Euler Tour Trees
Outline for Today

- **Dynamic Connectivity**
  - Figuring out what’s connected in a graph as the edges change.

- **Euler Tour Representations**
  - An inspired and clever way to represent trees.

- **Euler Tour Trees**
  - Encoding Euler tours in a creative way.

- **Extending ETTs**
  - Extending our basic structure.
The Dynamic Connectivity Problem
The Connectivity Problem

The graph connectivity problem is the following:

Given an undirected graph $G$, preprocess the graph so we can answer queries of the form “are nodes $u$ and $v$ connected?”

Using $\Theta(m + n)$ preprocessing, can preprocess the graph to answer queries in time $O(1)$. 
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![Graph Diagram](image-url)
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[Diagram of connected graphs]
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- The **dynamic connectivity problem** is the following:
  
  Maintain an undirected graph $G$ so that edges may be inserted and deleted and connectivity queries may be answered efficiently.

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Special Cases

• Last time, we covered the *incremental connectivity problem* in which edges can only be added and not removed.

• Today, we’ll cover *dynamic connectivity in forests*, a special case in which the graph is known to be a forest.

• Next time, we’ll cover *fully-dynamic connectivity*, in which there are no restrictions on which edges can be added and removed.
Dynamic Connectivity in Forests
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- Consider the following special-case of the dynamic connectivity problem:
  
  Maintain an undirected forest $F$ so that edges may be inserted and deleted and connectivity queries may be answered efficiently.

- Each deleted edge splits a tree in two; each added edge joins two trees and never closes a cycle.
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Dynamic Connectivity in Forests

- **Goal**: Support these three operations:
  - $\text{link}(u, v)$: Add in edge $uv$. The assumption is that $u$ and $v$ are in separate trees.
  - $\text{cut}(u, v)$: Cut the edge $uv$. The assumption is that the edge exists in the forest.
  - $\text{are-connected}(u, v)$: Return whether $u$ and $v$ are connected.

- The data structure we'll develop can perform these operations time $O(\log n)$ each.
Euler Tours
Euler Tours

• An *Euler tour* is a path through a graph $G$ that visits every edge exactly once.

• It mathematically formalizes the "trace this figure without picking up your pencil or redrawing any lines" puzzles.
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**Classic Theorem 1:** A graph $G$ has a closed Euler tour if and only if $G$ is connected and every node in $G$ has even degree.

**Classic Theorem 2:** A directed graph $G$ has a closed Euler tour if and only if $G$ is strongly connected and every node’s indegree equals its outdegree.
Euler Tours on Trees

- Trees do not have Euler tours.

- Technique: replace each undirected edge $uv$ with two directed edges $uv$ and $vu$.
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$ac$ $cd$ $db$ $bd$ $df$ $fd$ $dc$ $ce$ $ec$ $ca$
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ce ec cd db bd df fd dc ca ac
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```
fd db bd dc ca ac ce ec cd df
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Properties of Euler Tours

- **Fact:** Any cyclic shift of an Euler tour of a tree is also an Euler tour.

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ab ba ag gh hi id dc cd de ed di ij ji ih hg gf fg ga
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Rerooting a Tour

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- We will call this operation $\textit{reroot}(x)$. 
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![Graph](image)
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- To perform \textit{reroot}(x):
  - Pick any edge \textit{rx} leaving our new start node \textit{r}.
  - Split the tour into \textit{A} and \textit{B}, where \textit{A} consists of everything up to but not including \textit{rx} and \textit{B} consists of everything from \textit{rx} forward.
  - Concatenate \textit{B A}.

\begin{center}
\text{ij} \text{ ji} \text{ ih} \text{ hg} \text{ gf} \text{ fg} \text{ ga} \text{ ab} \text{ ba} \text{ ag} \text{ gh} \text{ hi} \text{ id} \text{ dc} \text{ cd} \text{ de} \text{ ed} \text{ di}
\end{center}
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• To perform \textit{reroot}(x):
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  • Concatenate \( B \ A \).

\[ \text{ij ji ih hg gf fg ga ab ba ag gh hi id dc cd de ed di} \]
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Rerooting a Tour

- To perform \texttt{rerooot}(x):
  - Pick any edge $r \times$ leaving our new start node $r$.
  - Split the tour into $A$ and $B$, where $A$ consists of everything up to but not including $r \times$ and $B$ consists of everything from $r \times$ forward.
  - Concatenate $B A$.
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  - Concatenate $BA$.

\begin{center}
\begin{tikzpicture}
% Graph Edge
\draw [thick, blue] (a) -- (b);
\draw [thick, blue] (a) -- (f);
\draw [thick, blue] (a) -- (g);
\draw [thick, blue] (b) -- (g);
\draw [thick, blue] (g) -- (h);
\draw [thick, blue] (h) -- (i);
\draw [thick, blue] (i) -- (j);
\draw [thick, blue] (d) -- (g);
\draw [thick, blue] (d) -- (c);
\draw [thick, blue] (c) -- (e);
\draw [thick, blue] (f) -- (g);
\draw [thick, blue] (i) -- (j);
\end{tikzpicture}
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\textit{gf fg ga ab ba ag gh hi id dc cd de ed di ij ji ih hg}
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Euler Tours and Dynamic Trees

- Given two trees $T_1$ and $T_2$, where $u \in T_1$ and $v \in T_2$, executing $\text{link}(u, v)$ links the trees together by adding edge $uv$.

- Watch what happens to the Euler tours:
Euler Tours and Dynamic Trees

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• To \textbf{link}(u, v):
  • Let $E_1$ and $E_2$ be Euler tours of $T_1$ and $T_2$, respectively.
  • \textbf{rerooot}(u).
  • \textbf{rerooot}(v).
  • Concatenate $E_1 uv E_2 vu$. 


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- Given a tree $T$, executing $\text{cut}(u, v)$ cuts the edge $uv$ from the tree (assuming it exists).
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```plaintext
ba ab bd db  ce ec cg gh hg gf fg gc  jk kj ji ij
```
Given a tree $T$, executing $\text{cut}(u, v)$ cuts the edge $uv$ from the tree (assuming it exists).

To perform $\text{cut}(u, v)$:

- Let $E$ be the Euler tour containing $uv$ and $vu$.
- Remove $uv$ and $vu$ from $E$ to form $E_1$, $E_2$, and $E_3$.
- Then $E_1E_3$ and $E_2$ are Euler tours of the two new trees.
Checking Connectivity

- We also need a way to answer queries of the form \( \text{are-connected}(u, v) \).
- This query focuses on nodes, but our Euler tours store edges.
- **Cute Trick:** Introduce a self-loop on each node that represents the node itself. Add that to each tour as a proxy for the node itself.
- Now, we can answer \( \text{are-connected}(x, y) \) by seeing if \( xx \) and \( yy \) are part of the same tour.
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- We also need a way to answer queries of the form \textit{are-connected}(u, v).
- This query focuses on nodes, but our Euler tours store edges.
- \textbf{Cute Trick:} Introduce a self-loop on each node that represents the node itself. Add that to each tour as a proxy for the node itself.
- Now, we can answer \textit{are-connected}(x, y) by seeing if xx and yy are part of the same tour.
Checking Connectivity

- This also makes it a lot easier to reroot a tour at a node $x$.
- We simply find $xx$, then rotate that edge to the front of the tour.
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Putting It All Together

- To **link**($x, y$):
  - Rotate $xx$ and $yy$ to the fronts of their tours $T_x$ and $T_y$.
  - Join the tours together as $T_x xy T_y yx$.

- To **cut**($x, y$):
  - Delete the edges $xy$ and $yx$ from the tour $T$ to form tours $T_1, T_2, T_3$.
  - Regroup the tours as $T_1 T_3$ and $T_2$.

- To answer **are-connected**($x, y$):
  - Determine whether $xx$ and $yy$ are in the same tour.
Putting It All Together

• To \textit{link}(x, y):
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  - Determine whether \(xx\) and \(yy\) are in the same tour.
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Implementing This Approach
The Story So Far

- We’ve seen how to implement *reroot*, *link*, *cut*, and *are-connected* in terms of operations on Euler tours.

- The efficiency of those operations depend on how we choose to encode our sequences.

- **Question:** What data structure should we use to store those sequences?
Representation Issues

- We need a representation that lets us perform the following operations:
  - Locate specific edges (reroot, link, cut, are-connected).
  - Split a sequence at a point (reroot, cut).
  - Join two sequences together (reroot, link).
  - Remove an edge from a sequence (cut).
  - Append an edge to a sequence (link).
  - Check if two edges are in the same sequence (are-connected).

- What data structures might be appropriate here?

Answer at

https://pollev.com/cs166spr23
Representation Issues

- **Idea 1:** Use doubly-linked lists, plus an auxiliary hash table / BST to locate edges.
  - Assuming we have a hash table telling us where edges are, we can split, join, and rotate tours in time $O(1)$.

![Diagram of graph with nodes labeled a, b, c, d, e and edges aa, ab, bb, bc, cc, cb, bd, dd, db, ba, ee]
Idea 1: Use doubly-linked lists, plus an auxiliary hash table / BST to locate edges.

Assuming we have a hash table telling us where edges are, we can split, join, and rotate tours in time $O(1)$.

Problem: There isn't an easy way to test whether two nodes are in the same tour. Scanning within the linked list may take time $\Theta(n)$.

Can we do better?
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![Graph with nodes and edges]
Representation Issues

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---

**Diagram:**

- Node $a$ is connected to node $b$.
- Node $b$ is connected to node $c$.

- Nodes $d$ and $e$ are separate and not directly connected to $a$, $b$, or $c$.

- Nodes labeled from $aa$ to $ee$ are connected in a linear sequence.
**Idea 1:** Use doubly-linked lists, plus an auxiliary hash table / BST to locate edges.

- Assuming we have a hash table telling us where edges are, we can split, join, and rotate tours in time $O(1)$.

**Problem:** There isn't an easy way to test whether two nodes are in the same tour. Scanning within the linked list make take time $\Theta(n)$.

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![Graph diagram]

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```
\begin{align*}
\text{a} & \leftrightarrow \text{bb} \leftrightarrow \text{bc} \leftrightarrow \text{cc} \leftrightarrow \text{cb} \leftrightarrow \text{ba} \leftrightarrow \text{dd} \\
\text{aa} & \leftrightarrow \text{ab} \leftrightarrow \text{bb} \leftrightarrow \text{bc} \leftrightarrow \text{cc} \leftrightarrow \text{cb} \leftrightarrow \text{ba} \leftrightarrow \text{dd} \leftrightarrow \text{ee}
\end{align*}
```
Representation Issues

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- Assuming we have a hash table telling us where edges are, we can split, join, and rotate tours in time $O(1)$.
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Can we do better?
• **Idea 1:** Use doubly-linked lists, plus an auxiliary hash table / BST to locate edges.

  • Assuming we have a hash table telling us where edges are, we can split, join, and rotate tours in time $O(1)$.

```plaintext
  a — b — c
d — e

aa ➔ ab ➔ bb ➔ bc ➔ cc ➔ cb ➔ ba ➔ dd ➔ de ➔ ee ➔ de
```
**Representation Issues**

- **Idea 1:** Use doubly-linked lists, plus an auxiliary hash table / BST to locate edges.
  - Assuming we have a hash table telling us where edges are, we can split, join, and rotate tours in time $O(1)$.

![Diagram of graph with nodes labeled 'a', 'b', 'c', 'd', 'e', and edges 'aa', 'ab', 'bb', 'bc', 'cc', 'cb', 'ba', 'dd', 'de', 'ee', and 'de']
**Representation Issues**

- **Idea 1:** Use doubly-linked lists, plus an auxiliary hash table / BST to locate edges.
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![Diagram showing nodes and edges labeled with characters: a, b, c, d, e, aa, ab, bb, bc, cc, cb, ba, dd, de, ee, de. Connections illustrate the linked lists and hash table concept.](image-url)
Representation Issues

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  - Assuming we have a hash table telling us where edges are, we can split, join, and rotate tours in time $O(1)$.
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![Diagram of a linked list with nodes and edges labeled from aa to de]
Representation Issues

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- **Problem:** There isn’t an easy way to test whether two nodes are in the same tour. Scanning within the linked list make take time $\Theta(n)$. 

```
aa  ab  bb  bc  cc  cb  ba  dd  de  ee  de
```
Representation Issues

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- **Problem:** There isn’t an easy way to test whether two nodes are in the same tour. Scanning within the linked list make take time $\Theta(n)$.

- Can we do better?
Representation Issues

• In incremental connectivity, we selected a representative for each CC.

• We then had elements store parent pointers that formed a path to the representative.

• Could we do something like that here?
The idea of using trees to store representatives is a good one.

- If the trees are wide and flat, it won’t take too long to find the representative.
- If we don’t have to update “too many” pointers when CC’s change, our operations can run quickly.

The trees we used last time won’t (immediately) work here.

- We have to store the elements of the tour in sequential order. There was no such notion of order in disjoint set forests.
- In disjoint-set forests, linked items can never be cut, allowing for some clever optimizations.
- What’s another tree we can use?
The idea of using trees to store representatives is a good one.

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In disjoint-set forests, linked items can never be cut, allowing for some clever optimizations.

What’s another tree we can use?
Binary Search(less) Trees

- **Idea 2:** Store our sequences in a balanced BST, sorted by their position within the sequence.
- We’ll use the *shape* and *algorithm* of a BST, but won’t have the ability to conventionally search the tree top-down.
- We’ll rely on the fact that we have external pointers that let us jump to items within the BST.

---

**Diagram:**

```
  a  d
   |
  b  c
   |
  e  f
```

```
  ab
 /  \
aa  bc
   |
  bb
```

```
  cb
 /  \
  dd
```

```
  ef
 /  \
  ee
```

```
  dd
 /  \
  db
```

```
  ba
 /  \
  bb
```

---

**List:**

- a
- b
- c
- d
- e
- f

- aa
- ab
- bb
- bc
- cc
- cb
- bd
- dd
- db
- ba

- ee
- ef
- ff
- fe
We can now answer \textit{are-connected}(x, y) in time $O(\log n)$.

- Find $xx$ and $yy$ using our auxiliary lookup table.
- Walk up from $xx$ and $yy$ to the roots of their trees.
- See if they’re the same root.
We can now answer *are-connected*(*x, y*) in time $O(\log n)$.

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We can now answer *are-connected* \((x, y)\) in time \(O(\log n)\).

- Find \(xx\) and \(yy\) using our auxiliary lookup table.
- Walk up from \(xx\) and \(yy\) to the roots of their trees.
- See if they’re the same root.
Binary Search(less) Trees

- **Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

Answer: Use splay trees! They support these operations in amortized time $O(\log n)$.
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```
aa ab bb bc cc cb bd dd db ba
```

```
ea
```

```
ef
```

```
ee ef ff fe
```

```
bb cc ba
```

```
ab cb dd
dd
ee ef
```

```
ba
```

```
bb cc ba
```

```
ab cb dd
dd
ee ef
```

```
ba
```

```
bb cc ba
```

```
ab cb dd
dd
ee ef
```

```
ba
```
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\[\begin{align*}
aa & \quad ab & \quad bb & \quad bc & \quad cc & \quad cb & \quad bd & \quad dd & \quad db & \quad ba \\
\quad e & \quad ee & \quad ef & \quad ff & \quad fe
\end{align*}\]
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```
aa ab bb bc cc cb bd dd db ba
 ee ef ff fe
```
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```plaintext
aa ab bb bc cc cb ba
  
dd

ee ef ff fe
```

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```plaintext
aa ab bb bc cc cb ba
dd
e
```

```
bb cc ba
db
cb
```

```
ee ef ff fe
```

```
ba
db
```

```
ab
dd
```

```
d
```

```
a
```

```
b
```

```
c
```

```
d
```

```
e
```

```
f
```
Binary Search(less) Trees

- **Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

![Diagram of binary search(less) trees](image)
Binary Search(less) Trees

- **Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
   a
  /  \
 b   c
   \
  a  b
   \
 f   e
```

```
   aa
  /  \
 ab  cb
   \
 cc
```

```
   dd
  /  \
 bb  cc
   \
 ba
```

```
   ee
  /  \
 ef  ff
   \
 fe
```
**Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

---

The image contains a diagram illustrating a binary search tree with nodes labeled from 'a' to 'f' and sequences of letters such as 'aa', 'ab', 'bb', 'bc', 'cc', 'cb', 'ba', 'dd', 'ee', 'ef', 'ff', 'fe'. The tree structure is shown with parent-child relationships indicated by lines connecting the nodes. The sequences are represented as labels on the nodes.
Binary Search(less) Trees

- **Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

Answer: Use splay trees! They support these operations in amortized time $O(\log n)$.
Binary Search(less) Trees

- **Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

```
  aa  ab  bb  bc  cc  cb  bd  dd  db  ba
  ee  ef  ff  fe
```

Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

- **Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$.

```
aa ab bb bc cc cb bd dd db ba
```
```
e e f
```
**Binary Search(less) Trees**

- **Challenge:** We need to be able to cut a sequence just before an edge, and we need to be able to join two sequences together efficiently.

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
a
  b
    d
  c

ea ab bb bc cc cb bd dd db ba

e
  f
  ee ef ff fe

ab
  bb
    cc
  ba

cb
  dd
    ee
  ff

ab
  bc
    bd
      db
        ba

ef
  ee
    ff
      fe
```
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
  a
  b
  c
  d

  aa ab bb bc cc cb bd dd db ba

  e
  f

  ee ef ff fe
```

```
  ab
  bc
  bd
  db
  dd

  aa
  bb
  cc
  ba

  cb
  ef
  ee
  ff
  fe
```
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
aa ab bb bc cc cb bd dd db ba ...
... ae ee ef ff fe ea
```
Answer: Use splay trees! They support these operations in amortized time $O(\log n)$. 
**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

---

```
aa ab bb bc cc cb bd dd db ba ...
... ae ee ef ff fe ea
```
**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

---

Binary Search(less) Trees.
Binary Search(less) Trees

• **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

• **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
  a
 b d
 c
 /
 e
```

```
  dd
 cb  ae
 ab  bd  ba
 a  bc db  ea
 aa bb cc db dd db ba ... ae ee ef ff fe ea
```
**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$.
**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
Answer: Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```plaintext
aa ab bb bc cc cb bd dd db ba ...
... ae ee ef ff fe ea
```
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

- **Answer**: Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
aa ab bb bc cc cb bd dd db ba ...
... ae ee ef ff fe ea
```
**Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
aa ab bb bc cc cb bd dd db ba ...
... ae ee ef ff fe ea
```
Answer: Use splay trees! They support these operations in amortized time $O(\log n)$.
Binary Search(less) Trees

• **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 
Binary Search(less) Trees

- **Answer:** Use splay trees! They support these operations in amortized time $O(\log n)$. 

```
a a ab bb bc cc cb bd dd db ba ...
  ... ae ee ef ff fe ea
d
f
c
b

```

```
  ea
  
  ef
  ae
  fe

  dd
  ee
  ff

  cb
  ba

  ab
  bd
db

  aa
  bc

  bb
  cc

```
Binary Search(less) Trees

• **Answer**: Use splay trees! They support these operations in amortized time $O(\log n)$. 

---

**Diagram**

```
aa ab bb bc cc cb bd dd db ba ... ae ee ef ff fe ea
```
Euler Tour Trees

- To answer \textit{are-connected}(x, y):

\begin{itemize}
  \item Splay \textit{xx}.
  \item Splay \textit{yy}.
  \item Return whether \textit{xx} was encountered on the second splay.
\end{itemize}

Amortized cost: $O(\log n)$. 
Euler Tour Trees

- To answer \texttt{are-connected}(x, y):

  - \texttt{Splay} \texttt{xx}
  - \texttt{Splay} \texttt{yy}
  - Return whether \texttt{xx} was encountered on the second splay.

Amortized cost: $O(\log n)$. 

\begin{tikzpicture}[level distance=1.5cm,sibling distance=1.5cm,edge from parent path={(	ikzparentnode.south) -- ++(0,-1.5cm) -- +(2cm,0) -- (	ikzchildnode.north)}]

  \node (cb) {cb}
  child {node (ab) {ab}
    child {node (aa) {aa}
      child {node (bb) {bb}}}
    child {node (bc) {bc}}}
  child {node (dd) {dd}
    child {node (bd) {bd}}
    child {node (db) {db}
      child {node (ba) {ba}}}};
\end{tikzpicture}
Euler Tour Trees

- To answer \textit{are-connected}(x, y):
  - Splay xx.
Euler Tour Trees

- To answer \textit{are-connected}(x, y):
  - Splay xx.

Amortized cost: $O(\log n)$. 

\begin{itemize}
  \item Euler Tour Trees
\end{itemize}
To answer *are-connected* \((x, y)\):

- Splay \(xx\).

Amortized cost: \(O(\log n)\).
Euler Tour Trees

- To answer \textcolor{blue}{\textit{are-connected}}(x, y):
  - Splay xx.

Amortized cost: $O(\log n)$. 

\begin{itemize}
  \item Euler Tour Trees
\end{itemize}
To answer *are-connected* \((x, y)\):

- Splay \(xx\).

Amortized cost: \(O(\log n)\).
Euler Tour Trees

• To answer \textit{are-connected}(x, y):
  • Splay xx.

[Diagram of Euler Tour Trees]

Amortized cost: \(O(\log n)\).
Euler Tour Trees

- To answer \textit{are-connected}(x, y):
  - Splay xx.
  - Splay yy.

Amortized cost: $O(\log n)$. 
Euler Tour Trees

- To answer \textit{are-connected}(x, y):
  - Splay xx.
  - Splay yy.

Amortized cost: $O(\log n)$. 

\textbf{Euler Tour Trees}
Euler Tour Trees

- To answer *are-connected*(x, y):
  - Splay xx.
  - Splay yy.

\[
\begin{align*}
\text{Splay xx.} \\
\text{Splay yy.}
\end{align*}
\]
Euler Tour Trees

- To answer *are-connected*(x, y):
  - Splay xx.
  - Splay yy.

Amortized cost: $O(\log n)$.
Euler Tour Trees

- To answer \textit{are-connected}(x, y):
  - Splay xx.
  - Splay yy.

**Amortized cost:** $O(\log n)$.
Euler Tour Trees

- To answer \textit{are-connected}(x, y):
  - Splay xx.
  - Splay yy.
  - Return whether xx was encountered on the second splay.
- Amortized cost: \(O(\log n)\).
Euler Tour Trees

• To \textit{reroot}(x):

\[\text{Splay } xx.\]
\[\text{Disconnect } xx \text{'s left child tree } T.\]
\[\text{Splay the rightmost node in } xx \text{'s subtree.}\]
\[\text{Make } T \text{ the right child of the root.}\]

Amortized cost: \(O(\log n)\).
Euler Tour Trees

- To reroot(x):
  - Splay xx.

Amortized cost: $O(\log n)$. 

```
aa ab bb bc cc cb bd dd db ba
```
Euler Tour Trees

• To \textit{reroot}(x):
  • Splay xx.

\begin{itemize}
  \item Splay xx.
  \item Disconnect xx's left child tree $T$.
  \item Splay the rightmost node in xx's subtree.
  \item Make $T$ the right child of the root.
\end{itemize}

Amortized cost: $O(\log n)$.
Euler Tour Trees

- To \textit{reroot}(x):
  - Splay xx.

\begin{itemize}
  \item Splay xx.
  \item Disconnect xx's left child tree T.
  \item Splay the rightmost node in xx's subtree.
  \item Make T the right child of the root.
\end{itemize}

Amortized cost: $O(\log n)$. 
Euler Tour Trees

- To \textit{rerooot}(x):
  - Splay xx.
Euler Tour Trees

• To \textit{rerooot}(x):
  • Splay xx.

\begin{itemize}
  \item Splay xx.
  \item Disconnect xx's left child tree \textit{T}.
  \item Splay the rightmost node in xx's subtree.
  \item Make \textit{T} the right child of the root.
\end{itemize}

Amortized cost: \(O(\log n)\).
Euler Tour Trees

- To *reroot*(*x*):
  - Splay *xx*.

- Amortized cost: $O(\log n)$.
Euler Tour Trees

- To \textit{reroot}(x):
  - Splay xx.

\begin{itemize}
  \item To \textit{reroot}(x):
  \item Splay xx.
\end{itemize}
Euler Tour Trees

- To \textit{reroot}(x):
  - Splay xx.
Euler Tour Trees

- To *reroot*(*x*):
  - Splay *xx*.

```plaintext
To reroot(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.
  - Make T the right child of the root.

Amortized cost: $O(\log n)$.
```
To \textit{reroot}($x$):

- Splay $xx$.
- Disconnect $xx$’s left child tree $T$.

\[ \text{Amortized cost: } O(\log n) \]
Euler Tour Trees

- To \textit{rroot}(x):
  - Splay xx.
  - Disconnect xx’s left child tree \( T \).

\[ cc \]
\[ cb \]
\[ bc \]
\[ cc cb bd dd db ba \]
\[ aa ab bb bc \]

\[ cc cb bd dd db ba \]
Euler Tour Trees

To \textit{reroot}(x):

- Splay xx.
- Disconnect xx’s left child tree $T$.

Amortized cost: $O(\log n)$. 
Euler Tour Trees

- To \textit{reroot}(x):
  - Splay \( xx \).
  - Disconnect \( xx \)'s left child tree \( T \).
  - Splay the rightmost node in \( xx \)'s subtree.

Amortized cost: \( O(\log n) \).
Euler Tour Trees

- To \textit{rerooot}(x):
  - Splay xx.
  - Disconnect xx’s left child tree $T$.
  - Splay the rightmost node in xx’s subtree.

- Amortized cost: $O(\log n)$. 
Euler Tour Trees

• To \textit{reroot}(x):
  • Splay xx.
  • Disconnect xx’s left child tree $T$.
  • Splay the rightmost node in xx’s subtree.

\begin{itemize}
  \item [Amortized cost:] $O(\log n)$.
\end{itemize}
Euler Tour Trees

• To \textit{reroot}(x):
  • Splay xx.
  • Disconnect xx’s left child tree \( T \).
  • Splay the rightmost node in xx’s subtree.

\[
\begin{align*}
\text{amortized cost: } & \mathcal{O}(\log n) \\
\end{align*}
\]
Euler Tour Trees

To **reroot** \(x\):

- Splay \(xx\).
- Disconnect \(xx\)’s left child tree \(T\).
- Splay the rightmost node in \(xx\)’s subtree.

Amortized cost: \(O(\log n)\).
Euler Tour Trees

• To \textit{reroot}(x):
  • Splay xx.
  • Disconnect xx’s left child tree T.
  • Splay the rightmost node in xx’s subtree.

\textbf{Amortized cost: }O(\log n).
Euler Tour Trees

- To \texttt{reroot}(x):
  - Splay xx.
  - Disconnect xx’s left child tree $T$.
  - Splay the rightmost node in xx’s subtree.

Amortized cost: $O(\log n)$.
Euler Tour Trees

- To **reroot** \(x\):
  - Splay \(xx\).
  - Disconnect \(xx\’s\) left child tree \(T\).
  - Splay the rightmost node in \(xx\’s\) subtree.

Amortized cost: \(O(\log n)\).
Euler Tour Trees

To \textit{rerooot}(x):

- Splay \textit{xx}.
- Disconnect \textit{xx}'s left child tree \( T \).
- Splay the rightmost node in \textit{xx}'s subtree.

Amortized cost: \( O(\log n) \).
Euler Tour Trees

• To \textit{rerooot}(x):
  • Splay \( xx \).
  • Disconnect \( xx \)'s left child tree \( T \).
  • Splay the rightmost node in \( xx \)'s subtree.
  • Make \( T \) the right child of the root.

Amortized cost: \( O(\log n) \).
Euler Tour Trees

To \textit{rerooot}(x):

- Splay xx.
- Disconnect xx’s left child tree $T$.
- Splay the rightmost node in xx’s subtree.
- Make $T$ the right child of the root.

Amortized cost: $O(\log n)$. 

\begin{center}
\textbf{cc cb bd dd db ba}
\end{center}
Euler Tour Trees

- To \textit{rerooot}(x):
  - Splay \( xx \).
  - Disconnect \( xx \)'s left child tree \( T \).
  - Splay the rightmost node in \( xx \)'s subtree.
  - Make \( T \) the right child of the root.

Amortized cost: \( O(\log n) \).
Euler Tour Trees

To \textit{rerooot}(x):

- Splay xx.
- Disconnect xx’s left child tree \( T \).
- Splay the rightmost node in xx’s subtree.
- Make \( T \) the right child of the root.

Amortized cost: \( O(\log n) \).
Euler Tour Trees

- **To reroot**(x):
  - Splay xx.
  - Disconnect xx’s left child tree \( T \).
  - Splay the rightmost node in xx’s subtree.
  - Make \( T \) the right child of the root.

- **Amortized cost:** \( O(\log n) \).
Euler Tour Trees

- To $\textit{link}(x, y)$:
  
  - reroot $(x)$ and reroot $(y)$.
  - Add $xy$ as the rightmost node of $x$'s tree.
  - Splay $xy$.
  - Make $y$'s right child $xy$.
  - Add $yx$ as the rightmost node of the tree.
  - Splay $yx$.

- Amortized cost: $O(\log n)$. 

```
xx  
```

```
cc cd ... dc
jj jk ... jk
```
Euler Tour Trees

- To \( \text{link}(x, y) \):
  - \( \text{rerooot}(x) \) and \( \text{rerooot}(y) \).
  - Add \( xy \) as the rightmost node of \( x \)'s tree.
  - Splay \( xy \).
  - Make \( y \)'s right child \( xy \).
  - Add \( yx \) as the rightmost node of the tree.
  - Splay \( yx \).

Amortized cost: \( O(\log n) \).
Euler Tour Trees

- To \textit{link}(x, y):
  - \textit{reroot}(x) and \textit{reroot}(y).

\[ xx \ x_a \ldots \ ax \quad yy \ y_f \ldots \ fy \]
Euler Tour Trees

- To \textit{link}(x, y):
  - \textit{reroot}(x) and \textit{reroot}(y).
  - Add \textit{xy} as the rightmost node of \textit{x}'s tree.
  - \text{Splay} \textit{xy}.
  - Make \textit{y}'s right child \textit{xy}.
  - Add \textit{yx} as the rightmost node of the tree.
  - \text{Splay} \textit{yx}.

Amortized cost: $O(\log n)$. 
Euler Tour Trees

- To \textit{link}(x, y):
  
  - \textit{reroot}(x) and \textit{reroot}(y).
  
  - Add \(xy\) as the rightmost node of \(x\)'s tree.

\[
\begin{align*}
&\text{xx xa ... ax xy} \\
&\text{yy yf ... fy}
\end{align*}
\]
Euler Tour Trees

To link \((x, y)\):  
- \textit{reroot}(x) and \textit{reroot}(y).
- Add \(xy\) as the rightmost node of \(x\)'s tree.
- Splay \(xy\).

Amortized cost: \(O(\log n)\).

\[
\begin{array}{c}
x \\
xx \ xa \ ax \ xy \\
\end{array}
\quad
\begin{array}{c}
y \\
yy \ yf \ fy \\
\end{array}
\]
Euler Tour Trees

• To \textit{link}(x, y):
  
  • \textit{reroot}(x) and \textit{reroot}(y).
  
  • Add xy as the rightmost node of x’s tree.
  
  • Splay xy.
Euler Tour Trees

- To **link**($x$, $y$):
  - **reroot**($x$) and **reroot**($y$).
  - Add $xy$ as the rightmost node of $x$’s tree.
  - Splay $xy$.
  - Set $yy$’s tree as $xy$’s right child.

**Amortized cost:** $O(\log n)$.
Euler Tour Trees

- To **link**(x, y):
  - **reroot**(x) and **reroot**(y).
  - Add xy as the rightmost node of x’s tree.
  - Splay xy.
  - Set yy’s tree as xy’s right child.

Amortized cost: $O(\log n)$. 

\[ xx \quad a \quad x \quad y \quad y \quad f \quad ... \quad f \]
Euler Tour Trees

- To $\text{link}(x, y)$:
  - $\text{rerooot}(x)$ and $\text{rerooot}(y)$.
  - Add $xy$ as the rightmost node of $x$’s tree.
  - Splay $xy$.
  - Set $yy$’s tree as $xy$’s right child.
  - Add $yx$ as the rightmost node of the tree.

$xx \ xa \ \ldots \ \ ax \ xy \ yy \ yf \ \ldots \ fy$
Euler Tour Trees

- To \textit{link}(x, y):
  - \textit{reroot}(x) and \textit{reroot}(y).
  - Add \(xy\) as the rightmost node of \(x\)'s tree.
  - Splay \(xy\).
  - Set \(yy\)'s tree as \(xy\)'s right child.
  - Add \(yx\) as the rightmost node of the tree.

\[ xx \ xa \ ... \ ax \ xy \ yy \ yf \ ... \ fy \ yx \]
Euler Tour Trees

- To $\textbf{link}(x, y)$:
  - $\textbf{reroot}(x)$ and $\textbf{reroot}(y)$.
  - Add $xy$ as the rightmost node of $x$’s tree.
  - Splay $xy$.
  - Set $yy$’s tree as $xy$’s right child.
  - Add $yx$ as the rightmost node of the tree.
  - Splay $yx$.

Amortized cost: $O(\log n)$. 

$xx \ xa \ ... \ ax \ xy \ yy \ yf \ ... \ fy \ yx$
Euler Tour Trees

- To **link**(x, y):
  - **reroot**(x) and **reroot**(y).
  - Add xy as the rightmost node of x’s tree.
  - Splay xy.
  - Set yy’s tree as xy’s right child.
  - Add yx as the rightmost node of the tree.
  - Splay yx.

Amortized cost: $O(\log n)$.  

xx xa ... ax xy yy yf ... fy yx
Euler Tour Trees

- To $\text{link}(x, y)$:
  - $\text{reroot}(x)$ and $\text{reroot}(y)$.
  - Add $xy$ as the rightmost node of $x$’s tree.
  - Splay $xy$.
  - Set $yy$’s tree as $xy$’s right child.
  - Add $yx$ as the rightmost node of the tree.
  - Splay $yx$.
- Amortized cost: $O(\log n)$. 

$x x a ... a x y y y f ... f y y x$
Euler Tour Trees

• To \textit{cut}(x, y):

\[ \text{Splay } xy. \]
\[ \text{Delete } xy. \]
\[ \text{Splay } yx. \]
\[ \text{Delete } yx. \]

Let \( T_1 \) and \( T_2 \) be the trees on the left and right.

\[ \text{Splay the rightmost node of } T_1. \]
\[ \text{Attach } T_2 \text{ as the right child of that node.} \]

Amortized cost: \( \mathcal{O}(\log n) \).
Euler Tour Trees

- To \textit{cut}(x, y):
  - Splay \(xy\).

\[\text{Amortized cost: } O(\log n)\]

\textit{Euler Tour Trees}

\[\text{aa ab } \ldots \text{ cx xy yy yf } \ldots \text{ fy yx xt } \ldots \text{ ba}\]
To \textit{cut}(x, y):

- Splay \textit{xy}.

Let $T_1$ and $T_2$ be the trees on the left and right.

- Splay the rightmost node of $T_1$.
- Attach $T_2$ as the right child of that node.

Amortized cost: $O(\log n)$. 

\textbf{Euler Tour Trees}

```
aa ab ... cx xy yy yf ... fy yx xt ... ba
```
Euler Tour Trees

- To \textbf{cut}(x, y):
  - Splay \texttt{xy}.
  - Delete \texttt{xy}.

\texttt{xy}

Amortized cost: \(O(\log n)\).

\texttt{aa ab \ldots cx xy yy yf \ldots fy yx xt \ldots ba}
Euler Tour Trees

- To **cut**($x, y$):
  - Splay $xy$.
  - Delete $xy$.

Let $T_1$ and $T_2$ be the trees on the left and right.
- Splay the rightmost node of $T_1$.
- Attach $T_2$ as the right child of that node.

Amortized cost: $O(\log n)$. 
Euler Tour Trees

- To \textbf{cut}(x, y):
  - Splay xy.
  - Delete xy.
  - Splay yx.

Amortized cost: \(O(\log n)\).

\begin{itemize}
  \item \(aa\ ab\ \ldots\ cx\)
  \item \(yy\ yf\ \ldots\ fy\ yx\ xt\ \ldots\ ba\)
\end{itemize}
Euler Tour Trees

- To \textit{cut}(x, y):
  - Splay \textit{xy}.
  - Delete \textit{xy}.
  - Splay \textit{yx}.

\textbf{Amortized cost: }\mathcal{O}(\log n).

Let $T_1$ and $T_2$ be the trees on the left and right.
- Splay the rightmost node of $T_1$.
- Attach $T_2$ as the right child of that node.

\begin{align*}
  \text{Euler Tour Trees} & \quad \begin{array}{c}
  aa \ ab \ \ldots \ cx \\
  yy \ yf \ \ldots \ fy \ yx \ xt \ \ldots \ ba
  \end{array}
\end{align*}
Euler Tour Trees

- To \textbf{cut}(x, y):
  - Splay xy.
  - Delete xy.
  - Splay yx.
  - Delete yx.

\textbf{Amortized cost:} $O(\log n)$.
Euler Tour Trees

- To **cut** \((x, y)\):
  - Splay \(xy\).
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  - Splay \(yx\).
  - Delete \(yx\).

Let \(T_1\) and \(T_2\) be the trees on the left and right.

- Splay the rightmost node of \(T_1\).
- Attach \(T_2\) as the right child of that node.

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\[ T_1 \quad \text{\[aa \ ab \ \ldots \ cx\]} \quad \text{\[yy \ yf \ \ldots \ fy\]} \quad \text{\[xt \ \ldots \ ba\]} \]
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\begin{center}
\begin{tikzpicture}
  \node (root) at (0,0) {\textit{T} \textsubscript{1}};
  \node (top) at (0,5) {\textit{T} \textsubscript{2}};
  \node (bottom) at (0,10) {aa \ ab \ ... \ cx \ yy \ yf \ ... \ fy \ xt \ ... \ ba};
  \draw[->] (root) -- (top);
\end{tikzpicture}
\end{center}
Euler Tour Trees

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Euler Tour Trees

- To \textit{cut}(x, y):
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  - Splay \textit{yx}.
  - Delete \textit{yx}.
  - Let \textit{T}_1 and \textit{T}_2 be the trees on the left and right.
  - Splay the rightmost node of \textit{T}_1.
  - Attach \textit{T}_2 as the right child of that node.

\begin{center}
\textit{T}_1 \hspace{2cm} \textit{T}_2
\end{center}

\textit{aa ab ... cx xt ... ba} \hspace{1cm} \textit{yy yf ... fy}
Euler Tour Trees

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- Amortized cost: \( O(\log n) \).
Euler Tour Trees

• With all things said and done, we get the following amortized runtimes for each operation:
  • \textit{are-connected}: O(\log n)
  • \textit{link}: O(\log n)
  • \textit{cut}: O(\log n)

• These bounds can be made worst-case efficient using different types of balanced BSTs instead of splay trees, but splaying is probably the fastest way to do this.
Extending Euler Tour Trees
Extending Euler Tour Trees

• We now have a (relatively) simple and fast data structure for solving dynamic connectivity in forests.

• What else can we do with them?
Extending Euler Tour Trees

• Suppose we want to add an operation \texttt{size}(x) that returns the number of nodes in the tree containing \( x \).

• How might we accomplish this?
Tree Sizes

- We can determine \textit{size}(x) as follows:
  - Figure out which Euler tour \( xx \) is in.
  - Count how many nodes of the form \( zz \) it contains.
- A naive implementation of this algorithm might take time \( \Theta(n) \) if all nodes are in the same tree. Can we do better?

\begin{itemize}
  \item \texttt{ab}
  \item \texttt{bb}
  \item \texttt{bc}
  \item \texttt{cc}
  \item \texttt{cb}
  \item \texttt{bd}
  \item \texttt{dd}
  \item \texttt{db}
  \item \texttt{ba}
\end{itemize}
Tree Sizes

- We’re storing our Euler tours in balanced BSTs.
- We want to be able to answer the following question about a given BST:
  
  **How many nodes of the form $xx$ are in this BST?**
- This can be done in time $O(\log n)$. How?
Tree Sizes

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• We want to be able to answer the following question about a given BST:
  
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• This can be done in time $O(\log n)$. How?

Answer at

https://pollev.com/cs166spr23
Tree Sizes

- **Idea:** Augment the BSTs holding our Euler tours.
- Specifically, each node stores the number of self-loops at or below it in the tree.
- This information can be maintained through rotations and after each splay tree operation.
Tree Sizes

- To determine $\text{size}(x)$:
Tree Sizes

• To determine $size(x)$:
  • Splay $xx$. 
Tree Sizes

- To determine $\text{size}(x)$:
  - Splay $xx$. 

\[ \text{Amortized cost: } O(\log n) \]
Tree Sizes

• To determine $size(x)$:
  • Splay $xx$.
  • Return the augmented value in the node for $xx$. 
Tree Sizes

• To determine $\text{size}(x)$:
  • Splay $xx$.
  • Return the augmented value in the node for $xx$.

• Amortized cost: $O(\log n)$. 
Suppose that each node represents a network router.

We want to add these two operations:

- **add-packet** \((x, p)\), which attaches packet \(p\) to node \(x\); and
- **remove-packet** \((x)\), which removes and returns some packet reachable from \(x\), chosen arbitrarily from all the options.

How might we do this?
Extending Euler Tour Trees

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• How might we do this?
Packet Finding

- Given the Euler tour representation of our trees, this essentially boils down to the following:

  *Augment a BST containing nodes and edges so that we can quickly identify a node with a packet.*

- How might we do this?
Packet Finding

• Augment each node with a list of the packets it stores.
Packet Finding

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- Augment each node with a list of the packets it stores.
- Augment each tree node with a bit indicating whether there’s a packet in its subtree.
Packet Finding

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Packet Finding

• Augment each node with a list of the packets it stores.
• Augment each tree node with a bit indicating whether there’s a packet in its subtree.
• We can use this latter information to quickly find nodes holding packets.
Packet Finding

- To find and remove a packet:
  - Walk from the root to any node containing a packet, using the augmentation to guide the search.
  - Splay that node to the root.
  - Remove a packet from it, updating the root's augmentation.

Amortized cost: $O(\log n)$. 
Packet Finding

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\[ O(\log n) \]
Packet Finding

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• Amortized cost: $O(\log n)$. 
Generalizing This Idea

- More generally, Euler tour trees play well with augmentations that care about global properties of individual trees.
- There’s another way to use splay trees to encode dynamic trees (st-trees, also called link/cut trees, though the later name is ambiguous) that works well for augmenting over paths in trees rather than trees as a whole.
- (Check out the Sleator/Tarjan paper for more details.)
Next Time

- **Fully-Dynamic Connectivity**
  - Solving connectivity in general graphs, not just forests.

- **“Blame It On The Little Guy”**
  - A surprisingly versatile algorithmic strategy.

- **Holm’s Structure**
  - An elegant way to solve dynamic connectivity by harnessing augmented ETTs.