Natural Language Processing with Deep Learning

CS224N/Ling284

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Lecture 3: Neural net learning: Gradients by hand (matrix calculus) and algorithmically (the backpropagation algorithm)
1. Introduction

Assignment 2 is all about making sure you really understand the math of neural networks ... then we’ll let the software do it!

We’ll go through it all quickly today, but this is the one week of quarter to most work through the readings!!!

This will be a tough week for some! →
Make sure to get help if you need it:
   Visit office hours! Read tutorial materials given in the syllabus!

Thursday will be mainly linguistics! Some people find that tough too 😊

PyTorch tutorial 1:30pm Friday – great chance to get an intro to PyTorch before Ass 3!
Deep Learning Classification Task: Named Entity Recognition (NER)

- The task: *find* and *classify* names in text, for example:

  Last night, Paris Hilton wowed in a sequin gown.
  
  \[ \text{PER} \quad \text{PER} \]

  Samuel Quinn was arrested in the Hilton Hotel in Paris in April 1989.
  
  \[ \text{PER} \quad \text{PER} \quad \text{LOC} \quad \text{LOC} \quad \text{LOC} \quad \text{DATE}\]

- Possible uses:
  - Tracking mentions of particular entities in documents
  - For question answering, answers are usually named entities
  - Relating sentiment analysis to the entity under discussion
  - Often followed by Entity Linking/Canonicalization into a Knowledge Base such as Wikidata
Simple NER: Window classification using binary logistic classifier

- **Idea:** classify each word in its context window of neighboring words
- Train logistic classifier on hand-labeled data to classify center word {yes/no} for each class based on a concatenation of word vectors in a window
  - Really, we usually use multi-class softmax, but we’re trying to keep it simple 😊
- **Example:** Classify “Paris” as +/- location in context of sentence with window length 2:

  the museums in Paris are amazing to see.

  \[
  X_{\text{window}} = [x_{\text{museums}} \ x_{\text{in}} \ x_{\text{Paris}} \ x_{\text{are}} \ x_{\text{amazing}}]^T
  \]

- Resulting vector $x_{\text{window}} = x \in \mathbb{R}^{5d}$, a column vector!
- To classify all words: run classifier for each class on the vector centered on each word in the sentence
Neural classification

• Typical ML/stats softmax classifier:
  \[ p(y|x) = \frac{\exp(W_y.x)}{\sum_{c=1}^{C} \exp(W_c.x)} \]

• Learned parameters \( \theta \) are just elements of \( W \) (not input representation \( x \), which has sparse symbolic features)

• Classifier gives linear decision boundary, which can be limiting

• A neural network classifier differs in that:
  • We learn both \( W \) and (distributed!) representations for words
  • The word vectors \( x \) re-represent one-hot vectors, moving them around in an intermediate layer vector space, for easy classification with a (linear) softmax classifier
    • Conceptually, we have an embedding layer: \( x = Le \)
  • We use deep networks—more layers—that let us re-represent and-compose our data multiple times, giving a non-linear classifier

But typically, it is linear relative to the pre-final layer representation
Training with “cross entropy loss” – you use this in PyTorch!

• Until now, our objective was stated as to maximize the probability of the correct class \( y \) or equivalently we can minimize the negative log probability of that class.

• Now restated in terms of cross entropy, a concept from information theory.

• Let the true probability distribution be \( p \); let our computed model probability be \( q \).

• The cross entropy is:

\[
H(p, q) = - \sum_{c=1}^{C} p(c) \log q(c)
\]

• Assuming a ground truth (or true or gold or target) probability distribution that is 1 at the right class and 0 everywhere else, \( p = [0, ..., 0, 1, 0, ..., 0] \), then:

• *Because of one-hot \( p \), the only term left is the negative log probability of the true class \( y_i \):* 

\[
- \log p(y_i | x_i)
\]

Cross entropy can be used in other ways with a more interesting \( p \), but for now just know that you’ll want to use it as the loss in PyTorch.
NER: Binary classification for center word being location

- We do supervised training and want high score if it’s a location

\[ J_t(\theta) = \sigma(s) = \frac{1}{1 + e^{-s}} \]

\[ s = u^T h \]

\[ h = f(\mathbf{W} \mathbf{x} + \mathbf{b}) \]

\[ \mathbf{x} \quad \text{(input)} \]

\[ \mathbf{x} = [ \mathbf{x}_{\text{museums}}, \mathbf{x}_{\text{in}}, \mathbf{x}_{\text{Paris}}, \mathbf{x}_{\text{are}}, \mathbf{x}_{\text{amazing}} ] \]

\( f = \text{Some element-wise non-linear function, e.g., logistic, tanh, ReLU} \)
Non-linearities (i.e., “f” on previous slide): Why they’re needed

• Neural networks do function approximation, e.g., regression or classification
  • Without non-linearities, deep neural networks can’t do anything more than a linear transform
  • Extra layers could just be compiled down into a single linear transform: $W_1 W_2 x = Wx$
  • But, with more layers that include non-linearities, they can approximate more complex functions!
Remember: Stochastic Gradient Descent

Update equation:

\[ \theta^{\text{new}} = \theta^{\text{old}} - \alpha \nabla_{\theta} J(\theta) \]

\[ \alpha = \text{step size or learning rate} \]

i.e., for each parameter: \[ \theta_j^{\text{new}} = \theta_j^{\text{old}} - \alpha \frac{\partial J(\theta)}{\partial \theta_j^{\text{old}}} \]

In deep learning, \( \theta \) includes the data representation (e.g., word vectors) too!

How can we compute \( \nabla_{\theta} J(\theta) \)?

1. By hand
2. Algorithmically: the backpropagation algorithm
Lecture Plan

Lecture 4: Gradients by hand and algorithmically

1. Introduction (10 mins)
2. Matrix calculus (40 mins)
3. Backpropagation (35 mins)
Computing Gradients by Hand

• **Matrix calculus**: Fully vectorized gradients
  • “Multivariable calculus is just like single-variable calculus if you use matrices”
  • Much faster and more useful than non-vectorized gradients
  • But doing a non-vectorized gradient can be good for intuition; recall the first lecture for an example
  • Lecture notes and matrix calculus notes cover this material in more detail
  • You might also review Math 51, which has an online textbook: [http://web.stanford.edu/class/math51/textbook.html](http://web.stanford.edu/class/math51/textbook.html)
Gradients

- Given a function with 1 output and 1 input
  \[ f(x) = x^3 \]
- It’s gradient (slope) is its derivative
  \[ \frac{df}{dx} = 3x^2 \]

“How much will the output change if we change the input a bit?”

At \( x = 1 \) it changes about 3 times as much: \( 1.01^3 = 1.03 \)
At \( x = 4 \) it changes about 48 times as much: \( 4.01^3 = 64.48 \)
Gradients

• Given a function with 1 output and \( n \) inputs

\[ f(x) = f(x_1, x_2, \ldots, x_n) \]

• Its gradient is a vector of partial derivatives with respect to each input

\[
\frac{\partial f}{\partial x} = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right]
\]
Jacobian Matrix: Generalization of the Gradient

- Given a function with \( m \) outputs and \( n \) inputs

\[
f(x) = [f_1(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)]
\]

- It’s Jacobian is an \( m \times n \) matrix of partial derivatives

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
\]

\[
\left(\frac{\partial f}{\partial x}\right)_{ij} = \frac{\partial f_i}{\partial x_j}
\]
Chain Rule

• For composition of one-variable functions: multiply derivatives
  
  \[
  \begin{align*}
  &z = 3y \\
  &y = x^2 \\
  &\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = (3)(2x) = 6x
  \end{align*}
  \]

• For multiple variables at once: multiply Jacobians
  
  \[
  \begin{align*}
  &h = f(z) \\
  &z = Wx + b \\
  &\frac{\partial h}{\partial x} = \frac{\partial h}{\partial z} \frac{\partial z}{\partial x} = \ldots
  \end{align*}
  \]
Example Jacobian: Elementwise activation Function

\[ h = f(z), \text{ what is } \frac{\partial h}{\partial z} ? \]

\[ h_i = f(z_i) \]

\( h, z \in \mathbb{R}^n \)
Example Jacobian: Elementwise activation Function

\[ h = f(z), \text{ what is } \frac{\partial h}{\partial z}? \quad h, z \in \mathbb{R}^n \]

\[ h_i = f(z_i) \]

Function has \( n \) outputs and \( n \) inputs \( \rightarrow \) \( n \) by \( n \) Jacobian
Example Jacobian: Elementwise activation Function

\[ h = f(z), \text{ what is } \frac{\partial h}{\partial z} ? \]
\[ h_i = f(z_i) \]

\[
\left( \frac{\partial h}{\partial z} \right)_{ij} = \frac{\partial h_i}{\partial z_j} = \frac{\partial}{\partial z_j} f(z_i) \]

definition of Jacobian
Example Jacobian: Elementwise activation Function

$h = f(z)$, what is \( \frac{\partial h}{\partial z} \)?

\( h_i = f(z_i) \)

\[
\begin{pmatrix}
\frac{\partial h}{\partial z}
\end{pmatrix}_{ij} = \frac{\partial h_i}{\partial z_j} = \frac{\partial}{\partial z_j} f(z_i)
\]

definition of Jacobian

\[
= \begin{cases} 
  f'(z_i) & \text{if } i = j \\
  0 & \text{if otherwise}
\end{cases}
\]

regular 1-variable derivative
Example Jacobian: Elementwise activation Function

\[ h = f(z), \text{ what is } \frac{\partial h}{\partial z} ? \quad h, z \in \mathbb{R}^n \]

\[ h_i = f(z_i) \]

\[
\left( \frac{\partial h}{\partial z} \right)_{i,j} = \frac{\partial h_i}{\partial z_j} = \frac{\partial}{\partial z_j} f(z_i) \quad \text{definition of Jacobian}
\]

\[ = \begin{cases} f'(z_i) & \text{if } i = j \\ 0 & \text{if otherwise} \end{cases} \quad \text{regular 1-variable derivative} \]

\[
\frac{\partial h}{\partial z} = \begin{pmatrix} f'(z_1) & 0 \\ \vdots & \ddots \\ 0 & \cdots & f'(z_n) \end{pmatrix} = \text{diag}(f'(z))
\]
Other Jacobians

\[
\frac{\partial}{\partial x} (Wx + b) = W
\]
Other Jacobians

\[
\frac{\partial}{\partial x} (Wx + b) = W
\]

\[
\frac{\partial}{\partial b} (Wx + b) = I \quad (\text{Identity matrix})
\]
Other Jacobians

\[ \frac{\partial}{\partial x} (Wx + b) = W \]

\[ \frac{\partial}{\partial b} (Wx + b) = I \quad \text{(Identity matrix)} \]

\[ \frac{\partial}{\partial u} (u^T h) = h^T \]

Fine print: This is the correct Jacobian. Later we discuss the “shape convention”; using it the answer would be \( h \).
Other Jacobians

\[ \frac{\partial}{\partial x} (Wx + b) = W \]
\[ \frac{\partial}{\partial b} (Wx + b) = I \quad \text{(Identity matrix)} \]
\[ \frac{\partial}{\partial u} (u^T h) = h^T \]

- Compute these at home for practice!
- Check your answers with the lecture notes
Back to our Neural Net!

\[ s = u^T h \]

\[ h = f(Wx + b) \]

\( x \) (input)

\( x = [x_{\text{museums}}, x_{\text{in}}, x_{\text{Paris}}, x_{\text{are}}, x_{\text{amazing}}] \)
Back to our Neural Net!

- Let’s find $\frac{\partial s}{\partial b}$
- Really, we care about the gradient of the loss $J_t$ but we will compute the gradient of the score for simplicity

$$s = u^T h$$

$$h = f(Wx + b)$$

$x$ (input)

$$x = \begin{bmatrix} x_{\text{museums}} & x_{\text{in}} & x_{\text{Paris}} & x_{\text{are}} & x_{\text{amazing}} \end{bmatrix}$$
1. Break up equations into simple pieces

\[ s = u^T h \]

\[ h = f(W x + b) \]

\[ z = W x + b \]

\[ x \quad \text{(input)} \]

Carefully define your variables and keep track of their dimensionality!
2. Apply the chain rule

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = Wx + b \]
\[ x \quad \text{(input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} \]
2. Apply the chain rule

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = Wx + b \]
\[ x \quad \text{(input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} \]
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\[ s = u^T h \]
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\[ z = Wx + b \]
\[ x \text{ (input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} \]
2. Apply the chain rule

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = Wx + b \]
\[ x \quad \text{(input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} \]
3. Write out the Jacobians

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = Wx + b \]
\[ x \text{ (input)} \]

Useful Jacobians from previous slide

\[ \frac{\partial}{\partial u} (u^T h) = h^T \]
\[ \frac{\partial}{\partial z} (f(z)) = \text{diag}(f'(z)) \]
\[ \frac{\partial}{\partial b} (Wx + b) = I \]
3. Write out the Jacobians

\[ s = u^T h \]

\[ h = f(z) \]

\[ z = Wx + b \]

\[ x \text{ (input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} \]

\[ u^T \]

Useful Jacobians from previous slide

\[ \frac{\partial}{\partial u}(u^T h) = h^T \]

\[ \frac{\partial}{\partial z}(f(z)) = \text{diag}(f'(z)) \]

\[ \frac{\partial}{\partial b}(Wx + b) = I \]
3. Write out the Jacobians

\[ s = u^T h \]

\[ h = f(z) \]

\[ z = Wx + b \]

\[ x \text{ (input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} \]

\[ u^T \text{diag}(f'(z)) \]

Useful Jacobians from previous slide

\[ \frac{\partial}{\partial u}(u^T h) = h^T \]

\[ \frac{\partial}{\partial z}(f(z)) = \text{diag}(f'(z)) \]

\[ \frac{\partial}{\partial b}(Wx + b) = I \]
3. Write out the Jacobians

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = Wx + b \]
\[ x \quad \text{(input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} = u^T \text{diag}(f'(z))I \]

Useful Jacobians from previous slide

\[ \frac{\partial}{\partial u}(u^T h) = h^T \]
\[ \frac{\partial}{\partial z}(f(z)) = \text{diag}(f'(z)) \]
\[ \frac{\partial}{\partial b}(Wx + b) = I \]
3. Write out the Jacobians

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = Wx + b \]
\[ x \text{ (input)} \]

\[ \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b} \]
\[ = u^T \text{diag}(f'(z)) I \]
\[ = u^T \odot f'(z) \]

Useful Jacobians from previous slide

\[ \frac{\partial}{\partial u}(u^T h) = h^T \]
\[ \frac{\partial}{\partial z}(f(z)) = \text{diag}(f'(z)) \]
\[ \frac{\partial}{\partial b}(Wx + b) = I \]

⊙ = Hadamard product = element-wise multiplication of 2 vectors to give vector
Re-using Computation

• Suppose we now want to compute \( \frac{\partial s}{\partial W} \)

• Using the chain rule again:

\[
\frac{\partial s}{\partial W} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial W}
\]
Re-using Computation

- Suppose we now want to compute \( \frac{\partial s}{\partial W} \).
  - Using the chain rule again:

    \[
    \frac{\partial s}{\partial W} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial W} \\
    \frac{\partial s}{\partial b} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial b}
    \]

    The same! Let’s avoid duplicated computation ...
Re-using Computation

• Suppose we now want to compute $\frac{\partial s}{\partial W}$.

• Using the chain rule again:

\[
\begin{align*}
\frac{\partial s}{\partial W} &= \delta \frac{\partial z}{\partial W} \\
\frac{\partial s}{\partial b} &= \delta \frac{\partial z}{\partial b} = \delta \\
\delta &= \frac{\partial s}{\partial h} \frac{\partial h}{\partial z} = u^T \circ f'(z)
\end{align*}
\]

$\delta$ is the local error signal.
Derivative with respect to Matrix: Output shape

- What does \( \frac{\partial s}{\partial W} \) look like? \( W \in \mathbb{R}^{n \times m} \)
- 1 output, \( nm \) inputs: 1 by \( nm \) Jacobian?
- Inconvenient to then do \( \theta^{new} = \theta^{old} - \alpha \nabla_{\theta} J(\theta) \)
Derivative with respect to Matrix: Output shape

- What does $\frac{\partial s}{\partial W}$ look like? $W \in \mathbb{R}^{n \times m}$
- 1 output, $nm$ inputs: 1 by $nm$ Jacobian?
  - Inconvenient to then do $\theta^{new} = \theta^{old} - \alpha \nabla_{\theta} J(\theta)$
- Instead, we leave pure math and use the **shape convention**: the shape of the gradient is the shape of the parameters!
  - So $\frac{\partial s}{\partial W}$ is $n$ by $m$:
    $$
    \begin{bmatrix}
    \frac{\partial s}{\partial W_{11}} & \cdots & \frac{\partial s}{\partial W_{1m}} \\
    \vdots & \ddots & \vdots \\
    \frac{\partial s}{\partial W_{n1}} & \cdots & \frac{\partial s}{\partial W_{nm}}
    \end{bmatrix}
    $$
Derivative with respect to Matrix

- What is \( \frac{\partial s}{\partial W} = \delta \frac{\partial z}{\partial W} \)
- \( \delta \) is going to be in our answer
- The other term should be \( \mathbf{x} \) because \( z = W \mathbf{x} + b \)
- Answer is: \( \frac{\partial s}{\partial W} = \delta^T \mathbf{x}^T \)

\( \delta \) is local error signal at \( z \)
\( \mathbf{x} \) is local input signal
Deriving local input gradient in backprop

- For $\frac{\partial z}{\partial W}$ in our equation:
  \[
  \frac{\partial s}{\partial W} = \delta \frac{\partial z}{\partial W} = \delta \frac{\partial}{\partial W}(Wx + b)
  \]

- Let’s consider the derivative of a single weight $W_{ij}$
  - $W_{ij}$ only contributes to $z_i$  
    - For example: $W_{23}$ is only used to compute $z_2$ not $z_1$

\[
\frac{\partial z_i}{\partial W_{ij}} = \frac{\partial}{\partial W_{ij}} W_{ij}x + b_i = \frac{\partial}{\partial W_{ij}} \sum_{k=1}^{d} W_{ik}x_k = x_j
\]
Why the Transposes?

\[ \frac{\partial s}{\partial W} = \delta^T x^T \]

\[ [n \times m] \quad [n \times 1][1 \times m] \]

\[ \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix} \begin{bmatrix} x_1, \ldots, x_m \end{bmatrix} = \begin{bmatrix} \delta_1 x_1 & \ldots & \delta_1 x_m \\ \vdots & \ddots & \vdots \\ \delta_n x_1 & \ldots & \delta_n x_m \end{bmatrix} \]

- Hacky answer: this makes the dimensions work out!
- Useful trick for checking your work!
- Full explanation in the lecture notes
- Each input goes to each output – you want to get outer product
What shape should derivatives be?

• Similarly, \( \frac{\partial s}{\partial b} = h^T \circ f'(z) \) is a row vector
  • But shape convention says our gradient should be a column vector because \( \mathbf{b} \) is a column vector ...

• Disagreement between Jacobian form (which makes the chain rule easy) and the shape convention (which makes implementing SGD easy)
  • We expect answers in the assignment to follow the shape convention
  • But Jacobian form is useful for computing the answers
What shape should derivatives be?

Two options:

1. Use Jacobian form as much as possible, reshape to follow the shape convention at the end:
   • What we just did. But at the end transpose $\frac{\partial s}{\partial b}$ to make the derivative a column vector, resulting in $\delta^T$.

2. Always follow the shape convention
   • Look at dimensions to figure out when to transpose and/or reorder terms
   • The error message $\delta$ that arrives at a hidden layer has the same dimensionality as that hidden layer.
3. Backpropagation

We’ve almost shown you backpropagation

It’s taking derivatives and using the (generalized, multivariate, or matrix) chain rule

Other trick:

We **re-use** derivatives computed for higher layers in computing derivatives for lower layers to minimize computation
Computation Graphs and Backpropagation

- Software represents our neural net equations as a graph
  - Source nodes: inputs
  - Interior nodes: operations

\[
\begin{align*}
  s &= u^T h \\
  h &= f(z) \\
  z &= W x + b \\
  x &= \text{(input)}
\end{align*}
\]
Computation Graphs and Backpropagation

- Software represents our neural net equations as a graph
  - Source nodes: inputs
  - Interior nodes: operations
  - Edges pass along result of the operation

\[
s = u^T h
\]
\[
h = f(z)
\]
\[
z = Wx + b
\]
\[
x \quad \text{(input)}
\]
Computation Graphs and Backpropagation

- Software represents our neural net equations as a graph

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = e + b \]

"Forward Propagation"

\[ s \]
Backpropagation

- Then go backwards along edges
- Pass along gradients

\( s = u^T h \)
\( h = f(z) \)
\( z = Wx + b \)
\( x \) (input)
Backpropagation: Single Node

- Node receives an “upstream gradient”
- Goal is to pass on the correct “downstream gradient”
Backpropagation: Single Node

- Each node has a **local gradient**
- The gradient of its output with respect to its input

\[ h = f(z) \]
Backpropagation: Single Node

- Each node has a **local gradient**
- The gradient of its output with respect to its input

\[ h = f(z) \]

![Diagram](image.png)

Downstream gradient

Local gradient

Upstream gradient

\[
\frac{\partial h}{\partial z} = \frac{\partial s}{\partial h} \frac{\partial h}{\partial z}
\]
Backpropagation: Single Node

- Each node has a **local gradient**
- The gradient of its output with respect to its input

\[
h = f(z)
\]

\[
\frac{\partial s}{\partial z} = \frac{\partial s}{\partial h} \cdot \frac{\partial h}{\partial z}
\]

**Downstream gradient**

**Local gradient**

**Upstream gradient**
Backpropagation: Single Node

- What about nodes with multiple inputs?

\[ z = Wx \]
Backpropagation: Single Node

- Multiple inputs $\rightarrow$ multiple local gradients

\[ z = Wx \]

\[ \frac{\partial s}{\partial W} = \frac{\partial s}{\partial z} \frac{\partial z}{\partial W} \]

\[ \frac{\partial s}{\partial x} = \frac{\partial s}{\partial z} \frac{\partial z}{\partial x} \]

Downstream gradients

Local gradients

Upstream gradient
An Example

\[ f(x, y, z) = (x + y) \max(y, z) \]

\[ x = 1, y = 2, z = 0 \]
An Example

Forward prop steps

\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

\[ f(x, y, z) = (x + y) \max(y, z) \]
\[ x = 1, y = 2, z = 0 \]
An Example

Forward prop steps

\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

\[ f(x, y, z) = (x + y) \max(y, z) \]
\[ x = 1, y = 2, z = 0 \]
An Example

Forward prop steps

\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

Local gradients

\[ \frac{\partial a}{\partial x} = 1 \quad \frac{\partial a}{\partial y} = 1 \]

\[ f(x, y, z) = (x + y) \max(y, z) \]
\[ x = 1, y = 2, z = 0 \]
An Example

Forward prop steps
\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

Local gradients
\[
\frac{\partial a}{\partial x} = 1 \quad \frac{\partial a}{\partial y} = 1 \\
\frac{\partial b}{\partial y} = 1(y > z) = 1 \quad \frac{\partial b}{\partial z} = 1(z > y) = 0
\]

\[ f(x, y, z) = (x + y) \max(y, z) \]
\[ x = 1, y = 2, z = 0 \]
An Example

\[ f(x, y, z) = (x + y) \max(y, z) \]
\[ x = 1, y = 2, z = 0 \]

Forward prop steps

\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

Local gradients

\[ \frac{\partial a}{\partial x} = 1 \quad \frac{\partial a}{\partial y} = 1 \]
\[ \frac{\partial b}{\partial y} = 1(y > z) = 1 \quad \frac{\partial b}{\partial z} = 1(z > y) = 0 \]
\[ \frac{\partial f}{\partial a} = b = 2 \quad \frac{\partial f}{\partial b} = a = 3 \]

```
+-----+      +-----+      +-----+      +-----+
| 1   |      | 3   |      | 6   |      | 2   |
| 2   |      |      |      |      |      | 0   |
| 2   |      | 2    |      |      |      |      |
|      |      |      |      |      |      |      |
```

\[ x, y, z \]
An Example

\[ f(x, y, z) = (x + y) \max(y, z) \]
\[ x = 1, y = 2, z = 0 \]

Forward prop steps:

\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

Local gradients:

\[ \frac{\partial a}{\partial x} = 1 \quad \frac{\partial a}{\partial y} = 1 \]
\[ \frac{\partial b}{\partial y} = 1(y > z) = 1 \quad \frac{\partial b}{\partial z} = 1(z > y) = 0 \]
\[ \frac{\partial f}{\partial a} = b = 2 \quad \frac{\partial f}{\partial b} = a = 3 \]

Diagram:

\[ \frac{\partial f}{\partial f} = 1 \]
An Example

Forward prop steps
\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

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upstream * local = downstream
An Example

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Gradients sum at outward branches
Gradients sum at outward branches

\[ a = x + y \]
\[ b = \max(y, z) \]
\[ f = ab \]

\[ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial a} \frac{\partial a}{\partial y} + \frac{\partial f}{\partial b} \frac{\partial b}{\partial y} \]
Node Intuitions

\[
f(x, y, z) = (x + y) \max(y, z)
\]

\[
x = 1, y = 2, z = 0
\]

- + “distributes” the upstream gradient to each summand

\[\begin{align*}
x &\rightarrow 1 \\
& \quad 2 \\
& \quad 2 \\
y &\rightarrow 2 \\
& \quad 2 \\
z &\rightarrow 0 \\
\end{align*}\]

\[\begin{align*}
&+ \rightarrow 3 \\
&\quad 3 \\
&\quad 2 \\
\max &\rightarrow 2 \\
&\quad 2 \\
* &\rightarrow 6 \\
&\quad 1
\end{align*}\]
Node Intuitions

\[
f(x, y, z) = (x + y) \max(y, z)
\]
\[
x = 1, y = 2, z = 0
\]

- + “distributes” the upstream gradient to each summand
- max “routes” the upstream gradient
Node Intuitions

\[ f(x, y, z) = (x + y) \max(y, z) \]
\[ x = 1, y = 2, z = 0 \]

- + “distributes” the upstream gradient
- \( \max \) “routes” the upstream gradient
- * “switches” the upstream gradient
Efficiency: compute all gradients at once

- Incorrect way of doing backprop:
  - First compute $\frac{\partial s}{\partial b}$

\[
\begin{align*}
  s &= u^T h \\
  h &= f(z) \\
  z &= W x + b \\
  x &\quad \text{(input)}
\end{align*}
\]
Efficiency: compute all gradients at once

- Incorrect way of doing backprop:
  - First compute $\frac{\partial s}{\partial b}$
  - Then independently compute $\frac{\partial s}{\partial W}$
  - Duplicated computation!

\[ s = u^T h \]
\[ h = f(z) \]
\[ z = Wx + b \]
\[ x \text{ (input)} \]
Efficiency: compute all gradients at once

- Correct way:
  - Compute all the gradients at once
  - Analogous to using $\delta$ when we computed gradients by hand

\[
\begin{align*}
  s &= u^T h \\
  h &= f(z) \\
  z &= Wx + b \\
  x &= \text{(input)}
\end{align*}
\]
1. Fprop: visit nodes in topological sort order
   - Compute value of node given predecessors
2. Bprop:
   - initialize output gradient = 1
   - visit nodes in reverse order:
     Compute gradient wrt each node using gradient wrt successors
     \( \{y_1, y_2, \ldots, y_n\} = \text{successors of } x \)
     \[
     \frac{\partial z}{\partial x} = \sum_{i=1}^{n} \frac{\partial z}{\partial y_i} \frac{\partial y_i}{\partial x}
     \]

Done correctly, big O() complexity of fprop and bprop is the same

In general, our nets have regular layer-structure and so we can use matrices and Jacobians...
The gradient computation can be automatically inferred from the symbolic expression of the fprop.

Each node type needs to know how to compute its output and how to compute the gradient wrt its inputs given the gradient wrt its output.

Modern DL frameworks (Tensorflow, PyTorch, etc.) do backpropagation for you but mainly leave layer/node writer to hand-calculate the local derivative.
```
class ComputationalGraph(object):
    #...
    def forward(inputs):
        # 1. [pass inputs to input gates...]
        # 2. forward the computational graph:
        for gate in self.graph.nodes_topologically_sorted():
            gate.forward()
        return loss # the final gate in the graph outputs the loss
    def backward():
        for gate in reversed(self.graph.nodes_topologically_sorted()):
            gate.backward() # little piece of backprop (chain rule applied)
        return inputs_gradients
```
Implementation: forward/backward API

\[
\begin{align*}
\partial L/\partial x & \quad \text{(partial derivative of loss with respect to } x) \\
\partial L/\partial z & \quad \text{(partial derivative of loss with respect to } z)
\end{align*}
\]

(x, y, z are scalars)

class MultiplyGate(object):
def forward(x, y):
z = x*y
return z
def backward(dz):
    # dx = ... #todo
    # dy = ... #todo
    return [dx, dy]
Implementation: forward/backward API

(x, y, z are scalars)
Manual Gradient checking: Numeric Gradient

- For small $h \approx 1\text{e-}4$,
  \[ f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} \]
- Easy to implement correctly
- But approximate and very slow:
  - You have to recompute $f$ for every parameter of our model
- Useful for checking your implementation
  - In the old days, we hand-wrote everything, doing this everywhere was the key test
  - Now much less needed; you can use it to check layers are correctly implemented
We’ve mastered the core technology of neural nets! 🎉

- Backpropagation: recursively (and hence efficiently) apply the chain rule along computation graph
  - [downstream gradient] = [upstream gradient] x [local gradient]
- Forward pass: compute results of operations and save intermediate values
- Backward pass: apply chain rule to compute gradients
Why learn all these details about gradients?

- **Modern deep learning frameworks compute gradients for you!**
  - Come to the PyTorch introduction this Friday!

- But why take a class on compilers or systems when they are implemented for you?
  - Understanding what is going on under the hood is useful!

- Backpropagation doesn’t always work perfectly out of the box
  - Understanding why is crucial for debugging and improving models
  - See Karpathy article (in syllabus):
    - https://medium.com/@karpathy/yes-you-should-understand-backprop-e2f06eab496b
    - Example in future lecture: exploding and vanishing gradients