Norms

- The most used vector norms are \( \|x\|_i = \sum_i |x_i| \), \( \|x\|_2 = \sqrt{\sum_i x_i^2} \), and \( \|x\|_\infty = \max_i |x_i| \)

- In some sense all norms are “equivalent”. That is, they are *interchangeable* for many theoretical pursuits

- In another sense, however, all norms are not equivalent. It is important to be aware of what a specific norm measures. All too often a misapplied norm will be used to legitimize undesirable results.
  - For example, minimizing an \( L_2 \) norm for the nodes on a piece of simulated cloth could yield wildly unsatisfactory results. One node on the moon and the rest on earth should NOT be considered a successful simulation.

- A vector norm induces a corresponding matrix norm via \( \|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \)

  - The corresponding matrix norms are as follows:
    - \( \|A\|_i = \max_j \sum_i |a_{ij}| \) which is the maximum absolute column sum
    - \( \|A\|_\infty = \max_i \sum_j |a_{ij}| \) which is the maximum absolute row sum
    - \( \|A\|_2 \) is the square root of the maximum eigenvalue of \( A^T A \)
      - i.e., the largest singular value of \( A \)

Condition Number (revisited)

- The **condition number** for solving the problem \( Ax=b \) for the matrix \( A \) is \( \|A\| \|A^{-1}\| \)
- Using \( \|A\|_2 \), or the Euclidean norm, this is \( \sigma_{\max}/\sigma_{\min} \) (as pointed out above)
- Note that it doesn’t matter what \( b \) is
- The condition number is always greater than or equal to 1
- The condition number of the identity is 1
- The condition number of a singular matrix is \( \infty \)