1 Solving linear systems

You can write a linear system into a matrix: $Ax = b$. The system can have a unique solution, no solution, or infinite solution. In later classes, no solution is treated as a least square problem. Infinite solution can be analyzed by Principle Component Analysis (PCA).

2 Zero

- Unit really matter in calculation. For example, even $1e7$ might be 0 ($1e23 − 1e7 = 1e23$).
- Question: how to avoid dropping large numbers, many digits. If you do not want to drop many digits, it will be very slow. Computers store in float. It means that everything is converted to float and then converted back, so just usually use double precision.
- If you lose digits in your inputs, the results may lose digits. For example: $\pi = 3.1415$ might be good enough. Do the best to make sure you do not lose a lot accuracy and also the answer does not go bad immediately. For example, the first digit should be correct if you care about many digits.

3 Non-Dimensionalization

- Row scaling and column scaling: Row scaling means take one row you have, multiply it with some number, and add to other rows. Then you can do column scaling. Column scaling is scaling one of the variables.
- People often use column scaling if they see one variable is in wrong scale. Then the output variable would be in good precision. For example: replace $x$ by $z$ with $z = 10^5x$ and then you can solve original variable $x$ on paper.
- You can do a combination of row scaling and column scaling.
- Imaginary number: you can multiply $i$ to get rid of the imaginary number in matrix.

4 Square matrix

5 Solvability

There are a lot of names in books for a matrix is singular.

1. A matrix is not invertible.
2. Rank of matrix \( < \min(m, n) \) (\( A \) is a \( m \times n \) matrix) means it is rank-deficient, thus singular.

3. Determinant 0.

4. Having a nonzero null space.

Note: for a \( 3 \times 3 \) matrix, both rank 1 and rank 2 matrices are singular, so the result that a matrix is singular does not imply its rank.

Here [Fig.5.1] is an example of how to write a linear system into a combination of column vectors equals \( b \). If the system \( Az = 0 \) has a nonzero solution, then this means the column vectors are linearly dependent, so \( A \) is a singular matrix.

![Figure 5.1: Az = 0](image)

Understanding and remembering some criteria of singular matrix is helpful for future study. Typically, singular matrix have infinite number of solutions; some have no solution.

6 Diagonal matrix

- Sometimes the problem is nasty, because it needs to be seen in the correct way. For example, you can perform diagonalization process to make the matrix a diagonal matrix, if the matrix is diagonalizable.

- Example of diagonal and non-singular matrix [Fig. 6.1]: A quick way to make it singular is to remove one number and change to 0. Then the system has no solution (0=1) and the matrix is degenerate. If \( b \) the right hand side also has 0 in that corresponding place, then it is infinite solution (0=0).

- For a linear system \( Ax = b \), we usually care about matrix \( A \), not right hand side \( b \). We will learn more about matrices and linear system in Unit 3. The reason of \( A \) has infinite solution because it is neglecting the changes in \( z \). A singular matrix ignores some sub-dimension of it. Any singular matrix can be written into a diagonal matrix with a 0 on one diagonal space.

- In solving real problems, we usually focus on infinite solution case because it gives information about \( A \), while 0 solution only tells about \( b \).
A non-singular matrix means it has an inverse. Matrix division can be calculated as multiplication with the inverse of the matrix. A square matrix $A$ has a 2-sided inverse $A^{-1}$, which means $AA^{-1} = A^{-1}A = I$. Then $Ax = b$ can be solved by $A^{-1}Ax = A^{-1}b \rightarrow x = A^{-1}b$. Note: we never compute inverse of matrix unless it is really a small matrix. We will exploit the existence of the inverse.

For diagonal matrix, equations are essentially decoupled, means $x$ and $y$ do not depend on each other. When you choose the variables and write down equations in the model, you can tell whether the variable matters from the matrix. If there is a 0 on diagonal in the matrix, then the corresponding variable does not matter.

7 Triangular matrix

- It takes a lot of efforts to transform a matrix into diagonal matrix, sometimes we cannot. Thus, usually we just use triangular matrices. Determinant of triangular matrix is still the multiplication of the numbers on diagonal.

- When you solve a linear system with a upper triangular matrix, start from bottom and solve each variable one by one (this method is called back substitution in Unit 2 notes). The opposite way (forward substitution) is for solving lower triangular matrix.

- Rearranging a matrix (switching rows or columns) does not change its determinant, nor whether the system has solution.

8 Gaussian Elimination Matrices

- Singular Value Decomposition (SVD) is an important method of automatically decomposing matrices to understand them. It will show up in later lectures.

- Read Unit 2 notes about examples of using Gaussian Elimination to transform a linear system into a system of triangular matrix. Then you can solve it by back or forward substitution. Figure 8.1 is the result of Gaussian Elimination.
Figure 8.1: Given \( A \) is not triangular, then use elimination matrices \( M_2, M_1 \) to make \( M_2M_1A \) a triangular matrix. In \( Ux = \hat{b}, x \) can be solved by back or forward substitution

9 LU factorization

As in Figure 8.1, \( U \) is an upper triangular matrix, and \( U = M_2M_1A \). Since \( M_2, M_1 \) are both lower triangular matrices, \( \hat{L} = M_2M_1 \) is also a lower triangular matrix. We can define \( L = \hat{L}^{-1} \), then

\[
M_2M_1A = U \implies \hat{L}A = U \implies A = LU
\]

By using gaussian elimination, you can apply LU factorization to get \( A = LU \). This helps in solving a linear system:

\[
Ax = b \implies LUx = b
\]

Define \( z = Ux \implies Lz = b \)

Use forward substitution to solve \( z \), and then use back substitution to solve \( x \) in \( Ux = z \).