CS205b/CME306

Lecture 14

1 Shallow Water Equations

Supplementary Reading: Osher and Fedkiw, §14.5.2

The shallow water equations are given by

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x = 0.$$

where h is the height of the water, and u is the velocity. The first equation is the equation for conservation of mass, and the second is the equation for conservation of momentum. The shallow water equations assume a constant density. In these equations, h is similar to the notion of mass, as can be seen by multiplying by the width of a column of water and the density column. The term hu on the right hand side is the advective term (conserved quantity times velocity). The hu term on the left hand side can be interpreted as momentum, with hu^2 the corresponding advective term. The final term in the momentum equation is $\frac{1}{2}gh^2$, which accounts for the extra force (change in momentum) due to gravity acting on columns of differing height.

In order to discretize the system using the procedure we described above, we must first find the Jacobian and its eigensystem analytically. In computing the Jacobian, it is very important to remember that we take the conserved variables (in this case h and hu) to be the independent variables. To make this fact more apparent, we can rewrite the equations as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} u_2 \\ u_2^2 u_1^{-1} + \frac{1}{2}gu_1^2 \end{pmatrix}_x = 0. \tag{1}$$

and then define $h = u_1$, $u = u_2 u_1^{-1}$. Below we compute the Jacobian matrix.

$$\begin{aligned}
\mathbf{J} &= \frac{\partial \mathbf{f}}{\partial \boldsymbol{\phi}} \\
&= \begin{pmatrix} \frac{\partial \mathbf{f}_1}{\partial u_1} & \frac{\partial \mathbf{f}_1}{\partial u_2} \\ \frac{\partial \mathbf{f}_2}{\partial u_1} & \frac{\partial \mathbf{f}_2}{\partial u_2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial (hu)}{\partial h} & \frac{\partial (hu)}{\partial (hu)} \\ \frac{\partial}{\partial h}\left((hu)^2 h^{-1} + \frac{1}{2}gh^2\right) & \frac{\partial}{\partial (hu)}\left((hu)^2 h^{-1} + \frac{1}{2}gh^2\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -(hu)^2 h^{-2} + gh & 2(hu)h^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix}
\end{aligned}$$

Note: if you find the treatment of h and hu as independent variables in the above computation confusing, you may prefer rewrite the system as in (1), compute the Jacobian in terms of u_1 and u_2, and then substitute for h and u at the end. Note that the Jacobian may also be obtained by simply expanding the spatial derivatives:

$$\begin{aligned}
\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x &= \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x \\
&= \begin{pmatrix} (hu) \\ (hu)^2 h^{-1} + \frac{1}{2}gh^2 \end{pmatrix}_x \\
&= \begin{pmatrix} (hu)_x \\ 2(hu)h^{-1}(hu)_x - (hu)^2 h^{-2} h_x + gh^2 h_x \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -u^2 + gh^2 & 2u \end{pmatrix} \begin{pmatrix} h \\ hu \end{pmatrix}_x \\
&= \mathbf{J} \begin{pmatrix} h \\ hu \end{pmatrix}_x .
\end{aligned}$$

Next we find the eigensystem for the Jacobian. We have

$$\det(\lambda \mathbf{I} - \mathbf{J}) = \begin{vmatrix} \lambda & -1 \\ -u^2 + gh & \lambda - 2u \end{vmatrix} = \lambda^2 - 2u\lambda + u^2 - gh.$$

$$\lambda = \frac{2u \pm \sqrt{4u^2 - 4u^2 + 4gh}}{2} = u \pm \sqrt{gh}.$$

The eigenvalues of our system tell us how how fast things are moving around. The u part may be thought of as the bulk velocity, and $\sqrt{gh}$ is the sound speed. Next we find the right eigenvectors.

$$\begin{aligned}
\mathbf{J} \begin{pmatrix} a \\ b \end{pmatrix} &= \left(u \pm \sqrt{gh}\right) \begin{pmatrix} a \\ b \end{pmatrix} \\
\Rightarrow \begin{pmatrix} b \\ (-u^2 + gh)a + 2ub \end{pmatrix} &= \left(u \pm \sqrt{gh}\right) \begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

Therefore, we have

$$\mathbf{R}^1 = \begin{pmatrix} 1 \\ u + \sqrt{gh} \end{pmatrix}, \qquad \mathbf{R}^2 = \begin{pmatrix} 1 \\ u - \sqrt{gh} \end{pmatrix}.$$

Then

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}^1 & \mathbf{R}^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ u + \sqrt{gh} & u - \sqrt{gh} \end{pmatrix}.$$

Hence

$$\mathbf{L} = \mathbf{R}^{-1} = \frac{1}{-2\sqrt{gh}} \begin{pmatrix} u - \sqrt{gh} & -1 \\ -u - \sqrt{gh} & 1 \end{pmatrix}$$

or

$$\mathbf{L}^1 = \begin{pmatrix} -\frac{u}{2\sqrt{gh}} + \frac{1}{2} & \frac{1}{2\sqrt{gh}} \end{pmatrix}, \qquad \mathbf{L}^2 = \begin{pmatrix} \frac{u}{2\sqrt{gh}} + \frac{1}{2} & -\frac{1}{2\sqrt{gh}} \end{pmatrix}.$$

Using $\mathbf{L}$ we may project into the characteristic fields, upwind in the characteristic variables, and get back using $\mathbf{R}$.

This approach has problems when $h = 0$. When this is the case, the Jacobian can be expressed in Jordan form as

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}^{-1}.$$

It is clear form this that $\mathbf{J}$ is defective, and thus the system is only *weakly hyperbolic*. The eigenvectors have coalesced. We cannot do upwinding, since we are do not have a full set of eigenvectors. There are other options, though, such as the complimentary projection method.

2 Compressible Flow

The inviscid Euler equations for one phase compressible flow in the absence of chemical reactions in one spatial dimension are

$$\boldsymbol{\phi}_t + \mathbf{f}(\boldsymbol{\phi})_x = 0$$

which can be written in detail as

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = 0$$

where ρ is the density, u are the velocities, E is the total energy per unit volume, and p is the pressure. The total energy is the sum of the internal energy and the kinetic energy,

$$E = \rho e + \rho(u^2)/2$$

where e is the internal energy per unit mass. The ρu, ρu^2, and Eu terms on the right are the usual advective terms. The two pressure terms apply forces to the system. This system still depends on the pressure, for which we do not yet have an equation. This is where an equation of state (EOS) is important.

2.1 Ideal Gas Equation of State

For an ideal gas

$$p = \rho RT,$$

where $R = R_u/M$ is the specific gas constant with $R_u \approx 8.31451 J/(molK)$ is the universal gas constant and M the molecular weight of the gas. Also valid for an ideal gas is

$$c_p - c_v = R,$$

where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume. Since R is known, only one of c_p and c_v needs to be measured. Gamma is the ratio of specific heats,

$$\gamma = c_p/c_v.$$

For an ideal gas, one can write

$$de = c_v\, dT,$$

and assuming that c_v does not depend on temperature (calorically perfect gas), integration yields

$$e = e_o + c_v T$$

where e_o is not uniquely determined, and one could choose any value for e at $0K$. We take $e_0 = 0$ arbitrarily for simplicity.

Note that

$$p = \rho RT = \frac{R}{c_v}\rho e = \frac{c_p - c_v}{c_v}\rho e = \rho(\gamma - 1)e = (\gamma - 1)\rho e,$$

so our equation of state is

$$p = (\gamma - 1)\rho e,$$

or

$$p = (\gamma - 1)\left(E - \frac{\rho u^2}{2}\right).$$

Noting that $\rho u^2 = (\rho u)^2 \rho^{-1}$, we can compute the partials of the pressure as

$$\frac{\partial p}{\partial \rho} = (\gamma - 1)\frac{(\rho u)^2}{2\rho^2} = (\gamma - 1)\frac{u^2}{2} \qquad \frac{\partial p}{\partial(\rho u)} = -(\gamma - 1)\frac{2(\rho u)}{2\rho} = -(\gamma - 1)u \qquad \frac{\partial p}{\partial E} = \gamma - 1.$$

2.2 Sound Speed

At this point we have sufficient information to compute the Jacobian, which is left as an exercise. We do, however, give its eigenvalues here

$$u, \quad u \pm \sqrt{\frac{p}{\rho^2}p_e + p_\rho}.$$

We may then define the sound speed as

$$c = \sqrt{\frac{p}{\rho^2}p_e + p_\rho}.$$

In the case of an ideal gas, this becomes

$$u, \quad u \pm \sqrt{\frac{\gamma p}{\rho}},$$

so that the sound speed is

$$c = \sqrt{\frac{\gamma p}{\rho}}.$$

The eigenvalues are now $\lambda_1 = u$, $\lambda_2 = u + c$, and $\lambda_3 = u - c$.