CS205b/CME306

Lecture 7

1 Finite Element Method

1.1 Geometric Calculation of Strain

This section uses a significant amount of text from Teran et al., Finite Volume Methods for the Simulation of Skeletal Muscle, 2003, with permission of the author.

A deformable object is characterized by a time dependent map ϕ from undeformed material coordinates $\mathbf{X}$ to deformed spatial coordinates $\mathbf{x}$. We use a tetrahedron mesh and assume that the deformation is piecewise linear, which implies $\phi(X) = \mathbf{FX} + \mathbf{b}$ in each tetrahedron. The linear part $\mathbf{F}$ is called the deformation gradient. In practice, obtaining a representation of the undeformed object in material space can be quite challenging in its own right, as materials are generally only observed under the pull of gravity and possibly other forces. It is necessary to remove the effects of such forces to obtain a material configuration.

For simplicity, consider two spatial dimensions where each element is a triangle. Figure 1 depicts a mapping ϕ from a triangle in material coordinates to the resulting triangle in spatial coordinate. We define edge vectors for each triangle as $\mathbf{d}_{m_1} = \mathbf{X}_1 - \mathbf{X}_0$, $\mathbf{d}_{m_2} = \mathbf{X}_2 - \mathbf{X}_0$, $\mathbf{d}_{s_1} = \mathbf{x}_1 - \mathbf{x}_0$, and $\mathbf{d}_{s_2} = \mathbf{x}_2 - \mathbf{x}_0$. Note that $\mathbf{d}_{s_1} = (\mathbf{FX}_1 + \mathbf{b}) - (\mathbf{FX}_0 + \mathbf{b}) = \mathbf{Fd}_{m_1}$ and likewise $\mathbf{d}_{s_2} = \mathbf{Fd}_{m_2}$ so that $\mathbf{F}$ maps the edges of the triangle in material coordinates to the edges of the triangle in spatial coordinates. Thus, if we construct 2×2 matrices $\mathbf{D}_m$ with columns $\mathbf{d}_{m_1}$ and $\mathbf{d}_{m_2}$, and $\mathbf{D}_s$ with columns $\mathbf{d}_{s_1}$ and $\mathbf{d}_{s_2}$, then $\mathbf{D}_s = \mathbf{FD}_m$ or $\mathbf{F} = \mathbf{D}_s\mathbf{D}_m^{-1}$.

The Green strain is defined as $\mathbf{G} = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2$. Note that $\mathbf{G}$ defined in this way is rotation invariant, since applying a rotation to the deformed object results in a new transformation $\hat{\phi}(X) = \mathbf{RFX} + \mathbf{Rb}$, and $\hat{\mathbf{G}} = ((\mathbf{RF})^T(\mathbf{RF}) - \mathbf{I})/2 = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2 = \mathbf{G}$. Note that $\mathbf{G}$ is missing some of the information contained in the actual mapping between material and spatial coordinates. In assuming that ϕ is piecewise linear, we lose nonlinear effects. By considering $\mathbf{F}$ instead of ϕ, we lose

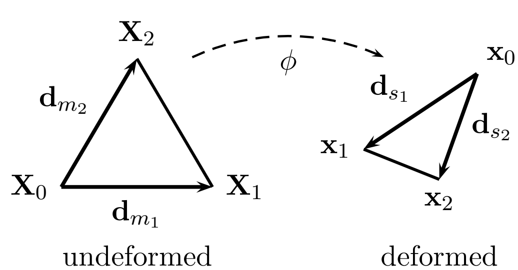
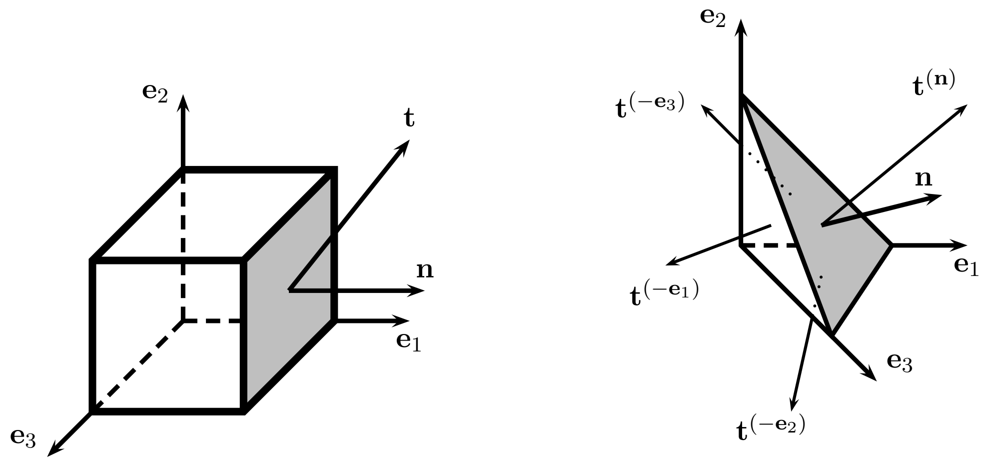


Figure 1: Undeformed and deformed triangle edges.



(a) Cauchy stress for unit cube.

(b) Cauchy stress in an arbitrary direction.

Figure 2: Cauchy stress tensor $\boldsymbol{\sigma}$.

discard translation. Finally, in forming the Green strain $\mathbf{G}$, we lose rotation and reflection. Since rotation and translation do not affect strain, invariance under them is advantageous. However, we have still lose reflection and nonlinear effects, which may be undesirable under some circumstances.

Using this definition of $\mathbf{F}$ the green strain is $\mathbf{G} = (\mathbf{D}_m^{-T}\mathbf{D}_s^T\mathbf{D}_s\mathbf{D}_m^{-1} - \mathbf{I})/2$, which can be rewritten to obtain

$$\mathbf{D}_m^T\mathbf{G}\mathbf{D}_m = \frac{1}{2}(\mathbf{D}_s^T\mathbf{D}_s - \mathbf{D}_m^T\mathbf{D}_m) = \frac{1}{2}\left[\begin{pmatrix} \mathbf{d}_{s_1}\cdot\mathbf{d}_{s_1} & \mathbf{d}_{s_1}\cdot\mathbf{d}_{s_2} \\ \mathbf{d}_{s_1}\cdot\mathbf{d}_{s_2} & \mathbf{d}_{s_2}\cdot\mathbf{d}_{s_2} \end{pmatrix} - \begin{pmatrix} \mathbf{d}_{m_1}\cdot\mathbf{d}_{m_1} & \mathbf{d}_{m_1}\cdot\mathbf{d}_{m_2} \\ \mathbf{d}_{m_1}\cdot\mathbf{d}_{m_2} & \mathbf{d}_{m_2}\cdot\mathbf{d}_{m_2} \end{pmatrix}\right]$$

in order to emphasize that we are simply measuring the change in the dot products of each edge with itself and the other edge.

The above discussion extends naturally to three spatial dimensions. Here, $\mathbf{D}_m$ and $\mathbf{D}_s$ are 3×3 matrices with columns equal to the edge vectors of the tetrahedra, and $\mathbf{D}_m^T\mathbf{G}\mathbf{D}_m$ is a measure of the difference between the dot products of each edge with itself and the other two edges. Note that $\mathbf{D}_m^{-1}$ can be computed and stored for efficiency.

The Green strain is not the only useful measure of strain. Another measure of strain that is often used is the Cauchy strain $\boldsymbol{\epsilon}$, which is obtained by linearizing the Green strain. This measure of strain has the advantage that it is linear in the deformation, but it is not rotation invariant.

1.2 Cauchy Stress

The stress an object is experiencing may be described by considering the relationship between a direction $\mathbf{n}$ and the traction (force per unit area) $\mathbf{t}^{(\mathbf{n})}$ applied to the plane cross section with normal $\mathbf{n}$. Consider a small cube inside a larger volume of material. The surrounding material applies stress to the cube through its faces, as illustrated in Figure 2(a). Forces in the direction of $\mathbf{n}$ are compressive or expansive forces. Forces orthogonal to $\mathbf{n}$ are sheer forces. Let $\mathbf{t}^{(\mathbf{e}_i)}$ be the traction applied to the plane of the unit cube with normal $\mathbf{e}_i$. Then, we can write these tractions

Stress Tensor	Symbol	Area-weighted normal	Force	Relations
Cauchy	$\boldsymbol{\sigma}$	spatial	spatial	
First Piola-Kirchhoff	$\mathbf{P}$	material	spatial	$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$
Second Piola-Kirchhoff	$\mathbf{S}$	material	material	$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$
Unknown	none	spatial	material	$\mathbf{F}^{-1}\boldsymbol{\sigma}$

Table 1: Four possibilities for stress tensors and their relationship to the Cauchy stress.

in terms of the basis as

$$\mathbf{t}^{(\mathbf{e}_1)} = \boldsymbol{\sigma}_{11}\mathbf{e}_1 + \boldsymbol{\sigma}_{21}\mathbf{e}_2 + \boldsymbol{\sigma}_{31}\mathbf{e}_3 \qquad \mathbf{t}^{(\mathbf{e}_2)} = \boldsymbol{\sigma}_{12}\mathbf{e}_1 + \boldsymbol{\sigma}_{22}\mathbf{e}_2 + \boldsymbol{\sigma}_{32}\mathbf{e}_3 \qquad \mathbf{t}^{(\mathbf{e}_3)} = \boldsymbol{\sigma}_{13}\mathbf{e}_1 + \boldsymbol{\sigma}_{23}\mathbf{e}_2 + \boldsymbol{\sigma}_{33}\mathbf{e}_3$$

That is, $\mathbf{t}^{(\mathbf{e}_i)} = \boldsymbol{\sigma}\mathbf{e}_i$. This defines a 3×3 tensor $\boldsymbol{\sigma}$ called the Cauchy stress tensor. This definition of $\boldsymbol{\sigma}$ is given in terms of the axis-aligned normal directions $\mathbf{e}_i$.

We can extend the Cauchy stress tensor's application to an arbitrary direction by considering a tetrahedron as situated in Figure 2(b). We begin by computing the areas for the four faces. Assume the shaded face has area a. The areas of the other three faces can then be expressed as $(\mathbf{n} \cdot \mathbf{e}_1)a$, $(\mathbf{n} \cdot \mathbf{e}_2)a$, and $(\mathbf{n} \cdot \mathbf{e}_3)a$. The tractions for the three orthogonal faces are $\mathbf{t}^{(-\mathbf{e}_1)} = -\boldsymbol{\sigma}\mathbf{e}_1$, $\mathbf{t}^{(-\mathbf{e}_2)} = -\boldsymbol{\sigma}\mathbf{e}_2$, and $\mathbf{t}^{(-\mathbf{e}_3)} = -\boldsymbol{\sigma}\mathbf{e}_3$. The traction of the shaded triangle is $\mathbf{t}^{(\mathbf{n})}$. Because the total force on the tetrahedron should cancel out (assuming the object is in equilibrium), the sum of the forces on the faces (traction times area) should sum to zero

$$a\mathbf{t}^{(\mathbf{n})} - (\mathbf{n} \cdot \mathbf{e}_1)a\boldsymbol{\sigma}\mathbf{e}_1 - (\mathbf{n} \cdot \mathbf{e}_2)a\boldsymbol{\sigma}\mathbf{e}_2 - (\mathbf{n} \cdot \mathbf{e}_3)a\boldsymbol{\sigma}\mathbf{e}_3 = 0.$$

Dividing off a and rewriting the dot products using transposes yields

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{e}_1(\mathbf{e}_1^T\mathbf{n}) + \boldsymbol{\sigma}\mathbf{e}_2(\mathbf{e}_2^T\mathbf{n}) + \boldsymbol{\sigma}\mathbf{e}_3(\mathbf{e}_3^T\mathbf{n}).$$

Factoring and simplifying finishes off the derivation

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}(\mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_2\mathbf{e}_2^T + \mathbf{e}_3\mathbf{e}_3^T)\mathbf{n} = \boldsymbol{\sigma}\mathbf{I}\mathbf{n} = \boldsymbol{\sigma}\mathbf{n}.$$

In particular, the traction on a plane with unit normal $\mathbf{n}$ is $\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$.

Two important identities are obtained by considering the Cauchy stress along with conservation of momentum. The first is sometimes called the Cauchy equation of motion and is derived from conservation of linear momentum. The second is symmetry, which is obtained from the first equation and conservation of angular momentum.

$$\rho\mathbf{v}' - \nabla \cdot \boldsymbol{\sigma} - \rho\mathbf{f} = 0 \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T.$$

Here, $\mathbf{f}$ is the external body force (force per unit mass).

The Cauchy stress defines a linear relationship between a unit normal and a traction. One may instead think of the Cauchy stress as the relationship between an area-weighted normal for a surface and the force exerted over that surface.

1.3 Force and Stress

Consider a tetrahedron and choose one of its faces. This face has the (outward-facing) area-weighted normal da. Using the Cauchy stress, the force exerted on this face is $\boldsymbol{\sigma}\, da$. Here, the area-weighted normal and the force are in the moving spatial coordinates of the object being simulated. This is not the only choice for measuring these quantities. One could instead imagine computing the area-weighted normal dA in material space from the initial material configuration of the tetrahedron. Similarly, one could compute the force on that face in the material configuration. This choice of where to measure the area-weighted normal and resulting force results in the four possible stress tensors shown in Table 1.

To understand how the stress tensors are related, it is important to understand how quantities transform between material and spatial coordinates. Force is a regular vector and transforms from material to spatial coordinates as $\mathbf{f}_s = \mathbf{F}\mathbf{f}_m$. It is easy to assume that the area-weighted normal may transform similarly, but this is not the case. If we consider the tetrahedron to have edge vectors dU, dV, and dW in material coordinates, the area-weighted normal is $dA = 1/2 dU \times dV$. Further, the volume of the tetrahedron is $1/6(dU \times dV) \cdot dW$, and the transformation rule for volumes is $1/6(du \times dv) \cdot dw = 1/6J(dU \times dV) \cdot dW$ or $dw^T da = J dW^T dA$, where $J = \det(\mathbf{F})$ is the Jacobian. Further, dW is a regular vector, so that $dw = \mathbf{F} dW$. Substituting this in yields $dW^T \mathbf{F}^T da = J dW^T dA$. Because dW does not affect dA and could have been chosen arbitrarily, we must have $\mathbf{F}^T da = J dA$ or $da = J\mathbf{F}^{-T} dA$.

The First Piola-Kirchhoff stress uses an area-weighted normal measured in material coordinates and yields a force in spatial coordinates. That is, $\mathbf{f} = \boldsymbol{\sigma} da = \boldsymbol{\sigma} J\mathbf{F}^{-T} dA = \mathbf{P} dA$ expresses the action of the first Piola-Kirchhoff stress tensor, so that $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$ or $\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T$. Note that $\mathbf{P}$ is not symmetric.

The Second Piola-Kirchhoff stress also uses an area-weighted normal measured in material coordinates but also yields a force in material coordinates. $\mathbf{S} dA = \mathbf{f}_m = \mathbf{F}^{-1}\mathbf{f} = \mathbf{F}^{-1}\mathbf{P} dA$. From this we obtain the relations $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$, $\mathbf{P} = \mathbf{F}\mathbf{S}$ and $\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^T$. Note that the symmetry of $\boldsymbol{\sigma}$ also implies that $\mathbf{P}$ is symmetric.

The fourth possibility relates area-weighted normals in world space to force in material space. This is not a particularly useful combination, since it is both asymmetric and yields a force in material coordinates, which must be converted to spatial coordinates to be useful. For this reason, it is not used and is as far as we know unnamed.