x-Fast and y-Fast Tries
Outline for Today

- **Data Structures on Integers**
  - How can we speed up operations that work on integer data?
- **x-Fast Tries**
  - Bit manipulation meets tries and hashing.
- **y-Fast Tries**
  - Combining RMQ, strings, balanced trees, amortization, and randomization!
Many practical problems involve working specifically with integer values.

- **CPU Scheduling:** Each thread has some associated integer priority, and we need to maintain those priorities in sorted order.
- **Network Routing:** Each computer has an associated IP address, and we need to figure out which connections are active.
- **ID Management:** We need to store social security numbers, zip codes, phone numbers, credit card numbers, etc. and perform basic lookups and range searches on them.

We’ve seen many general-purpose data structures for keeping things in order and looking things up.

**Question:** Can we improve those data structures if we know in advance that we’re working with integer data?
Working with Integers

- Integers are interesting objects to work with:
  - Their values can directly be used as indices in lookup tables.
  - They can be treated as strings of bits, so we can use techniques from string processing.
  - They fit into machine words, so we can process the bits in parallel with individual word operations.
- The data structures we’ll explore over the next few lectures will give you a sense of what sorts of techniques are possible with integer data.
An Auxiliary Motive

• Integer data structures are also a great place to see just how much you’ve learned over the quarter!

• Today’s data structures cover every single unit from the quarter (RMQ, strings, balanced trees, amortization, and randomization).

• I hope this gives you a chance to pause and reflect on just how far you’ve come!
The Setup
Our Machine Model

• We will assume we’re working on a machine where memory is segmented into $w$-bit words.
  • Although on any one fixed machine $w$ is a constant, in general, don’t assume this is the case. 32-bit was the norm until fairly recently, and before that 16-bit was standard.

• We will assume that, if we build a data structure that holds $n$ elements, then $w = \Omega(\log n)$.
  • This is called the transdichotomous machine model. Essentially, your word size has to be big enough to hold the size of your input.

• We’ll assume C integer operators work in constant time, and won’t assume other integer operations (say, finding most significant bits, counting 1 bits set) are available.
  
  +  -  *  /  %  <<  >>  &  |  ^  =  <=
Our Machine Model

• Rank the following from smallest to largest, in an asymptotic sense, in the transdichotomous machine model. Assume $n$ is the number of items stored in the data structure.

$$\log n \quad w \quad \log_w n \quad \log w$$

Formulate a hypothesis!
Our Machine Model

- Rank the following from smallest to largest, in an asymptotic sense, in the transdichotomous machine model. Assume $n$ is the number of items stored in the data structure.

$$\log n \quad w \quad \log_w n \quad \log w$$

Discuss with your neighbors!
Besting BSTs
Besting BSTs

- BSTs store their elements in sorted order, which enables them to quickly answer each of the following queries:
  - **lookup**(x), which returns whether x ∈ S;
  - **insert**(x), which adds x to S;
  - **delete**(x), which removes x from S;
  - **max() / min()**, which return the max/min element of S;
  - **successor**(x), which returns the smallest element of S greater than x; and
  - **predecessor**(x), which returns the largest element of S smaller than x.

**Question:** Can we build another data structure that answers these same queries, but does so faster than a BST?
A Start: *Bitwise Tries*
**Recall:** A trie is a simple data structure for storing strings.

- Integers can be thought of as strings of bits.

**Idea:** Store integers in a **bitwise trie**.
Finding Successors

- To compute \textit{successor}(x), do the following:
  - Search for x.
  - If x is a leaf node, its successor is the next leaf.
  - If you don't find x, back up until you find a node with a 1 child not already followed, follow the 1, then take the cheapest path down.
Bitwise Trie Efficiency

- All operations on bitwise tries take time proportional to the number of bits in each number.
- Runtime for each operation: $O(w)$.
  - This is worse than a BST’s $O(\log n)$.
- *Can we do better?*
Speeding up Successors

- There are two independent pieces that contribute to the $O(w)$ runtime:
  - Need to walk down the trie following the bits of $x$, and there are $\Theta(w)$ of those.
  - From there, need to back up to a branching node where we can find the successor.
- Can we speed up those operations? Or at least work around them?
A Quick Algorithms Problem

• You’re given an array consisting of some number of Y’s followed by some number of N’s. There are \( k \) total letters. 

    YYYYYYY...YYYYNNNN...NN

• How quickly can you find the last Y?

Formulate a hypothesis!
A Quick Algorithms Problem

- You’re given an array consisting of some number of Y’s followed by some number of N’s. There are $k$ total letters.

  YYYYYYY...YYYYNNNN...NN

- How quickly can you find the last Y?

Discuss with your neighbors!
A Quick Algorithms Problem

• You’re given an array consisting of some number of Y’s followed by some number of N’s. There are $k$ total letters.

YYYYYYY...YYYYNNNNN...NN

• How quickly can you find the last Y?

• Answer: Can be done in time $O(\log k)$ using a binary search.
Search for 010011
Search for 010011

All layers above this line have a node on the search path.

No layers below this line have a node on the search path.
Search for 010011
This assumes that we can tell whether there’s a trie node on each level that we’d be searching for. How can we determine that?
Search for 010011
Store the contents of each level of the tree in a cuckoo hash table.

This gives worst-case O(1) lookups of the prefixes on each level.

That lets us use a binary search to find the longest matching prefix!
One Speedup

- **Goal**: Encode the trie so that we can do a binary search over its layers.

- **One Solution**: Store an array of cuckoo hash tables, one per layer of the trie, that holds all the nodes in that layer.

- Can now query, in worst-case time $O(1)$, whether a node's prefix is present on a given layer.

- There are $O(w)$ layers in the trie.

- Binary search will take worst-case time $O(\log w)$. This is much better than $O(\log n)$ for any reasonable value of $n$. 
Performing the Binary Search

- This binary search assumes that, given a number $x$ and a length $k$, we can extract the first $k$ bits of $x$ in time $O(1)$.

- Despite reading a non-constant number of bits, we can do this by using the fact that individual arithmetic operations transform multiple bits at once.
Performing the Binary Search

- This binary search assumes that, given a number \( x \) and a length \( k \), we can extract the first \( k \) bits of \( x \) in time \( O(1) \).
- Despite reading a non-constant number of bits, we can do this by using the fact that individual arithmetic operations transform multiple bits at once.

\[
x = 1101100 \ 1011101 \ 11000100 \ 11010110 \ 11110011 \ 01111011 \ 11110000 \ 10001100
\]

\[
\text{uint64_t prefix} = x \gg (64 - k);
\]
Performing the Binary Search

● This binary search assumes that, given a number $x$ and a length $k$, we can extract the first $k$ bits of $x$ in time $O(1)$.

● Despite reading a non-constant number of bits, we can do this by using the fact that individual arithmetic operations transform multiple bits at once.

```
uint64_t prefix = x >> (64 - k);
```
Finding Successors

• We can now find the node where the successor search would initially arrive in time $O(\log w)$.

• At this point, we’d normally back up until we find a branching node where we can follow a 1 child pointer, then descend from there to the leaves.

• This will take time $O(w)$.

• *Can we do better?*
Finding Successors

- **Claim**: If the binary search terminates at a node \( v \), that node must have at most one child.

Why?
Formulate a hypothesis!
Finding Successors

- **Claim**: If the binary search terminates at a node $v$, that node must have at most one child.

Why?

Discuss with your neighbors!
Finding Successors

- **Claim:** If the binary search terminates at a node $v$, that node must have at most one child.

- If it doesn't, it has both a 0 child and a 1 child, so there's a longer prefix that can be matched.

- **Idea:** Steal the missing pointers and use them to speed up successor and predecessor searches.
x-Fast Tries

- An **x-fast trie** is a modified binary trie.
- Each missing 1 pointer points to a node’s successor.
- Each missing 0 pointer points to a node’s predecessor.
- Each layer of the tree is stored in a cuckoo hash table for fast binary search.
x-Fast Tries

- **Claim:** Can determine \textit{successor}(x) in time $O(\log w)$.
- Binary search for the longest prefix of $x$.
- If that node has a missing 1 pointer, it points to the successor.
- Otherwise, it has a missing 0 pointer. Follow it to a leaf, then follow the leaf’s 1 pointer.
x-Fast Trie Maintenance

• Based on what we've seen:
  • **lookup** takes worst-case time O(1).
  • **successor** and **predecessor** queries take worst-case time O(log \(w\)).
  • **min** and **max** can be done in time O(1), assuming we cache those values.

• How efficiently can we support **insert** and **delete**?
x-Fast Tries

- If we \textit{insert}(x), we need to
  - add some new nodes to the trie, and
  - update the thread pointers to include $x$.
- Worst-case will be $\Omega(w)$ due to the first and third steps.
Here is an (amortized, expected) $O(w)$-time algorithm for $\textbf{insert}(x)$:

- Find $\textbf{successor}(x)$.
- Add $x$ to the trie.
- Walk up from $x$, its successor, and its predecessor and update threads.
Here is an (amortized, expected) $O(w)$-time algorithm for $\text{insert}(x)$:

- Find $\text{successor}(x)$.
- Add $x$ to the trie.
- Walk up from $x$, its successor, and its predecessor and update threads.
x-Fast Tries

- Here is an (amortized, expected) $O(w)$-time algorithm for $\text{insert}(x)$:
  - Find $\text{successor}(x)$.
  - Add $x$ to the trie.
  - Walk up from $x$, its successor, and its predecessor and update threads.
x-Fast Tries

- Here is an (amortized, expected) $O(w)$-time algorithm for $\text{insert}(x)$:
  - Find $\text{successor}(x)$.
  - Add $x$ to the trie.
  - Walk up from $x$, its successor, and its predecessor and update threads.
x-Fast Tries

- Here is an (amortized, expected) $O(w)$-time algorithm for \texttt{insert}(x):
  - Find \texttt{successor}(x).
  - Add $x$ to the trie.
  - Walk up from $x$, its successor, and its predecessor and update threads.
x-Fast Tries

- Here is an (amortized, expected) $O(w)$-time algorithm for \texttt{insert}(x):
  - Find \texttt{successor}(x).
  - Add $x$ to the trie.
  - Walk up from $x$, its successor, and its predecessor and update threads.
x-Fast Tries

Here is an (amortized, expected) $O(w)$-time algorithm for $\text{insert}(x)$:

- Find $\text{successor}(x)$.
- Add $x$ to the trie.
- Walk up from $x$, its successor, and its predecessor and update threads.
Here is an (amortized, expected) $O(w)$-time algorithm for $\text{insert}(x)$:

- Find $\text{successor}(x)$.
- Add $x$ to the trie.
- Walk up from $x$, its successor, and its predecessor and update threads.
Deletion

• To delete \( x \), we need to
  • Remove \( x \) from the trie.
  • Splice \( x \) out of its linked list.
  • Update thread pointers from \( x \)'s former predecessor and successor.

• Runs in expected, amortized time \( O(w) \).

• Full details are left as a proverbial Exercise to the Reader. 😊
Space Usage

- Each leaf node (item stored in the x-fast trie) contributes at most $O(w)$ nodes in the trie and at most $O(w)$ entries into the hash tables.
- Total space: $O(nw)$. 

![Trie Diagram]
Where We Stand

- Right now, we have a reasonably fast data structure for storing a sorted set of integers.
- If we have a static set of integers that we want to make lots of queries on, this is pretty good as-is!
- As you’ll see, though, we can make this even better with some kitchen sink techniques.

\(x\)-Fast Trie:
- **lookup**: \(O(1)\)
- **insert**: \(O(w)^*\)
- **delete**: \(O(w)^*\)
- **max**: \(O(1)\)
- **succ**: \(O(\log w)\)
- **is-empty**: \(O(1)\)
- Space: \(O(nw)\) words

\(^*\) Expected, amortized
Where We Stand

- Where is there room for improvement in this data structure?
- Ideally, we’d like to improve these highlighted costs, which are places where this structure currently is beaten by a standard BST.

\[
\text{\textbf{x-Fast Trie:}}
\]

- **lookup**: $O(1)$
- **insert**: $O(w)$*
- **delete**: $O(w)$*
- **max**: $O(1)$
- **succ**: $O(\log w)$
- **is-empty**: $O(1)$
- Space: $O(nw)$ words

* Expected, amortized
Shaving Off Logs

• We’re essentially at a spot where we need to shave off a factor of $w$ from a couple of operations.

• Figure that $w$ is kinda sorta ish like log $n$, so this is like shaving off a log factor.

• **Question:** What techniques have we developed so far to do this?

---

**X-Fast Trie:**

- **lookup:** $O(1)$
- **insert:** $O(w)^*$
- **delete:** $O(w)^*$
- **max:** $O(1)$
- **succ:** $O(\log w)$
- **is-empty:** $O(1)$

• Space: $O(nw)$ words

* Expected, amortized
Two-Level Structures

- Think back to the hybrid approach we used for solving RMQ.
- It consisted of a two-tiered structure:
  - A bunch of small, lower-level structures that each solve the problem in small cases.
  - A single, larger, top-level structure that helps aggregate those solutions together.
Two-Level Structures

- **Main Idea:** Partition the input into blocks that are really, really small.
- Small blocks make the block-level structures run quickly.
- Assuming they’re not “too small,” small blocks reduce the size of the inputs to the summary as well.
The Idea

- Build a two-level ordered dictionary out of existing ordered dictionaries.
- Split the keys apart into logarithmic-sized blocks.
- Build ordered dictionaries for each of the block-level dictionaries.
- Build a summary dictionary to aggregate the blocks together.
The y-Fast Trie
The Setup

- For now, assume all keys are given to us in advance, in sorted order.
- Split the keys apart into blocks of size $\Theta(w)$ and store them in balanced BSTs.

| 1  | 3  | 7  | 27 | 31 | 41 | 59 | 79 | 90 | 103 | 106 | 107 | 109 | 110 | 161 |
The Setup

- For now, assume all keys are given to us in advance, in sorted order.
- Split the keys apart into blocks of size $\Theta(w)$ and store them in balanced BSTs.
- Create a summary $x$-fast trie that stores the maximum key from each block but the last.
The Setup

This summary structure is used for routing information so we know which BSTs to look at.

These trees are used for storage of the actual items.
Performing a Lookup

- Suppose we want to perform *lookup*(90).
- **Idea:** figure out which block 90 would belong to, then search within the BST in that block.
- Cost: $O(\log w)$.

- Search this BST in time $O(\log w)$.
- Ask for *successor*(89) up here in time $O(\log w)$. 

*Supplementary Diagram*
Performing a Lookup

- Suppose we want to perform \textit{lookup}(110).
- \textbf{Idea:} figure out which block 109 would belong to, then search within the BST in that block.
- Cost: \(O(\log w)\).

\[\text{x-Fast Trie}\]

Ask for successor(109) in time \(O(\log w)\). (Oops, doesn’t exist!)

Search this BST in time \(O(\log w)\).
Successor Queries

- How might we perform *successor* queries?
- Here’s how we’d determine *successor*(59).

- Ask for *successor*(58) up here in time $O(\log w)$.
- Find successor in time $O(\log w)$. 

![x-Fast Trie Diagram]
Successor Queries

- How might we perform successor queries?
- Here’s how we’d determine successor(107).
- Cost: $O(\log w)$.

![Diagram of x-Fast Trie]

Ask for successor(106) up here in time $O(\log w)$.

Find successor in time $O(\log w)$. (Oops, doesn’t exist!)

Find min in time $O(\log w)$. 
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here’s how we’d **insert**(6)

  **insert** into this BST in time \(O(\log w)\)

  Ask for **successor**(5) in time \(O(\log w)\).
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here’s how we’d **insert** (6)

Ask for **successor** (5) in time $O(\log w)$. 

**insert** into this BST in time $O(\log w)$
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here’s how we’d **insert** (4) into this BST in time $O(\log w)$.

Ask for successor (3) in time $O(\log w)$.
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here’s how we’d **insert**(4)

```
3
  1
  6
  4

31
  27
  41

79
  59
  90

106
  103
  107

110
  109
  161
```

**insert** into this BST in time \(O(\log w)\)

**x-Fast Trie**

- 7
- 41
- 90
- 107

Ask for **successor**(3) in time \(O(\log w)\).
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here’s how we’d **insert** (2)

The x-Fast Trie is a data structure that allows for efficient insertion and search operations. The diagram illustrates the structure of the x-Fast Trie with the keys 7, 41, 90, and 107. The `successor` operation is used to find the successor of a given node in time $O(\log w)$. To insert a new node, `insert` into this BST in time $O(\log w)$. The diagram also shows the structure of the binary search tree (BST) with keys 1, 3, 6, 4, 27, 41, 59, 90, 103, 107, 109, 110, 161.
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here’s how we’d **insert**(2)

  **insert** into this BST in time $O(\log w)$

  **successor**(1) in time $O(\log w)$. 

  Ask for **successor**(1) in time $O(\log w)$. 

  x-Fast Trie

  7  41  90  107

  insert into this BST in time $O(\log w)$
The Problem

- If our trees get too big, we may lose our $O(\log w)$ time bound. \textit{(Why?)}
- \textbf{Idea:} Require each tree to have at most $2w$ elements. If it gets too big, split it and update the $x$-fast trie.
The Problem

- If our trees get too big, we may lose our $O(\log w)$ time bound. *(Why?)*

- **Idea:** Require each tree to have at most $2w$ elements. If it gets too big, split it and update the $x$-fast trie.
The Problem

- If our trees get too big, we may lose our $O(\log w)$ time bound. \textit{(Why?)}

- **Idea:** Require each tree to have at most $2w$ elements. If it gets too big, split it and update the $x$-fast trie.
The Problem

- If our trees get too big, we may lose our $O(\log w)$ time bound. *Why?*
- **Idea:** Require each tree to have at most $2w$ elements. If it gets too big, split it and update the $x$-fast trie.
The Problem

- If our trees get too big, we may lose our $O(\log w)$ time bound. *(Why?)*

- **Idea:** Require each tree to have at most $2w$ elements. If it gets too big, split it and update the $x$-fast trie.
Analyzing an Insertion

- If we perform an insert and don’t end up doing a resize, the cost is $O(\log w)$.
- If we perform an insert and do have to do a resize, the extra work done is
  - $O(w)$ to split the binary search tree, and
  - $O(w)$ to insert into the $x$-fast trie.
- Total work: $O(w)$. 
Analyzing an Insertion

• If we perform an *insert* and don’t end up doing a resize, the cost is $O(\log w)$.

• If we perform an *insert* and do have to do a resize, the extra work done is
  • $O(w)$ to split the binary search tree, and
  • $O(w)$ to insert into the $x$-fast trie.

• Total work: $O(w)$. 

But this is uncommon! We only do this if a tree got way too big.
An Amortized Analysis

• Set our potential $\Phi$ to be the number of items in each BST above $w$.
  • Cost of a “light” insert still $O(\log w)$.
• If we have to split a tree, the tree size was above $2w$ and each new tree has size $w$.
  • Therefore, $\Delta\Phi = -w$.
• The amortized cost of a “heavy” insert is then $O(\log w) + O(w) - kw = O(\log w)$. 

Cost of a regular insert.  
Splitting a BST and adding to the x-fast trie.  
$\Delta\Phi$
A Nice Side-Effect

- We can now abandon our assumption that we’re given all the keys in sorted order in advance.
- Each insertion takes amortized time $O(\log w)$, so we can build the structure up from scratch!
We can now abandon our assumption that we’re given all the keys in sorted order in advance.

Each insertion takes amortized time $O(\log w)$, so we can build the structure up from scratch!
A Nice Side-Effect

- We can now abandon our assumption that we’re given all the keys in sorted order in advance.
- Each insertion takes amortized time $O(\log w)$, so we can build the structure up from scratch!

This is an (expected) $O(n \log w)$-time sorting algorithm!
Making Edits

- With a major caveat, deletions follow the same procedure as insertions.
- Here’s how we’d `delete(7)`.
Making Edits

- With a major caveat, deletions follow the same procedure as insertions.
- Here’s how we’d **delete** (7).

`delete` from this BST in time $O(\log w)$

Ask for **successor** (6) in time $O(\log w)$. 
Making Edits

- Our x-fast trie still holds 7, even though 7 is no longer present.
- That’s not a problem – those keys just serve as “routing information” to tell us which BSTs to look at.
- **Intuition:** The x-fast trie keys act as partitions between BSTs. They don’t need to actually be present in our data structure.
Shrinking our Structure

• What happens if we remove all the elements from our structure without touching the x-fast trie?
Shrinking our Structure

• What happens if we remove all the elements from our structure without touching the $x$-fast trie?

$x$-Fast Trie

\[
\begin{array}{c|c|c|c}
7 & 41 & 90 & 107 \\
\end{array}
\]
Shrinking our Structure

- What happens if we remove all the elements from our structure without touching the x-fast trie?
- Each operation still takes time $O(\log w)$.
- But now our space usage depends on the maximum size we reached, not the current size!
Achieving a Balance

- **Invariant:** Require each tree to have between $\frac{1}{2}w$ and $2w$ elements.
- If a tree gets too small:
  - Merge with the tree next to you, editing the $x$-fast trie as appropriate.
  - If the resulting tree is too big, split it in half, editing the $x$-fast trie.
- This does $O(w)$ on the $x$-fast trie only once every $\Theta(w)$ operations, so this amortizes out to $O(\log w)$ work per operation.
Achieving a Balance

- If each tree has $\Theta(w)$ elements in it, then our space usage is
  - $\Theta(n)$ words for all the trees, plus
  - $\Theta((n / w) w) = \Theta(n)$ words for the x-fast trie,
- This uses $\Theta(n)$ words of memory.
What We’ve Seen

• Here’s the final scorecard for the $y$-fast trie.

• Assuming $n = \omega(w)$, which it probably is, this is faster than a binary search tree!

• And it gives rise to an $O(n \log w)$-expected-time sorting algorithm!

The $y$-Fast Trie:

• $\textit{lookup}$: $O(\log w)$

• $\textit{insert}$: $O(\log w)^*$

• $\textit{delete}$: $O(\log w)^*$

• $\textit{max}$: $O(\log w)$

• $\textit{succ}$: $O(\log w)$

• Space: $\Theta(n)$ words

* Expected, amortized.
What We Needed

• An $x$-fast trie requires *tries* and *cuckoo hashing*.

• The $y$-fast trie requires *amortized analysis* and *balanced BSTs*.

• $y$-fast tries also use the “blocking” technique from *RMQ* we used to shave off log factors.
What’s Missing

• There’s still a little gap between where BSTs dominate and where $y$-fast tries take over.
  • Specifically, what if $n = \Theta(w)$?
• Our solution still involves randomness.
  • We need that in the cuckoo hash tables at each level.
• **Question:** Can we build a solution with neither of these weaknesses?
Next Time

- **Word-Level Parallelism**
  - Treating arithmetic as parallel computation.

- **Sardine Trees**
  - A fast ordered dictionary for truly tiny integers.

- **Finding the Most Significant Bit**
  - An astonishing algorithm for a deceptively tricky problem.