CS 205b / CME 306

Application Track

Homework 9

1. Let $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle \mathbf{u}, \mathbf{v} \rangle_\rho$ denote the two inner products defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dV \qquad \langle \mathbf{u}, \mathbf{v} \rangle_\rho = \int_\Omega \rho \mathbf{u} \cdot \mathbf{v} \, dV,$$

where $\mathbf{u} \cdot \mathbf{v}$ denotes the standard pointwise dot product, and ρ is the density. The two new inner products take two vector fields to produce a single scalar. The dot product is well-defined for vectors of any dimension. You may assume for this assignment that all fields have as many derivatives defined as desired. By regarding scalars as 1D vectors, the above inner products can also be defined for scalar fields such as pressure. If $\mathbf{u}$ is the velocity, what is $\frac{1}{2}\langle \mathbf{u}, \mathbf{u} \rangle_\rho$?

It is the kinetic energy of the system.

2. Let $\mathbf{G}$ and $\mathbf{D}$ be the operators defined by

$$\mathbf{G} : \phi \to \frac{1}{\rho}\nabla\phi \qquad \mathbf{D} : \mathbf{u} \to \frac{1}{\rho}\nabla \cdot \mathbf{u}.$$

Show that the operators $\mathbf{G}$ and $\mathbf{D}$ are linear operators.

Let α and β be scalar constants.

$$\begin{aligned}
\mathbf{G}(\alpha\phi + \beta\tau) &= \frac{1}{\rho}\nabla(\alpha\phi + \beta\tau) \\
&= \alpha\left(\frac{1}{\rho}\nabla\phi\right) + \beta\left(\frac{1}{\rho}\nabla\tau\right) \\
&= \alpha\mathbf{G}(\phi) + \beta\mathbf{G}(\tau) \\
\mathbf{D}(\alpha\mathbf{u} + \beta\mathbf{v}) &= \frac{1}{\rho}\nabla \cdot (\alpha\mathbf{u} + \beta\mathbf{v}) \\
&= \alpha\left(\frac{1}{\rho}\nabla \cdot \mathbf{u}\right) + \beta\left(\frac{1}{\rho}\nabla \cdot \mathbf{v}\right) \\
&= \alpha\mathbf{D}(\mathbf{u}) + \beta\mathbf{D}(\mathbf{v})
\end{aligned}$$

3. The linear operator $\mathbf{B}$ is said to be the transpose of the linear operator $\mathbf{A}$ with respect to an inner product $(\cdot, \cdot)$ if for any vectors $\mathbf{u}$ and $\mathbf{v}$, it is true that $(\mathbf{Bu}, \mathbf{v}) = (\mathbf{u}, \mathbf{Av})$. Show that this definition of a transpose corresponds precisely to the definition of a matrix transpose (even for non-square matrices) when the standard inner product is used: $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T\mathbf{v}$.

$$\begin{aligned}
(\mathbf{Bu}, \mathbf{v}) &= (\mathbf{u}, \mathbf{Av}) \\
(\mathbf{Bu})^T\mathbf{v} &= \mathbf{u}^T\mathbf{Av} \\
\mathbf{u}^T\mathbf{B}^T\mathbf{v} &= \mathbf{u}^T\mathbf{Av} \\
\mathbf{B}^T &= \mathbf{A}
\end{aligned}$$

The last step is because the vectors $\mathbf{u}$ *and* $\mathbf{v}$ *are arbitrary.*

4. Show that the operator $-\mathbf{D}$ is the transpose of $\mathbf{G}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\rho$ if and only if a particular boundary condition is satisfied, and find that boundary condition. The boundary condition should not contain any volume integrals.

$$\begin{aligned}
\langle -\mathbf{Du}, \phi \rangle_\rho &= \langle \mathbf{u}, \mathbf{G}\phi \rangle_\rho \\
\int_\Omega \rho(-\mathbf{Du})\phi \, dV &= \int_\Omega \rho\mathbf{u} \cdot (\mathbf{G}\phi) \, dV \\
\int_\Omega \rho\left(-\frac{1}{\rho}\nabla \cdot \mathbf{u}\right)\phi \, dV &= \int_\Omega \rho\mathbf{u} \cdot \left(\frac{1}{\rho}\nabla\phi\right) dV \\
-\int_\Omega (\nabla \cdot \mathbf{u})\phi \, dV &= \int_\Omega \mathbf{u} \cdot \nabla\phi \, dV \\
0 &= \int_\Omega \mathbf{u} \cdot \nabla\phi + (\nabla \cdot \mathbf{u})\phi \, dV \\
&= \int_\Omega \nabla \cdot (\phi\mathbf{u}) \, dV \\
&= \int_{\partial\Omega} \phi\mathbf{u} \cdot dS
\end{aligned}$$

5. Show that both Direchlet and Neumann boundary conditions satisfy this boundary condition. (Hint: write out the boundary condition obtained in the previous question with pressure and velocity as the two arbitrary fields. Then, show that Direchlet and Neumann boundary each suffice by considering what they mean for velocity and pressure.)

Written with velocity and pressure, the boundary condition is

$$\int_{\partial\Omega} p\mathbf{u} \cdot dS.$$

If Direchlet boundray conditions are assumed, then $p = 0$ at the boundary, and the integral vanishes. For Neumann boundary conditions, $\mathbf{u} \cdot dS = \mathbf{u} \cdot \mathbf{n}\, dA = 0$, so the integral vanishes.

6. A linear operator is said to be symmetric with respect to an inner product if it equals its transpose with respect to that inner product. Show that $\mathbf{DG}$ is symmetric with respect to $\langle \cdot, \cdot \rangle_\rho$. You may assume for this and all subsequent problems that suitable boundary conditions will be met.

   Assuming the boundary condition is satisfied, $-\langle \mathbf{Du}, \phi \rangle_\rho = \langle \mathbf{u}, \mathbf{G}\phi \rangle_\rho$.

$$\begin{aligned}
\langle \mathbf{DG}\tau, \phi \rangle_\rho &= -\langle \mathbf{G}\tau, \mathbf{G}\phi \rangle_\rho \\
&= -\langle \mathbf{G}\phi, \mathbf{G}\tau \rangle_\rho \\
&= \langle \mathbf{DG}\phi, \tau \rangle_\rho \\
&= \langle \tau, \mathbf{DG}\phi \rangle_\rho
\end{aligned}$$

7. Show that the operator $\mathbf{L}$ defined by

$$\mathbf{L} : \phi \rightarrow \nabla \cdot \left(\frac{1}{\rho} \nabla \phi \right)$$

   is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$.

   Observe that $\rho\mathbf{DG} = \mathbf{L}$.

$$\begin{aligned}
\langle \mathbf{DG}\tau, \phi \rangle_\rho &= \langle \tau, \mathbf{DG}\phi \rangle_\rho \\
\int_\Omega \rho(\mathbf{DG}\tau)\phi \, dV &= \int_\Omega \rho\tau(\mathbf{DG}\phi) \, dV \\
\int_\Omega (\rho\mathbf{DG}\tau)\phi \, dV &= \int_\Omega \tau(\rho\mathbf{DG}\phi) \, dV \\
\int_\Omega (\mathbf{L}\tau)\phi \, dV &= \int_\Omega \tau(\mathbf{L}\phi) \, dV \\
\langle \mathbf{L}\tau, \phi \rangle &= \langle \tau, \mathbf{L}\phi \rangle
\end{aligned}$$

A few *rules of thumb* might be taken from what has been shown above, even though you have not considered discretizations.

- It was not an accident that the Poisson equation obtained in class was symmetric negative semidefinite. In particular, when discretized, the gradient and divergence operators will be matrices.

- If suitable boundary conditions are not used, the Poisson operator will not be symmetric.
- Provided suitable boundary conditions are applied, there will be a *preferred* discretization of divergence corresponding to any discretization of gradient which will result in a symmetric negative definite system, and the two discretizations, written as matrices, will be negative transposes of each other.
- Once you have chosen a discretization for gradient or divergence, the negative transpose relationship can be used to derive the other discretization.

8. What are you doing for your final project? (This question is optional if you are not taking the application track or receive project approval from the CA by email prior to the due date of this assignment.) This question will be graded.

9. In 3D, there are three *special* spatial derivatives: gradient, curl, and divergence.

    - Show that the curl of a gradient is identically zero.

      Using Cartesian tensor notation, the curl can be expressed as $e_{ijk}u_{i,j}$.

$$\begin{aligned} \nabla\times\nabla\phi &\Rightarrow e_{ijk}\phi_{,ij} \\ &= e_{ijk}\phi_{,ji} \\ &= -e_{jik}\phi_{,ji} \\ &= -e_{ijk}\phi_{,ij} \\ e_{ijk}\phi_{,ij} &= 0 \\ \nabla\times\nabla\phi &= \mathbf{0} \end{aligned}$$

    - Show that the divergence of a curl is identically zero.

$$\begin{aligned} \nabla\cdot(\nabla\times\mathbf{u}) &\Rightarrow (e_{ijk}u_{i,j})_{,k} \\ &= e_{ijk}u_{i,jk} \\ &= e_{ijk}u_{i,kj} \\ &= -e_{ikj}u_{i,kj} \\ &= -e_{ijk}u_{i,jk} \\ e_{ijk}u_{i,jk} &= 0 \\ \nabla\cdot(\nabla\times\mathbf{u}) &= \mathbf{0} \end{aligned}$$

- Assume that a vector field $\mathbf{u}$ can be decomposed as $\mathbf{u} = \nabla\phi + \nabla \times \mathbf{w}$. Show that ϕ can be obtained by solving Poisson's equation.

$$\begin{aligned}
\mathbf{u} &= \nabla\phi + \nabla \times \mathbf{w} \\
\nabla \cdot \mathbf{u} &= \nabla \cdot \nabla\phi + \nabla \cdot (\nabla \times \mathbf{w}) \\
\nabla \cdot \mathbf{u} &= \nabla \cdot \nabla\phi \\
\nabla^2\phi &= \nabla \cdot \mathbf{u}
\end{aligned}$$

- This question is optional and will not be graded. Initialize a (2D) grid with random (2D) vectors and decompose the perform the decomposition by solving the Poisson equation. The divergence-free part should look rather like a fluid flow field. Also visualize the curl-free part and the scalar field ϕ.