Advertising on the Web
Online Algorithms

- **Classic model of algorithms**
  - You get to see the entire input, then compute some function of it
  - In this context, “offline algorithm”

- **Online Algorithms**
  - You get to see the input one piece at a time, and need to make irrevocable decisions along the way
  - Similar to the data stream model
Sponsored Search: Ads

Query-to-advertiser graph:

[Andersen, Lang: Communities from seed sets, 2006]
This is an online problem: We have to make decisions as queries/topics show up. We do not know what topics will show up in the future.
Online Bipartite Matching
Example: Bipartite Matching

Nodes: Boys and Girls; Links: Preferences

Goal: Match boys to girls so that the most preferences are satisfied

Note: edges are only preferences with no weight or order.
Example: Bipartite Matching

$M = \{(1,a),(2,b),(3,d)\}$ is a matching

Cardinality of matching $= |M| = 3$

Matching means that we are not using any vertex twice
Example: Bipartite Matching

\[ M = \{(1,c),(2,b),(3,d),(4,a)\} \] is a perfect matching

**Perfect matching** … all vertices of the graph are matched

**Maximum matching** … matching that contains the largest possible number of matches
Problem: Find a maximum matching for a given bipartite graph
- A perfect one if it exists

There is a polynomial-time offline algorithm based on augmenting paths (Hopcroft & Karp 1973, see http://en.wikipedia.org/wiki/Hopcroft-Karp_algorithm)

But what if we do not know the entire graph upfront?
Initially, we are given the set **boys**

In each **round**, **one girl’s choices are revealed**

- That is, the girl’s **edges** are revealed

At that time, we have to decide to either:

- Pair the **girl** with a **boy**
- Do not pair the **girl** with any **boy**

**Example of application:**
Assigning tasks to servers

**Note:** Matching means that we are not using any girl or boy twice
Online Graph Matching: Example

\[ (1,a) \]
\[ (2,b) \]
\[ (3,d) \]
Greedy Algorithm

- Greedy algorithm for the online graph matching problem:
  - Pair the new girl with any eligible boy
    - If there is none, do not pair the girl

- How good is the algorithm?
For input \( I \), suppose greedy produces matching \( M_{\text{greedy}} \) while an optimal matching is \( M_{\text{opt}} \)

**Competitive ratio** = 

\[
\min_{\text{all possible inputs } I} \left( \frac{|M_{\text{greedy}}|}{|M_{\text{opt}}|} \right)
\]

(what is greedy’s worst performance over all possible inputs \( I \))
Consider a case: \( M_{\text{greedy}} \neq M_{\text{opt}} \)

- Consider the set \( G \) of girls matched in \( M_{\text{opt}} \) but not in \( M_{\text{greedy}} \)

1. By definition of \( G \):
   \[ |M_{\text{opt}}| \leq |M_{\text{greedy}}| + |G| \]

2. Define set \( B \) of boys linked to girls in \( G \)
   - Notice boys in \( B \) are already matched in \( M_{\text{greedy}} \). Why?
     - If there would exist such non-matched (by \( M_{\text{greedy}} \)) boy adjacent to a non-matched girl then greedy would have matched them
   
   So: \( |M_{\text{greedy}}| \geq |B| \)
Analyzing the Greedy Algorithm

- **Summary so far:**
  - Girls $G$ matched in $M_{opt}$ but not in $M_{greedy}$
  - Boys $B$ adjacent to girls in $G$
  - (1) $|M_{opt}| \leq |M_{greedy}| + |G|$
  - (2) $|M_{greedy}| \geq |B|$

- Optimal matches all girls in $G$ to (some) boys in $B$
  - (3) $|G| \leq |B|$

- Combining (2) and (3):
  - (4) $|G| \leq |B| \leq |M_{greedy}|$
So we have:

1. \( |M_{opt}| \leq |M_{greedy}| + |G| \)
2. \( |G| \leq |B| \leq |M_{greedy}| \)

Combining (1) and (4):

- Worst case is when \( |G| = |B| = |M_{greedy}| \)
- \( |M_{opt}| \leq |M_{greedy}| + |M_{greedy}| \)
- Then \( |M_{greedy}| / |M_{opt}| \geq 1/2 \)
Worst-case Scenario

(1,a)
(2,b)
Web Advertising
History of Web Advertising

- **Banner ads (1995-2001)**
  - Initial form of web advertising
  - Popular websites charged $X for every 1,000 “impressions” of the ad
    - Called “**CPM**” rate (Cost per thousand impressions)
  - Modeled similar to TV, magazine ads
  - From **untargeted** to **demographically targeted**
  - **Low click-through rates**
  - Low ROI for advertisers

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Performance-based Advertising

- Introduced by Overture around 2000
  - Advertisers bid on search keywords
  - When someone searches for that keyword, the highest bidder’s ad is shown
  - Advertiser is charged only if the ad is clicked on

- Similar model adopted by Google with some changes around 2002
  - Called Adwords
Ads vs. Search Results

Web

GEICO Car Insurance. Get an auto insurance quote and save today...
GEICO auto insurance, online car insurance quote, motorcycle insurance quote, online insurance sales and service from a leading insurance company.
www.geico.com/ - 21k - Sep 22, 2005 - Cached - Similar pages
  - Auto Insurance - Buy Auto Insurance
  - Contact Us - Make a Payment
More results from www.geico.com »

Geico, Google Settle Trademark Dispute
The case was resolved out of court, so advertisers are still left without legal guidance on use of trademarks within ads or as keywords.

Google and GEICO settle AdWords dispute | The Register
Google and car insurance firm GEICO have settled a trade mark dispute over ... Car insurance firm GEICO sued both Google and Yahoo! subsidiary Overture in ...
www.theregister.co.uk/2005/09/09/google_geico_settlement/ - 21k - Cached - Similar pages

GEICO v. Google
... involving a lawsuit filed by Government Employees Insurance Company (GEICO). GEICO has filed suit against two major Internet search engine operators, ...
www.consumeraffairs.com/news04/geico_google.html - 19k - Cached - Similar pages

Sponsored Links

Great Car Insurance Rates
Simplify Buying Insurance at Safeco
See Your Rate with an Instant Quote
www.Safeco.com

Free Insurance Quotes
Fill out one simple form to get multiple quotes from local agents.
www.HometownQuotes.com

5 Free Quotes. 1 Form.
Get 5 Free Quotes In Minutes!
You Have Nothing To Lose. It's Free
sayyessoftware.com/Insurance
Missouri
Performance-based advertising works!
- Multi-billion-dollar industry

Interesting problem:
Which ads to show for a given query?
- (Today’s lecture)

If I am an advertiser, which search terms should I bid on and how much should I bid?
- (Not focus of today’s lecture)
A stream of queries arrives at the search engine: $q_1, q_2, \ldots$

Several advertisers bid on each query

When query $q_i$ arrives, search engine must pick a subset of advertisers to show their ads

**Goal:** Maximize search engine’s revenues

- **Simple solution:** Instead of raw bids, use the “expected revenue per click” (i.e., $\text{Bid} \times \text{CTR}$)

- **Clearly we need an online algorithm!**
### The Adwords Innovation

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>Bid</th>
<th>CTR</th>
<th>Bid * CTR</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$1.00</td>
<td>1%</td>
<td>1 cent</td>
</tr>
<tr>
<td>B</td>
<td>$0.75</td>
<td>2%</td>
<td>1.5 cents</td>
</tr>
<tr>
<td>C</td>
<td>$0.50</td>
<td>2.5%</td>
<td>1.25 cents</td>
</tr>
</tbody>
</table>

Click through rate

Expected revenue
Instead of sorting advertisers by bid, sort by expected revenue
### Limitations of Simple Algorithm

Instead of sorting advertisers by bid, sort by expected revenue

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**Challenges:**
- CTR of an ad is unknown
- Advertisers have limited budgets and bid on multiple queries
Complications: Budget

Two complications:

- Budget
- CTR of an ad is unknown

1) Budget: Each advertiser has a limited budget

- Search engine guarantees that the advertiser will not be charged more than their daily budget
2) CTR (Click-Through Rate): Each ad-query pair has a different likelihood of being clicked

- **Advertiser 1** bids $2 on query A, click probability = 0.1
- **Advertiser 2** bids $1 on query B, click probability = 0.5

**CTR** is predicted or measured historically
- Averaged over a time period

**Some complications we will not cover:**

- 1) CTR is position dependent:
  - Ad #1 is clicked more than Ad #2
Some complications we will cover (next lecture):

- 2) Exploration vs. exploitation
  - **Exploit**: Should we keep showing an ad for which we have good estimates of click-through rate? or
  - **Explore**: Shall we show a brand new ad to get a better sense of its click-through rate?
Online Algorithms
The BALANCE Algorithm
Given:

1. A set of bids by advertisers for search queries
2. A click-through rate for each advertiser-query pair
3. A budget for each advertiser (say for 1 month)
4. A limit on the number of ads to be displayed with each search query

Respond to each search query with a set of advertisers such that:

1. The size of the set is no larger than the limit on the number of ads per query
2. Each advertiser has bid on the search query
3. Each advertiser has enough budget left to pay for the ad if it is clicked upon
Our setting: Simplified environment

- There is 1 ad shown for each query
- All advertisers have the same budget $B$
- All ads are equally likely to be clicked
- Bid value of each ad is the same ($=\$1$)

Simplest algorithm is greedy:

- For a query pick any advertiser who has bid 1 for that query
- Competitive ratio of greedy is $1/2$
Two advertisers A and B

- A bids on query \( x \), B bids on \( x \) and \( y \)
- Both have budgets of $4

Query stream: \( x \ x \ x \ x \ y \ y \ y \ y \)

- Worst case greedy choice: \( B \ B \ B \ B \ _ \ _ \ _ \ _ \)
- Optimal: \( A \ A \ A \ A \ B \ B \ B \ B \)

Competitive ratio = \( \frac{1}{2} \)

This is the worst case!

Note: Greedy algorithm is deterministic – it always resolves draws in the same way.
**BALANCE Algorithm [MSVV]**

- **BALANCE** Algorithm by Mehta, Saberi, Vazirani, and Vazirani
  - For each query, pick the advertiser with the largest unspent budget
    - Break ties arbitrarily (*but in a deterministic way*)
Two advertisers A and B
- A bids on query $x$, B bids on $x$ and $y$
- Both have budgets of $4$

Query stream: $x x x x y y y y$

**BALANCE choice:** A B A B B B _ _
- Optimal: A A A A B B B B

In general: For BALANCE on 2 advertisers
**Competitive ratio = $\frac{3}{4}$**
Consider simple case (w.l.o.g.):

- 2 advertisers, $A_1$ and $A_2$, each with budget $B \geq 1$
- Optimal solution exhausts both advertisers’ budgets

**BALANCE must exhaust at least one budget:**

- If not, we can allocate more queries
  - Whenever BALANCE makes a mistake (both advertisers bid on the query), advertiser’s unspent budget only decreases
  - Since optimal exhausts both budgets, one will for sure get exhausted

- Assume BALANCE exhausts $A_2$’s budget, but allocates $x$ queries fewer than the optimal
  - So revenue of $BALANCE = 2B - x$ (where OPT is $2B$)

Let’s work out what $x$ is!
Opt revenue = 2B

Balance revenue = 2B - x = B + y

We claim y ≥ x (next slide).
Balance revenue is minimum for x = y = B/2.
Minimum Balance revenue = 3B/2.
Competitive Ratio = 3/4.
Analyzing \textbf{BALANCE: What’s x?}

Optimal revenue = \(2B\)
Assume Balance gives revenue = \(2B-x = B+y\)
Assume we exhausted \(A_2\)’s budget

Notice: Unassigned queries should be assigned to \(A_2\) (since if we could assign to \(A_1\) we would since we still have the budget)

**Goal:** Show we have \(y \geq \frac{B}{2}\)

Case 1) BALANCE assigns \(\geq B/2\) blue queries to \(A_1\). Then trivially, \(y \geq \frac{B}{2}\)

Queries allocated to \(A_1\) in the optimal solution
Queries allocated to \(A_2\) in the optimal solution
Optimal revenue = $2B$
Assume Balance gives revenue $= 2B - x = B + y$
Assume we exhausted $A_2$’s budget

Unassigned queries should be assigned to $A_2$
(if we could assign to $A_1$ we would since we still have the budget)

**Goal:** Show we have $y \geq B/2$

**Case 2) BALANCE assigns $\geq B/2$ blue queries to $A_2$.**

Consider the last blue query assigned to $A_2$.
At that time, $A_2$’s unspent budget must have been at least as big as $A_1$’s.
That means at least as many queries have been assigned to $A_1$ as to $A_2$.
At this point, we have already assigned at least $B/2$ queries to $A_2$.
So, $x \leq B/2$ and $x + y = B$ then $y > B/2$
In the general case, worst competitive ratio of BALANCE is $1 - 1/e = \text{approx. 0.63}$

- $e = 2.7182$

  Interestingly, no online algorithm has a better competitive ratio!

Let’s see the worst case example that gives this ratio
Worst case for BALANCE

- **N advertisers:** $A_1, A_2, \ldots, A_N$
  - Each with budget $B > N$
- **Queries:**
  - $N \cdot B$ queries appear in $N$ rounds of $B$ queries each
- **Bidding:**
  - Round 1 queries: bidders $A_1, A_2, \ldots, A_N$
  - Round 2 queries: bidders $A_2, A_3, \ldots, A_N$
  - Round $i$ queries: bidders $A_i, \ldots, A_N$
- **Optimum allocation:**
  - Allocate all round $i$ queries to $A_i$
  - Optimum revenue $N \cdot B$
BALANCE assigns each of the queries in round 1 to \( N \) advertisers. After \( k \) rounds, sum of allocations \( S_k \) to each of advertisers \( A_k, \ldots, A_N \) is

\[
S_k = S_{k+1} = \cdots = S_N = \sum_{i=1}^{k} \frac{B}{N-(i-1)}
\]

If we find the smallest \( k \) such that \( S_k \geq B \), then after \( k \) rounds we cannot allocate any queries to any advertiser.
BALANCE: Analysis

Can divide everything by B:

\[ \frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \ldots \quad \frac{1}{(N-(k-1))} \quad \ldots \quad \frac{1}{(N-1)} \quad \frac{1}{N} \]

\[ S_1 \quad S_2 \]

\[ S_k = B \]

\[ S_1 \quad S_2 \]

\[ S_k = 1 \]
Fact: $H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln(n)$ for large $n$

- Result due to Euler

\[ \frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \ldots \quad \frac{1}{(N-(k-1))} \quad \ldots \quad \frac{1}{(N-1)} \quad \frac{1}{N} \]

\[ \frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \ldots \quad \frac{1}{(N-(k-1))} \quad \ldots \quad \frac{1}{(N-1)} \quad \frac{1}{N} \]

**$S_k = 1$ implies:** $H_{N-k} = \ln(N) - 1 = \ln\left(\frac{N}{e}\right)$

**We also know:** $H_{N-k} = \ln(N - k)$

**So:** $N - k = \frac{N}{e}$

**Then:** $k = N\left(1 - \frac{1}{e}\right)$

$N$ terms sum to $\ln(N)$.  
Last $k$ terms sum to $1$.  
First $N-k$ terms sum to $\ln(N-k)$ but also to $\ln(N-1)$
So after the first $k=N(1-1/e)$ rounds, we cannot allocate a query to any advertiser

Revenue = $B \cdot N \cdot (1-1/e)$

Competitive ratio = $1-1/e$

Note: So far we assumed:
- All advertisers have the same budget $B$
- All advertisers bid 1 for the ad
- (but each advertiser can bid on any subset of ads)
Arbitrary bids and arbitrary budgets!

Consider we have 1 query $q$, advertiser $i$

- Bid = $x_i$
- Budget = $b_i$

In a general setting BALANCE can be terrible

Consider two advertisers $A_1$ and $A_2$

- $A_1$: $x_1 = 1$, $b_1 = 110$
- $A_2$: $x_2 = 10$, $b_2 = 100$

Consider we see 10 instances of $q$

- BALANCE always selects $A_1$ and earns 10
- Optimal earns 100
**Arbitrary bids:** consider query $q$, bidder $i$

- Bid = $x_i$
- Budget = $b_i$
- Amount spent so far = $m_i$
- Fraction of budget left over $f_i = 1-m_i/b_i$
- Define $\psi_i(q) = x_i(1-e^{-f_i})$

- Allocate query $q$ to bidder $i$ with largest value of $\psi_i(q)$

- **Same competitive ratio** $(1-1/e) = 0.63$