CS231a HW2

Fei Xia

*Adapted from last year’s slides*
Overview

Focus on

1. Problem 2 - Image Rectification
2. Problem 4 - Structure from Motion

Will briefly cover Problem 1 - Fundamental Matrix Estimation

Will not cover Problem 3 - Factorization Method
(Enough details covered in class. Please visit our office hour if you need help)
Problem 1 - Fundamental Matrix Estimation

Fundamental Matrix

- A matrix which maps the relationship of correspondences between stereo images

\[ p^T F p' = 0 \]

\[ F = K^{-T} \cdot [T_x] \cdot R \cdot K'^{-1} \]
Problem 1 - Fundamental Matrix Estimation

Ex)

Image and correspondences given in the homework
Problem 1 - Fundamental Matrix Estimation

How to compute F?

- Eight point algorithm
Problem 1 - Fundamental Matrix Estimation

How to compute F?

- Eight point algorithm

Problem?

- W is highly unbalanced (not well conditioned)
Problem 1 - Fundamental Matrix Estimation

Possible improvement?

Pre-condition our linear system to get more stable result

- origin = centroid of the image
- mean square distance of the image points from origin is ~2px
Problem 1 - Fundamental Matrix Estimation

Final step

- Reduce rank(F) to 2
Problem 1 - Fundamental Matrix Estimation

Epipolar lines
Computing epipolar lines from $F$

$I = Fp'$

$I' = F^Tp$
Problem 2 - Image Rectification

Image Rectification
Making two images “parallel”
All correspondences lie on the same y axis
Problem 2 - Image Rectification

Application

Novel view synthesis (view morphing) based on rectified images
Problem 2 - Image Rectification

Find a homography $H_1$ and $H_2$ such that corresponding points of image 1 and image 2 lie on the same y axis.
Problem 2 - Image Rectification

Overview

- Find $H_2$: morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ($e_2$)
  - Find homography $H_2$ which brings $e_2$ to a point at infinity
- Find $H_1$: minimize square distance from rectified image 2 to image 1
Problem 2 - Image Rectification

Find a homography $H_1$ and $H_2$ such that corresponding points of image 1 and image 2 lie on the same y axis

1. Find $H_2$ such that epipolar lines are all horizontally aligned (parallel)
2. Find $H_1$ which minimizes square distance between corresponding points of the image
Problem 2 - Image Rectification

Find a homography $H_1$ and $H_2$ such that corresponding points of image 1 and image 2 lie on the same y axis

1. **Find $H_2$ such that epipolar lines are all horizontally aligned (parallel)**
2. **Find $H_1$ which minimizes square distance between corresponding points of the image**
Problem 2 - Image Rectification

Find a homography $H_1$ and $H_2$ such that corresponding points of image 1 and image 2 lie on the same $y$ axis

1. Find $H_2$ such that epipolar lines are all horizontally aligned (parallel)
2. Find $H_1$ which minimizes square distance between corresponding points of the image
Problem 2 - Image Rectification

Overview

- **Find** $H_2$: morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ($e_2$)
  - Find homography $H_2$ which brings $e_2$ to a point at infinity
- **Find** $H_1$: minimize square distance from rectified image 2 to image 1
Problem 2 - Image Rectification

1. Find epipole of image 2, \( e_2 \)
2. Find homography \( H_2 \) which brings \( e_2 \) to a point at infinity

\[ \Rightarrow H_2 e_2 \]
Problem 2 - Image Rectification

Overview

- Find $H_2$: morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ($e_2$)
  - Find homography $H_2$ which brings $e_2$ to a point at infinity
- Find $H_1$: minimize square distance from rectified image 2 to image 1
Problem 2 - Image Rectification

Find epipole of image 2 ($e_2$)

- Epipole is an intersection of epipolar lines
  - epipole lies on every epipolar line
  - defining epipolar line $\ell$ such that all points on the line are in set $\{x|\ell^T x = 0\}$, formulate a linear system of equations on the right
  - Solve using SVD
Problem 2 - Image Rectification

Computing epipolar lines from $F$

\[
\begin{align*}
    l &= Fp' \\
    l' &= F^T p
\end{align*}
\]

• $l = Fp'$ is the epipolar line associated with $p'$
• $l' = F^T p$ is the epipolar line associated with $p$
Problem 2 - Image Rectification

Overview

- Find $H_2$: morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ($e_2$)
  - Find homography $H_2$ which brings $e_2$ to a point at infinity
- Find $H_1$: minimize square distance from rectified image 2 to image 1
Problem 2 - Image Rectification

Find homography $H_2$ which brings $e_2$ to a point at infinity

Intuition (from Hartley, Zisserman):

To keep image as realistic as possible after transformation, we want to keep $H_2$ to act as a rigid transformation in the neighborhood of a given selected point $x_0$ of the image

Center of the image is often a good choice for $x_0$
Problem 2 - Image Rectification

Find homography $H_2$ which brings $e_2$ to a point at infinity

1. **Translate image coordinate so that the origin will be at the image center**
2. Rotate the image so that $e_2$ will lie horizontal axis at some point $(f, 0, 1)$
3. Bring $e_2$ to infinity on $(f, 0, 0)$
4. Translate image coordinate back to the original origin

\[
T = \begin{bmatrix}
1 & 0 & -\frac{\text{width}}{2} \\
0 & 1 & -\frac{\text{height}}{2} \\
0 & 0 & 1
\end{bmatrix}
\]
Problem 2 - Image Rectification

Find homography $H_2$ which brings $e_2$ to a point at infinity

1. Translate image coordinate so that the origin will be at the image center
2. **Rotate the image so that $e_2$ will lie horizontal axis at some point ($f$, 0, 1)**
3. Bring $e_2$ to infinity on ($f$, 0, 0)
4. Translate image coordinate back to the original origin
Problem 2 - Image Rectification

Find homography $H_2$ which brings $e_2$ to a point at infinity

1. Translate image coordinate so that the origin will be at the image center
2. Rotate the image so that $e_2$ will lie horizontal axis at some point $(f, 0, 1)$
3. Bring $e_2$ to infinity on $(f, 0, 0)$
4. Translate image coordinate back to the original origin
Problem 2 - Image Rectification

Find homography $H_2$ which brings $e_2$ to a point at infinity

1. Translate image coordinate so that the origin will be at the image center
2. Rotate the image so that $e_2$ will lie horizontal axis at some point $(f, 0, 1)$
3. Bring $e_2$ to infinity on $(f, 0, 0)$
4. Translate image coordinate back to the original origin
Problem 2 - Image Rectification

Find homography $H_2$ which brings $e_2$ to a point at infinity

$$H_2 = T^{-1} GRT$$
Problem 2 - Image Rectification

Overview

- Find $H_2$: morph image 2 to make all epipolar lines parallel
  - Find epipole of image 2 ($e_2$)
  - Find homography $H_2$ which brings $e_2$ to a point at infinity
- Find $H_1$: minimize square distance from rectified image 2 to image 1
Problem 2 - Image Rectification

Find $H_1$: minimize square distance from rectified image 2 to image 1

$$\arg \min_{H_1} \sum_i \|H_1 p_i - H_2 p'_i\|^2$$

Why this? $H_1^{-T} \ell_i = H_2^{-T} \ell'_i$  
Reduce matching lines to matching points

We will take a simpler approach for this problem
Problem 2 - Image Rectification

Find $H_1$: minimize square distance from rectified image 2 to image 1

We can show that

$$H_1 = H_A H_2 M$$

where

$$H_A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This form of $H_A$ allows a transformation of $H_1$ where its epipole is at $(1, 0, 0)$ - a page long proof of this in Chapter 11 of Hartley & Zisserman’s textbook
Problem 2 - Image Rectification

Find $H_1$: minimize square distance from rectified image 2 to image 1

We can also show that $M = [e] \times F + e v^T$

because 1) $M = [e] \times F$

- we know $F$ up to scale
- any skew-symmetric matrix $X$ (including $[e]_x$) is $A = A^3$ up to scale

$F = [e] \times M = [e] \times [e] \times [e] \times M = [e] \times [e] \times [e] \times F$
Problem 2 - Image Rectification

Find $H_1$: minimize square distance from rectified image 2 to image 1

We can also show that

$$M = [e]_x F + ev^T$$

because 2) if columns of M are added by any scalar multiple of $e$,

up to scale,

$$F = [e]_x M$$

Therefore,

$$M = [e]_x F + ev^T$$

is more general case of defining $M$

Where in practice, we use

$$v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$
Problem 2 - Image Rectification

Find $H_1$: minimize square distance from rectified image 2 to image 1

Which reduces to

$$\arg \min_{H_A} \sum_i \| H_A \hat{p}_i - \hat{p}'_i \|^2$$

where $\hat{p}_i = H_2 M p_i$, $\hat{p}'_i = H_2 p_i$
Problem 2 - Image Rectification

Find $H_1$: minimize square distance from rectified image 2 to image 1

Which reduces down to solving least squares $W[a_1, a_2, a_3]^T = b$

Where

$$W = \begin{bmatrix} \hat{x}_1 & \hat{y}_1 & 1 \\ \vdots & \vdots & \vdots \\ \hat{x}_n & \hat{y}_n & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \hat{x}_1' \\ \vdots \\ \hat{x}_n' \end{bmatrix}$$

and

$$\hat{p}_i = H_2M p_i \quad \hat{p}_i' = H_2 p_i$$
Problem 2 - Image Rectification

Find a homography $H_1$ and $H_2$ such that corresponding points of image 1 and image 2 lie on the same $y$ axis

1. Find $H_2$ such that epipolar lines are all horizontally aligned (parallel)
2. Find $H_1$ which minimizes square distance between corresponding points of the image
Problem 4: Structure from Motion

Structure from Motion (SfM)

Estimating 3D structure from 2D images that may be coupled with local motions

Input: 2D images

Output: 3D structure (+ camera extrinsic)
Problem 4: Structure from Motion

In this homework, we explore two different approaches for SfM

- Factorization Method (problem 3) - Tomasi & Kanade algorithm
- Bundle Adjustment (problem 4)
Problem 4: Structure from Motion

1. Compute essential matrix $E$ from two views
2. Use $E$ to make initial estimate of relative rotation $R$ and translation $T$
3. Estimate 3D location of the reconstruction given $RT$
4. Optimize (bundle adjustment)
   - Jointly optimize all relative camera motions ($R$’s and $T$’s)
   - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames
Problem 4: Structure from Motion

1. Compute essential matrix $E$ from two views
2. **Use $E$ to make initial estimate of relative rotation $R$ and translation $T$**
3. Estimate 3D location of the reconstruction given $RT$
4. Optimize (bundle adjustment)
   - Jointly optimize all relative camera motions ($R$’s and $T$’s)
   - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames
Problem 4: Structure from Motion

1. Compute essential matrix $E$ from two views
2. Use $E$ to make initial estimate of relative rotation $R$ and translation $T$
3. **Estimate 3D location of the reconstruction given $RT$**
4. Optimize (bundle adjustment)
   - Jointly optimize all relative camera motions ($R$’s and $T$’s)
   - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames
Problem 4: Structure from Motion

Estimate 3D location of the reconstruction given 1. projective camera matrix 2. their image coordinates

Two different possible approaches:

1. Formulating a linear equation to solve
2. Nonlinear optimization to minimize reprojection error
Problem 4: Structure from Motion

Estimate 3D location of the reconstruction given 1. projective camera matrix 2. their image coordinates

Formulating a linear equation to solve, using \([p=MP]\)
Solve for \(AP=0\) (using SVD) where:

\[
A = \begin{bmatrix}
p_{1,1}m^{3T} - m^{1T} \\
p_{1,2}m^{3T} - m^{2T} \\
\vdots \\
p_{n,1}m^{3T} - m^{1T} \\
p_{n,2}m^{3T} - m^{2T}
\end{bmatrix}
\]

\(p_{ij}: (x, y)[j]\) coordinate of \(i\)th image
\(m^{kT}\): \(k\)-th row of \(M\)
Problem 4: Structure from Motion

Estimate 3D location of the reconstruction given 1. projective camera matrix 2. their image coordinates

Nonlinear optimization to minimize reprojection error

Gauss-Newton algorithm

\[ \hat{P} = \hat{P} - (J^T J)^{-1} J^T e \]

Begin from linear estimation for better initialization
How to define error and Jacobian?
Problem 4: Structure from Motion

(reprojection) error: difference between the projected point \((M_iP)\) and ground-truth image coordinate \(p_i\)

Jacobian:

\[
J = \begin{bmatrix}
\frac{\partial e_1}{\partial P_1} & \frac{\partial e_1}{\partial P_2} & \frac{\partial e_1}{\partial P_3} \\
\vdots & \vdots & \vdots \\
\frac{\partial e_m}{\partial P_1} & \frac{\partial e_m}{\partial P_2} & \frac{\partial e_m}{\partial P_3}
\end{bmatrix}
\]
Problem 4: Structure from Motion

1. Compute essential matrix $E$ from two views
2. Use $E$ to make initial estimate of relative rotation $R$ and translation $T$
3. Estimate 3D location of the reconstruction given $RT$
4. Optimize (bundle adjustment)
   - Jointly optimize all relative camera motions ($R$’s and $T$’s)
   - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames
Problem 4: Structure from Motion

Use $E$ to make initial estimate of relative rotation $R$ and translation $T$

$$E = \begin{bmatrix} T_x \end{bmatrix} \cdot R$$

1. To compute $R$: Given the singular value decomposition $E = UDV^T$, we can rewrite $E = MQ$ where $M = UZU^T$ and $Q = UWV^T$ or $UW^TV^T$, where

$$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that this factorization of $E$ only guarantees that $Q$ is orthogonal. To find a rotation, we simply compute $R = (\det Q)Q$.

2. To compute $T$: Given that $E = U\Sigma V^T$, $T$ is simply either $u_3$ or $-u_3$, where $u_3$ is the third column vector of $U$. 
Problem 4: Structure from Motion

Use E to make initial estimate of relative rotation $R$ and translation $T$

However, this gives four pairs of rotation and translation, $(R_1, R_2) \times (T, -T)$

How do we find out which $R$ and $T$ is the correct one?
Problem 4: Structure from Motion

There exists only one solution that will consistently produce 3D points which are both in front of the camera.

Compute 3D point’s location in the RT frame!

- Find 3D location of the image points given RT frame
- Chose the one which has the most 3D points with positive depth (z-coordinate) with respect to both camera frame
Problem 4: Structure from Motion

1. Compute fundamental matrix $F$ from two views
2. **Use $F$ to make initial estimate of relative rotation $R$ and translation $T$**
3. **Estimate 3D location of the reconstruction given $RT$**
4. Optimize (bundle adjustment)
   - Jointly optimize all relative camera motions ($R$’s and $T$’s)
   - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames