?
Frequency Estimation
Frequency Estimators

- A frequency estimator is a data structure supporting the following operations:
  - \textit{increment}(x), which increments the number of times that \textit{x} has been seen, and
  - \textit{estimate}(x), which returns an estimate of the frequency of \textit{x}.

- Using BSTs, we can solve this in space $\Theta(n)$ with worst-case $O(\log n)$ costs on the operations.

- Using hash tables, we can solve this in space $\Theta(n)$ with expected $O(1)$ costs on the operations.
Frequency Estimators

• Frequency estimation has many applications:
  • Search engines: Finding frequent search queries.
  • Network routing: Finding common source and destination addresses.

• In these applications, $\Theta(n)$ memory can be impractical.

• **Goal**: Get approximate answers to these queries in sublinear space.
The Count-Min Sketch
How to Build an Estimator

1. Design a simple data structure that, intuitively, gives you a good estimate.

2. Use a *sum of indicator variables* and *linearity of expectation* to prove that, on expectation, the data structure is pretty close to correct.

3. Use a *concentration inequality* to show that, with decent probability, the data structure’s output is close to its expectation.

4. Run multiple copies of the data structure in parallel to amplify the success probability.
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Revisiting the Exact Solution

- In the exact solution to the frequency estimation problem, we maintained a single counter for each distinct element. This is too space-inefficient.

- **Idea:** Store a fixed number of counters and assign a counter to each $x_i \in \mathcal{U}$. Multiple $x_i$'s might be assigned to the same counter.

- To **increment**($x$), increment the counter for $x$.

- To **estimate**($x$), read the value of the counter for $x$. 

---

![Diagram with counters and numbers: 11, 6, 4, 7]
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• **Idea:** Store a fixed number of counters and assign a counter to each \( x_i \in \mathcal{U} \). Multiple \( x_i \)'s might be assigned to the same counter.

• To **increment** \( (x) \), increment the counter for \( x \).

• To **estimate** \( (x) \), read the value of the counter for \( x \).
Our Initial Structure

- We can model “assigning each $x_i$ to a counter” by using hash functions.
- Choose, from a family of 2-independent hash functions $\mathcal{H}$, a uniformly-random hash function $h : \mathcal{U} \rightarrow [w]$.
- Create an array `count` of $w$ counters, each initially zero.
  - We'll choose $w$ later on.
- To `increment`(x), increment `count`[$h(x)$].
- To `estimate`(x), return `count`[$h(x)$].
Analyzing our Structure
For each \( x_i \in \mathcal{U} \), let \( a_i \) denote the number of times we’ve seen \( x_i \).

Similarly, let \( \hat{a}_i \) denote our estimated value of the frequency of \( x_i \).

**Goal:** Bound the probability that the error \((\hat{a}_i - a_i)\) is too high.
**Question:** Intuitively, what should we expect our approximation error to be?

**Idea:** Think of our element frequencies $a_1, a_2, a_3, \ldots$ as a vector $a = [a_1, a_2, a_3, \ldots]$. The total number of objects is the sum of the vector entries. This is called the **$L_1$ norm** of $a$, and is denoted $\|a\|_1$:

$$\|a\|_1 = \sum_i |a_i|$$

There are $\|a\|_1$ total elements distributed across $w$ buckets. We’re using a 2-independent hash family.

**Reasonable guess:** each bin has $\|a\|_1 / w$ elements in it, so $\hat{a}_i - a_i \leq \|a\|_1 / w$
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Analyzing this Structure

• Let's look at $\hat{a}_i = \text{count}[h(x_i)]$ for some choice of $x_i$.
• For each element $x_j$:
  • If $h(x_i) = h(x_j)$, then $x_j$ contributes $a_j$ to $\text{count}[h(x_i)]$.
  • If $h(x_i) \neq h(x_j)$, then $x_j$ contributes 0 to $\text{count}[h(x_i)]$.
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  - If $h(x_i) \neq h(x_j)$, then $x_j$ contributes 0 to $\text{count}[h(x_i)]$.
- To pin this down precisely, let’s define a set of random variables $X_1, X_2, \ldots$, as follows:
  
  $$X_j = \begin{cases} 
    1 & \text{if } h(x_i) = h(x_j) \\
    0 & \text{otherwise}
  \end{cases}$$

Each of these variables is called an *indicator random variable*, since it “indicates” whether some event occurs.
Analyzing this Structure

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• For each element $x_j$:
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$$X_j = \begin{cases} 
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\end{cases}$$

• The value of $\hat{a}_i - a_i$ is then given by

$$\hat{a}_i - a_i = \sum_{j \neq i} a_j X_j$$
\[ E[\hat{a}_i - a_i] = E[\sum_{j \neq i} a_j X_j] \]
\[
E[\hat{a}_i - a_i] = E[\sum_{j \neq i} a_j X_j] \\
= \sum_{j \neq i} E[a_j X_j]
\]

This follows from **linearity of expectation**. We’ll use this property extensively over the next few days.
\[ E[\hat{a}_i - a_i] = \mathbb{E}[\sum_{j \neq i} a_j X_j] \]

\[ = \sum_{j \neq i} \mathbb{E}[a_j X_j] \]

\[ = \sum_{j \neq i} a_j \mathbb{E}[X_j] \]

The values of \( a_j \) are not random. The randomness comes from our choice of hash function.
\[
E[\hat{a}_i - a_i] = E[\sum_{j \neq i} a_j X_j] \\
= \sum_{j \neq i} E[a_j X_j] \\
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\]

\[
E[X_j] =
\]
\[ \mathbb{E}[\hat{a}_i - a_i] = \mathbb{E}\left[ \sum_{j \neq i} a_j X_j \right] \]

\[ = \sum_{j \neq i} \mathbb{E}[a_j X_j] \]

\[ = \sum_{j \neq i} a_j \mathbb{E}[X_j] \]

\[ \mathbb{E}[X_j] = \begin{cases} 
1 & \text{if } h(x_i) = h(x_j) \\
0 & \text{otherwise}
\end{cases} \]
$$E[\hat{a}_i - a_i] = E[\sum_{j \neq i} a_j X_j]$$

$$= \sum_{j \neq i} E[a_j X_j]$$

$$= \sum_{j \neq i} a_j E[X_j]$$

$$E[X_j] = 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)]$$

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\[ E[X_j] = 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)] = \Pr[h(x_i) = h(x_j)] \]

If \( X \) is an indicator variable for some event \( \mathcal{E} \), then \( E[X] = \Pr[\mathcal{E}] \). This is really useful when using linearity of expectation!
\[
\mathbb{E}[\hat{a}_i - a_i] = \mathbb{E}\left[\sum_{j \neq i} a_j X_j \right] \\
= \sum_{j \neq i} \mathbb{E}[a_j X_j] \\
= \sum_{j \neq i} a_j \mathbb{E}[X_j]
\]

\[
\mathbb{E}[X_j] = 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)] \\
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$$E[X_j] = 1 \cdot \Pr[h(x_i)=h(x_j)] + 0 \cdot \Pr[h(x_i)\neq h(x_j)]$$

$$= \Pr[h(x_i)=h(x_j)]$$

$$= \frac{1}{w}$$

Hey, we saw this earlier!
$$E[\hat{a}_i-a_i] = E[\sum_{j \neq i} a_j X_j]$$

$$= \sum_{j \neq i} E[a_j X_j]$$

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$$= \sum_{j \neq i} \frac{a_j}{w}$$

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\[ = \sum_{j \neq i} a_j E[X_j] \]
\[ = \sum_{j \neq i} \frac{a_j}{w} \]
\[ \leq \frac{\|a\|_1}{w} \]

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How to Build an Estimator

1. Design a simple data structure that, intuitively, gives you a good estimate.

2. Use a *sum of indicator variables* and *linearity of expectation* to prove that, on expectation, the data structure is pretty close to correct.

3. Use a *concentration inequality* to show that, with decent probability, the data structure’s output is close to its expectation.

4. Run multiple copies of the data structure in parallel to amplify the success probability.
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Goal: Make an estimator $\hat{a}$ for some quantity $a$ where

With probability at least $1 - \delta$,

$$|\hat{a} - a| \leq \varepsilon \cdot \text{size}(\text{input})$$

for some measure of the size of the input.

How do we tune $w$ so we’re likely to fall in this range?
\[ \text{Pr}[\hat{a}_i - a_i > \varepsilon \|a\|_1] \]
\[
\Pr \left[ \hat{a}_i - a_i > \varepsilon \|a\|_1 \right]
\]
\[
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\]

We don’t know the exact distribution of this random variable.

However, we have a one-sided error: our estimate can never be lower than the true value. This means that \( \hat{a}_i - a_i \geq 0 \).

**Markov’s inequality** says that if \( X \) is a nonnegative random variable, then

\[
\Pr[ X \geq c ] \leq \frac{\mathbb{E}[X]}{c}.
\]
\[ \Pr \left[ \hat{a}_i - a_i > \varepsilon \| a \|_1 \right] \]

\[ \leq \frac{E \left[ \hat{a}_i - a_i \right]}{\varepsilon \| a \|_1} \]

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\[ \Pr \left[ \hat{a}_i - a_i > \varepsilon \| a \|_1 \right] \leq \frac{\mathbb{E} \left[ \hat{a}_i - a_i \right]}{\varepsilon \| a \|_1} \]

\[ \mathbb{E} \left[ \hat{a}_i - a_i \right] \leq \frac{\| a \|_1}{w} \]
\[
\Pr [\hat{a}_i - a_i > \varepsilon \|a\|_1] \\
\leq \frac{E [\hat{a}_i - a_i]}{\varepsilon \|a\|_1} \\
\leq \frac{\|a\|_1}{w} \cdot \frac{1}{\varepsilon \|a\|_1}
\]

\[
E [\hat{a}_i - a_i] \leq \frac{\|a\|_1}{w}
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\[
\Pr \left[ \hat{a}_i - a_i > \varepsilon \| a \|_1 \right] \\
\leq \frac{E \left[ \hat{a}_i - a_i \right]}{\varepsilon \| a \|_1} \\
\leq \frac{\| a \|_1}{\mathcal{W}} \cdot \frac{1}{\varepsilon \| a \|_1}
\]
\[ \Pr [ \hat{a}_i - a_i > \varepsilon \| a \|_1 ] \leq \frac{E [ \hat{a}_i - a_i ]}{\varepsilon \| a \|_1} \leq \frac{\| a \|_1 \cdot 1}{w \varepsilon \| a \|_1} = \frac{1}{\varepsilon w} \]
**Goal:** Make an estimator \( \hat{a} \) for some quantity \( a \) where

With probability at least \( 1 - \delta \),

\[
|\hat{a} - a| \leq \varepsilon \cdot \text{size(input)}
\]

for some measure of input size.

\[
\Pr[\hat{a}_i - a_i > \varepsilon \|a\|_1] \leq \frac{1}{\varepsilon w}
\]

**Initial Idea:**
Pick \( w = \varepsilon^{-1} \cdot \delta^{-1} \). Then

\[
\Pr[\hat{a}_i - a_i > \varepsilon \|a\|_1] \leq \delta
\]

Suppose we’re counting 1,000 distinct items.

If we want our estimate to be within \( \varepsilon \|a\|_1 \) of the true value with 99.9% probability, how much memory do we need?

**Answer:** \( 1,000 \cdot \varepsilon^{-1} \).

*Can we do better?*
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for some measure of input size.

\[
\Pr[\hat{a}_i - a_i > \varepsilon \|a\|_1] \leq \frac{1}{\varepsilon w}
\]

**Revised Idea**: Pick $w = e \cdot \varepsilon^{-1}$. Then

\[
\Pr[\hat{a}_i - a_i > \varepsilon \|a\|_1] < e^{-1}
\]

This simple data structure, by itself, is likely to be wrong.

What happens if we run a bunch of copies of this approach in parallel?
Running in Parallel

- Let's suppose that we run $d$ independent copies of this data structure. Each has its own independently randomly chosen hash function.
- To $\text{increment}(x)$ in the overall structure, we call $\text{increment}(x)$ on each of the underlying data structures.
- The probability that at least one of them provides a good estimate is quite high.
- **Question:** How do you know which one?

<table>
<thead>
<tr>
<th>Estimator 1:</th>
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<th>Estimator 3:</th>
<th>Estimator 4:</th>
<th>Estimator 5:</th>
</tr>
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<tr>
<td>137</td>
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<td>261</td>
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- The probability that at least one of them provides a good estimate is quite high.
- Question: How do you know which one?

Intuition: The smallest estimate returned has the least "noise," and that's the best guess for the frequency.

Estimator 1: 137
Estimator 2: 271
Estimator 3: 166
Estimator 4: 103
Estimator 5: 261
Let $\hat{a}_{ij}$ be the estimate from the $j$th copy of the data structure.

Our final estimate is $\min \{ \hat{a}_{ij} \}$.
\[
\Pr \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \| a \|_1 \right] = \prod_j \Pr \left[ \hat{a}_{ij} - a_i > \varepsilon \| a \|_1 \right] \leq \prod_j e^{-1} = e^{-d}
\]

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Our final estimate is

\[
\min \{ \hat{a}_{ij} \}
\]
\[ \Pr \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \|a\|_1 \right] \]

The only way the minimum estimate is inaccurate is if every estimate is inaccurate.

Let \( \hat{a}_{ij} \) be the estimate from the \( j \)th copy of the data structure.

Our final estimate is \( \min \{ \hat{a}_{ij} \} \)
\[ \text{Pr} \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \| a \|_1 \right] \]

\[ = \text{Pr} \left[ \bigwedge_{j=1}^{d} \left( \hat{a}_{ij} - a_i > \varepsilon \| a \|_1 \right) \right] \]

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\]
\[
\leq \prod_{j=1}^{d} e^{-1}
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\[
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**Goal:** Make an estimator \( \hat{a} \) for some quantity \( a \) where

\[
\text{With probability at least } 1 - \delta,
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\]

for some measure of input size.

\[
\Pr \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \|a\|_1 \right] \leq e^{-d}
\]

**Idea:** Choose \( d = -\ln \delta \).

(Equivalently: \( d = \ln \delta^{-1} \).) Then

\[
\Pr \left[ \min \{ \hat{a}_{ij} \} - a_i > \varepsilon \|a\|_1 \right] \leq \delta
\]
The Count-Min Sketch

Sampled uniformly and independently from a 2-independent family of hash functions

\[ w = \lceil e \cdot \varepsilon^{-1} \rceil \]

\[ d = \lceil \ln \delta^{-1} \rceil \]

<table>
<thead>
<tr>
<th>(h_1)</th>
<th>(h_2)</th>
<th>(h_3)</th>
<th>...</th>
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| 31 | 27 | 16 | ...
| 41 | 18 | 18 | ...
| 59 | 28 | 3 | ...
| 26 | 18 | 39 | ...
| 53 | 28 | 88 | ...
| ... | ... | ... | ... |
| 58 | 45 | 75 | 59 |
The Count-Min Sketch

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
h_1 & 31 & 41 & 59 & 26 & 53 & \ldots & 58 \\
\hline
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\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\hline
h_d & 69 & 31 & 47 & 18 & 5 & \ldots & 59 \\
\hline
\end{array}
\]

\texttt{increment}(x):
\begin{verbatim}
    for i = 1 \ldots d:
        count[i][h_i(x)]++
\end{verbatim}
The Count-Min Sketch

\[
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\]

**increment**\( (x) \):

\[
\text{for } i = 1 \ldots d:\n\quad \text{count}[i][h_i(x)]++
\]
The Count-Min Sketch

```
increment(x):
    for i = 1 ... d:
        count[i][h_i(x)]++
```
The Count-Min Sketch

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**increment**($x$):
```
for i = 1 ... d:
    count[i][h_i(x)]++
```

**estimate**($x$):
```
result = ∞
for i = 1 ... d:
    result = min(result, count[i][h_i(x)])
return result
```
The Count-Min Sketch

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The Count-Min Sketch

- Update and query times are \( \Theta(d) \), which is \( \Theta(\log \delta^{-1}) \).
- Space usage: \( \Theta(\varepsilon^{-1} \cdot \log \delta^{-1}) \) counters.
  - This is a major improvement over our earlier approach that used \( \Theta(\varepsilon^{-1} \cdot \delta^{-1}) \) counters.
  - This can be significantly better than just storing a raw frequency count!
- Provides an estimate to within \( \varepsilon \|a\|_1 \) with probability at least \( 1 - \delta \).
Major Ideas From Today

- **2-independent hash families** are useful when we want to keep collisions low.
- A “good” approximation of some quantity should have tunable confidence and accuracy parameters.
- **Sums of indicator variables** are useful for deriving expected values of estimators.
- **Concentration inequalities** like Markov’s inequality are useful for showing estimators don’t stay too much from their expected values.
- Good estimators can be built from multiple parallel copies of weaker estimators.
Next Time

- **Count Sketches**
  - An alternative frequency estimator with different time/space bounds.

- **Bloom Filters**
  - Storing data in close to theoretically optimal space.