Diagonally Dominant Matrices
- A matrix is **diagonally dominant** if the magnitude of the diagonal element is strictly larger than the sum of the magnitudes of all the other elements in its row, and strictly larger than the sum of the magnitudes of all the other elements its column
- Guaranteed to be full rank, and thus have a unique solution
- Do not need pivoting, so can skip that step!

Symmetric Positive Definite (SPD) Matrices
- A matrix is symmetric if $A=A^T$
- A matrix is positive definite if $x^T A x > 0$ for all $x \neq 0$
- All the eigenvalues are positive
- Guaranteed to be full rank, and thus have a unique solution
- Do not need pivoting, so can skip that step!

Cholesky Factorization
- SPD matrices have an LU factorization of the form $A=LL^T$ due to symmetry
- Don’t need Gaussian Elimination to find it
- For example, suppose that we write $\begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$ to represent $A=LL^T$ for a 2x2 matrix
- Then multiplying out the right hand side gives $\begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (l_{11})^2 & l_{11}l_{21} \\ l_{11}l_{21} & (l_{11})^2 + (l_{22})^2 \end{bmatrix}$
- Thus, $l_{11} = \sqrt{a_{11}}$, $l_{21} = a_{21}/l_{11}$, and $l_{22} = \sqrt{a_{22}-(l_{21})^2}$
- The general algorithm is as follows:
  - for(j=1,n) {
    - for(k=1,j-1) for(i=j,n) $a_{ij} = a_{ik}a_{kj}$;
    - $a_{jj} = \sqrt{a_{jj}}$; for(k=j+1,n) $a_{kj}/a_{jj}$
  }
- In other words:
  - For each column $j$ of the matrix
    - Loop over all previous columns $k$, and subtract a multiple of column $k$ from the current column $j$
    - Take the square root of the diagonal entry, and scale column $j$ by that value
- Note that this algorithm above factors the matrix “in place”
Symmetric Approximation

- One way of symmetrizing a matrix $A$ is by taking $(A + A^T)/2$
  - This does not change the diagonal entries, and simply averages the off diagonal entries
  - For example, in a seminal cloth simulation paper for computer graphics by Baraff and Witkin they encountered a matrix equation, $Ax = b$, which was difficult to solve. They simply discarded the non-symmetric part of $A$ and symmetrized it reducing their problem to one that could utilize more effective solution techniques such as Cholesky decomposition

- One might pretend a matrix is SPD just to estimate an inverse to be used as a preconditioner
  - E.g., **Incomplete Cholesky preconditioning** uses a Cholesky factorization with the caveat that only the nonzero entries are modified, i.e. all the zeros remain zeroes