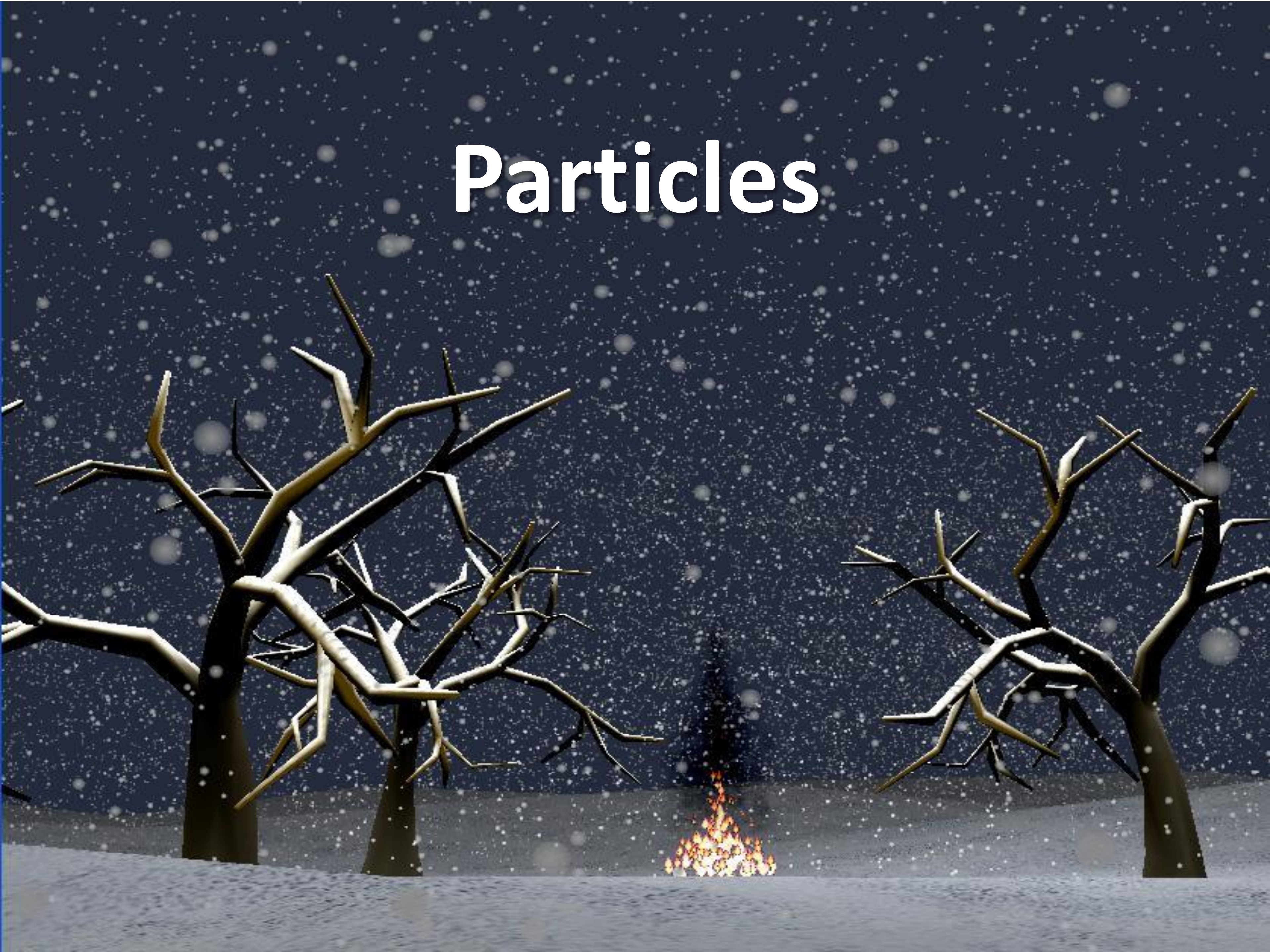
Particles



Simulation Homework

- Build a particle system based either on F=ma or procedural simulation
  - Examples: Smoke, Fire, Water, Wind, Leaves, Cloth, Magnets, Flocks, Fish, Insects, Crowds, etc.
- Simulate a rigid body
  - Examples: Angry birds, Bodies tumbling, bouncing, moving around in a room and colliding, Explosions & Fracture, Drop the camera, Etc…

Particle

- A particle is simply a point in space with some attributes
- The attributes are what makes different kinds of particles
  - Mass (m)
  - Position (x)
  - Velocity (v)
  - External Force (F)
  - Color, Animal type, Etc.

Particle Motion

- Dynamic
  - A particle with a non-zero initial velocity tends to keep moving with that velocity (Newton's 1st Law)
  - Its motion changes whenever unbalanced external forces are applied to it (Newton's 2nd Law)
- Kinematic
  - An "infinite mass" particle can move along a prescribed path or animated curve directed by an artist
    - Static – a kinematic particle with zero velocity


Dynamics

Newton's Second Law

- The net force on an object is equal to the rate of change of its linear momentum P=mv

$$F = \frac{dP}{dt} = \frac{d(mv)}{dt} = ma$$

- The last equality holds if the object has constant mass

Newton’s Second Law

$$F = ma = m\ddot{x}$$

- This is a second order differential equation in position
- Higher order differential equations can be analyzed and solved by rewriting them as a system of first order equations:

$$\dot{x} = v$$
$$\dot{v} = F/m$$

Forces

Types of Forces

- A particle system that simulates water requires gravity as an external force, as well as internal forces for incompressibility and advection
- Dust particles (or leaves) require air currents and wind as external forces
- If particles are used to model cloth, we require elastic or spring forces between them
- If each particle is fish, they need attractive forces to school and repulsive forces to avoid collisions
- Etc…

Types of forces

- Constant forces (e.g. gravity)
- Time dependent forces (e.g. wind)
- Position dependent forces (e.g. force fields, spatially varying wind)
- Velocity dependent forces (e.g. drag, friction)
- Position & Velocity dependent forces (e.g. springs)

Gravity

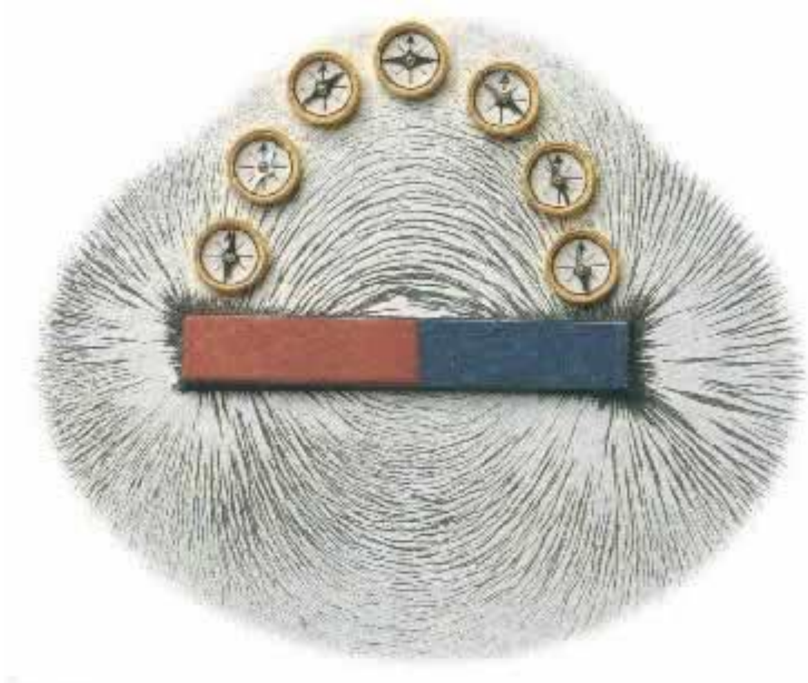
- $F_{grav} = -mg$
  - $g = 9.8\ m/s^2$ is a constant
  - m is the mass of the body/particle
- Simple ballistic motion…

Wind

- Position and time dependent force
- $f_{wind} = f(\vec{x}, t)$

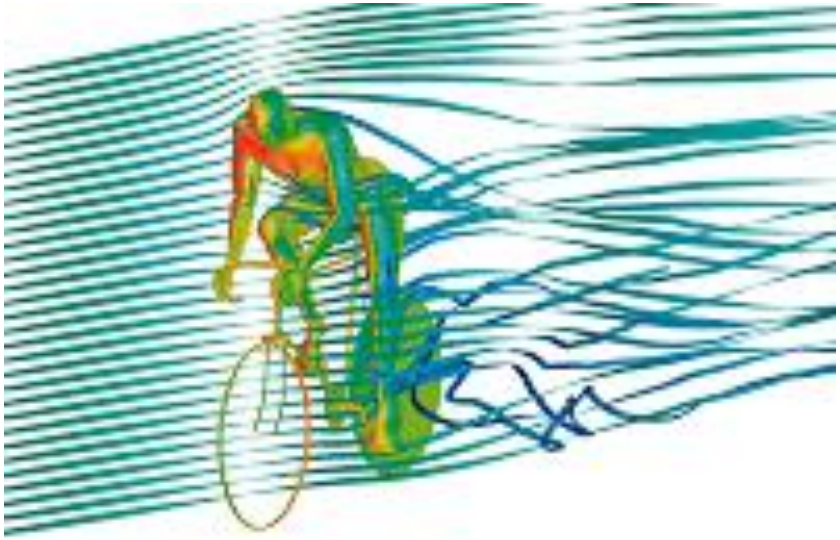
Magnetism

- Assign the particles a magnetic monopole attribute q
- $\left|f_{magnet}\right| = \frac{\mu q_1 q_2}{4\pi r^2}$
  - q_1 and q_2 are magnitudes of magnetic monopoles, r is the distance between the poles, and μ is a constant
  - Also need to add a direction between particles
  - Like poles repel and unlike poles attract



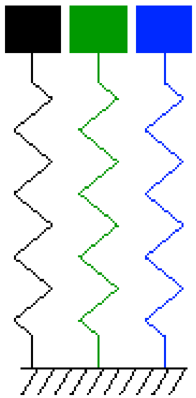
Drag Force

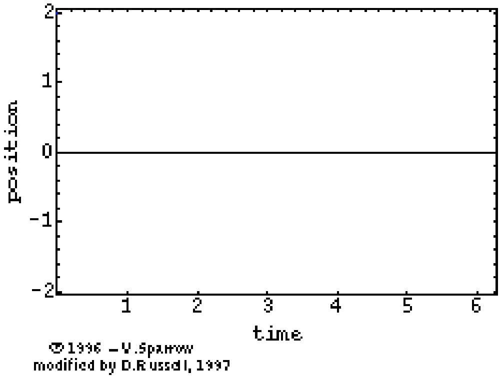
- Velocity dependent force (linear in velocity)
- The faster the velocity, the larger the drag
  - think molasses or honey
- $f_{drag} = -k_{drag}\, v_{rel}$
  - where k_{drag} is the drag coefficient, and v_{rel} is the particle's velocity relative to the fluid it is in

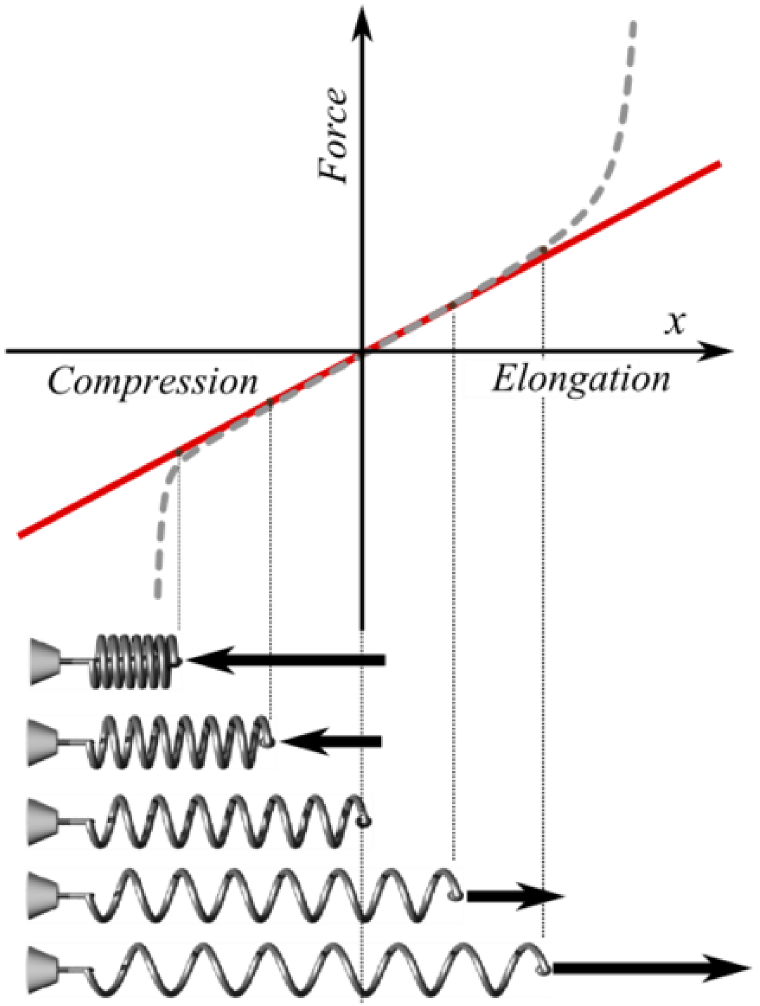


Spring Force (no damping)

- Hooke's Law
- $F_{spring} = -kx$
- Linearization of the spring forces for small displacements

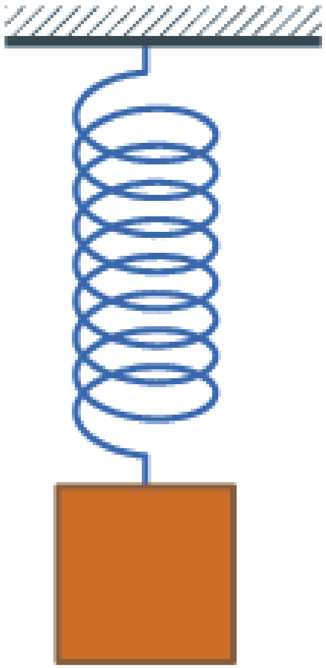






Spring Force (with damping)

- $F_{spring} = -kx - k_d\dot{x}$
- Adds an exponential decay to the amplitude of oscillation
- It is a good practice to add some damping to physical systems to keep them form going unstable
  - and for realism



Question #1

LONG FORM:

- Briefly discuss various types of forces that can be used in video game simulations
- Answer short form question below

SHORT FORM:

- Can you think of a use for forces in <u>your</u> video game? Briefly explain
- (Notice I'm starting to assume you have a video game idea. Do you? ☺)

Collision Detection

Collisions

- As particles move around under the influence of gravity, drag, and other forces, how do they interact with other objects?
- This is where collisions come into play
- How do we detect collisions?
  - Check to see if a particle is inside some object

Example: Plane

- Consider for example using a plane to represent the ground (or a wall)
- Define the plane by a point $\vec{p}$ and normal $\vec{n}$
- Given our particle position $\vec{x}$, we calculate

$$\mathrm{s} = (\vec{x} - \vec{p}) \cdot \vec{n}$$

- $\vec{x}$ is outside the plane if $s > 0$ and inside if $s < 0$

- Normal to the plane/object is given by $\vec{n}$

Example: Box

- Use a plane for each of the six faces of the box
- If the particle is inside all 6 faces, it is inside the box

- To find the normal, one has to identify the closest of the 6 planes
- This is given by the value of s closest to zero

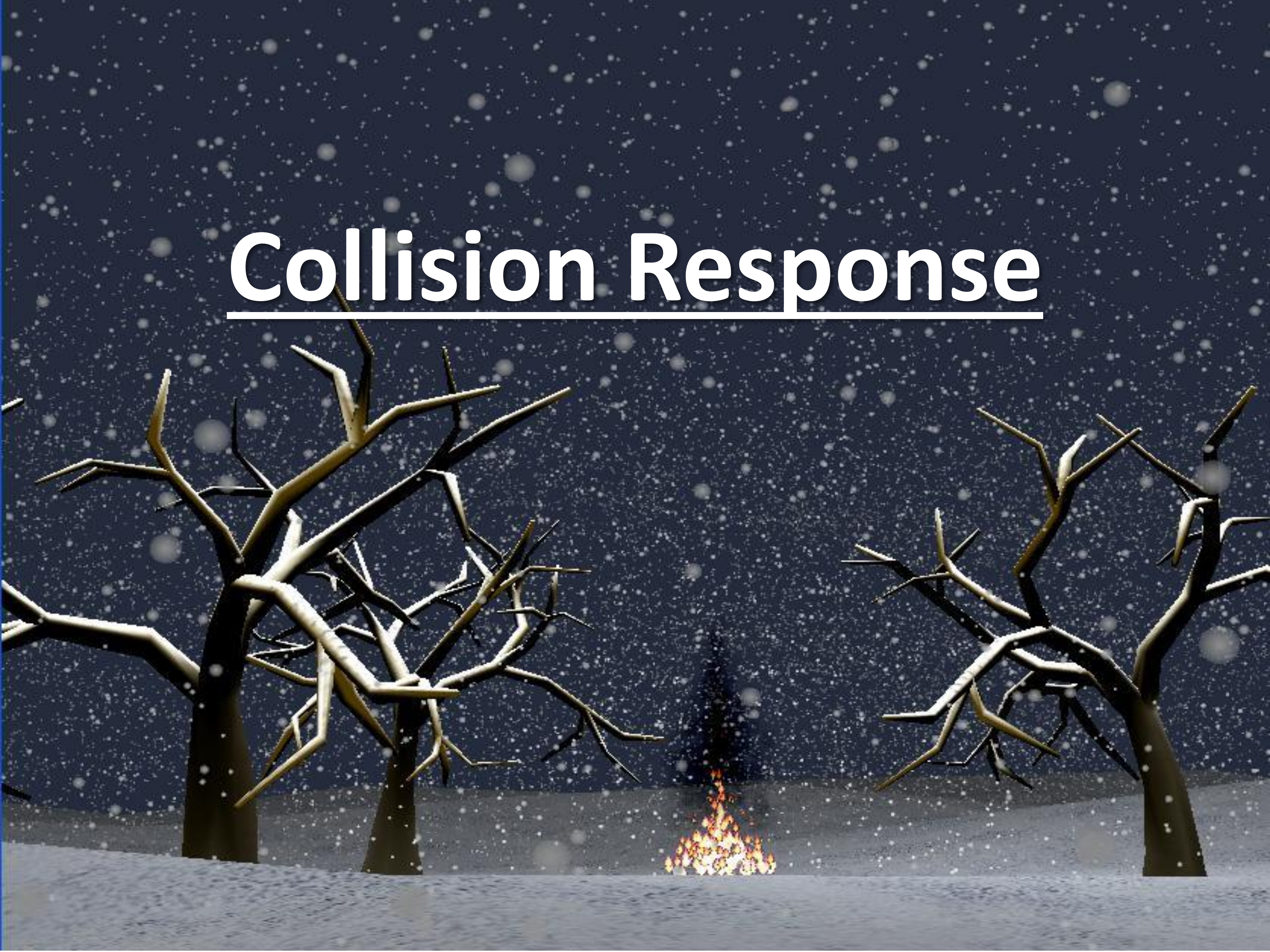
- Can be used for other convex polyhedra as well

Example: Sphere

- Define a sphere with a center $\vec{c}$ and a radius r
- Given a point $\vec{q}$, calculate $s = |\vec{q} - \vec{c}| - r$.
- $\vec{q}$ is outside the sphere if $s > 0$ and inside if $s < 0$

- Normal at the point is $(\vec{q} - \vec{c})/|\vec{q} - \vec{c}|$

Collision Response



Collision Response

- Do something to take the bodies from a “colliding state” to a “non-colliding state”
- What properties should a good response algorithm have?
  - Remove interpenetrations
  - Conserve linear and angular momentum
  - Have the correct relative velocities based on the material properties of the colliding bodies
  - Should look plausible!

Collision Response (Notation Key)

- c_R is the coefficient of restitution
  - 0 is completely inelastic; objects stick together
  - 1 is completely elastic; objects bounce without losing any kinetic energy
  - Between 0 and 1 means some energy is lost due to deformation, damage, sound, heat, etc.
- m_a is the mass of the first object
- m_b is the mass of the second object
- u_a is the velocity of the first object before impact
- u_b is the velocity of the second object before impact
- v_a is the velocity of the first object after impact
- v_b is the velocity of the second object after impact

Collision Response (Formulas)

$$c_R = -\frac{v_b - v_a}{u_b - u_a} \quad \text{(definition)}$$

$$m_a u_a + m_b u_b = m_a v_a + m_b v_b \quad \text{(momentum conservation)}$$

- Two equations in two unknowns, solve…
  - $v_a = \frac{(m_a\, u_a + m_b\, u_b - m_b\, c_R(u_a - u_b))}{(m_a + m_b)}$
  - $v_b = \frac{(m_a\, u_a + m_b\, u_b - m_a\, c_R(u_b - u_a))}{(m_a + m_b)}$

- We can also look at this in terms of an impulse. The impulse required to change the velocity of object a is

$$j = m_a(v_a - u_a)$$

- An equal and opposite impulse is applied to object b

If one object is infinitely heavy…

- Useful for kinematic objects (stationary or moving)
- Make m_b infinite
- $v_b = u_b$ (doesn't change)
- $v_a = \frac{(m_b u_b - m_b c_R(u_a - u_b))}{m_b} = u_b - c_R(u_a - u_b)$
- If $u_b = 0$ (stationary object, e.g. ground plane), then this further simplifies to $v_a = -c_R u_a$

Collisions in 3D

Higher Spatial Dimensions

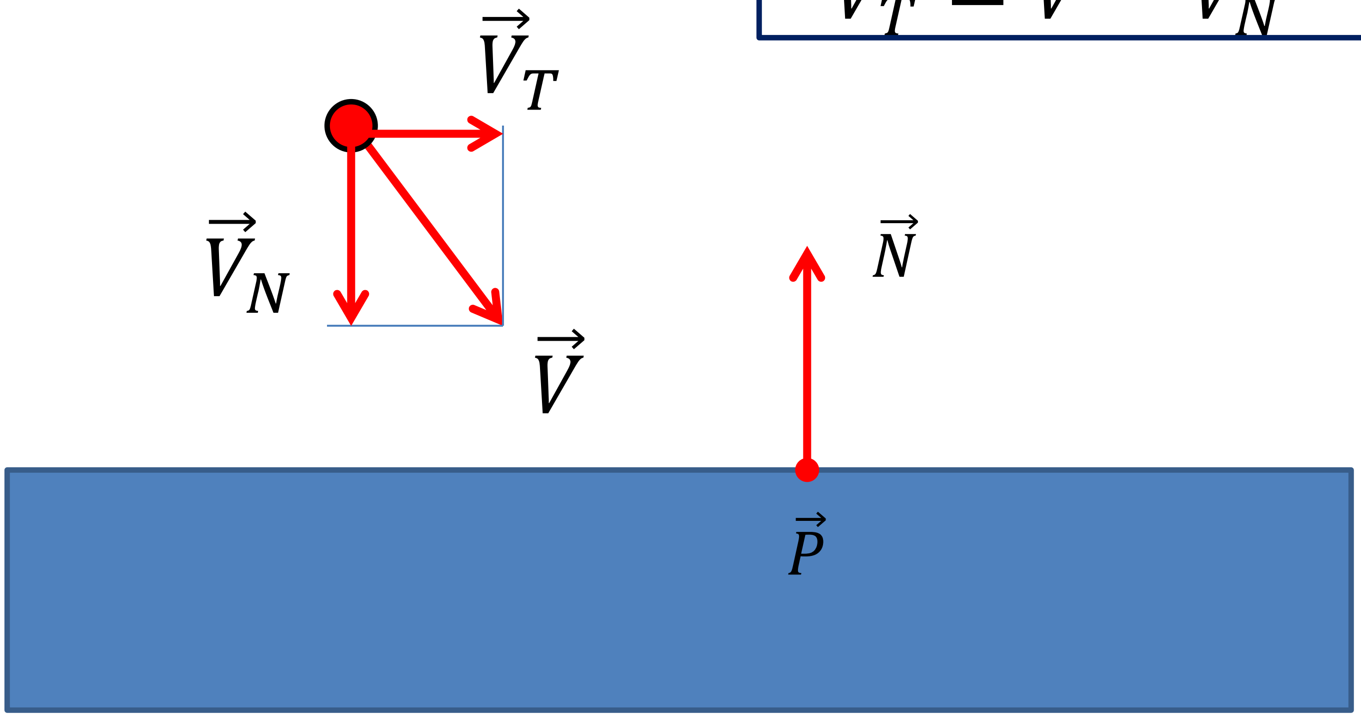
- The prior equations describe collision in 1D only
- In 3D, they describe the collision in the normal direction, i.e. on the components of velocity (dot product-ed) into the normal direction
- The tangential components of the velocity do not change, unless there is collisional friction

- Since most surfaces can be locally approximated as being planar, let's consider point plane collisions….

Point-Plane Collision Response

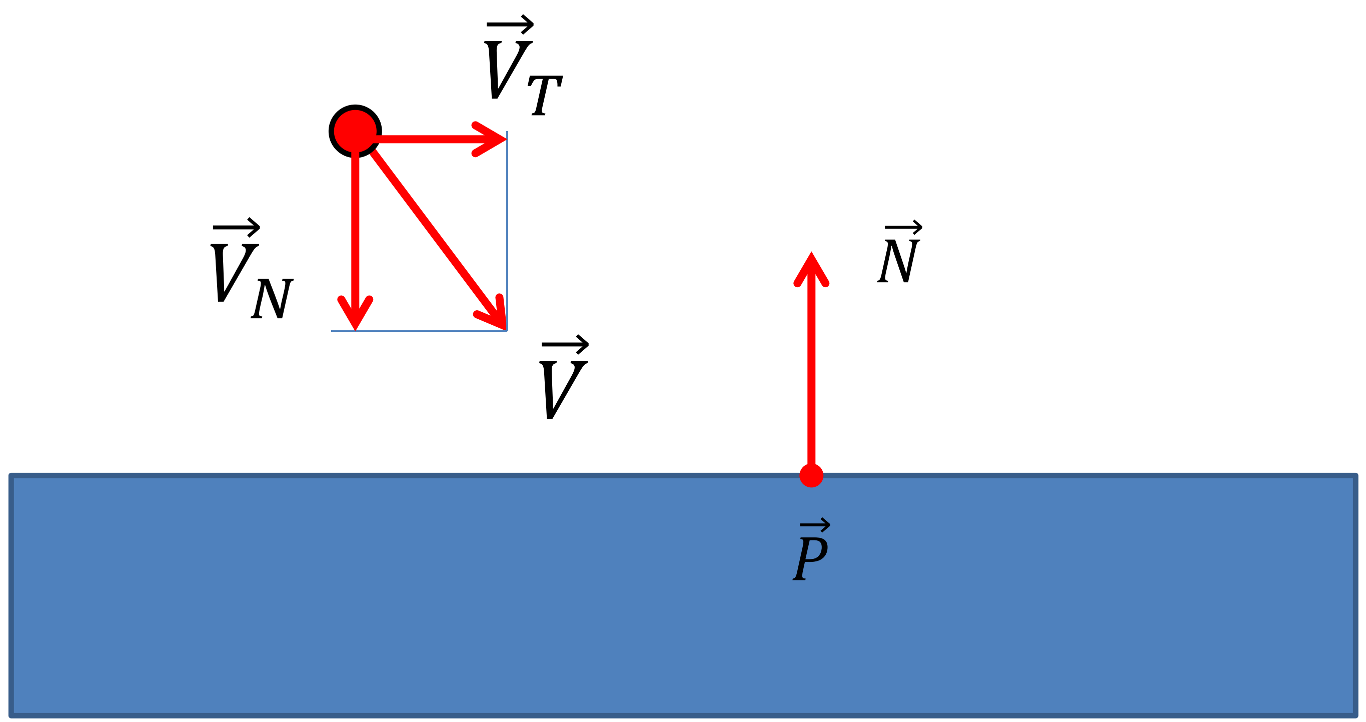
- Collision only affects the normal component of velocity
- As such, split the velocity into a normal and tangent component:

$$\vec{V}_N = \left(\vec{V} \cdot \vec{N}\right)\vec{N}$$
$$\vec{V}_T = \vec{V} - \vec{V}_N$$



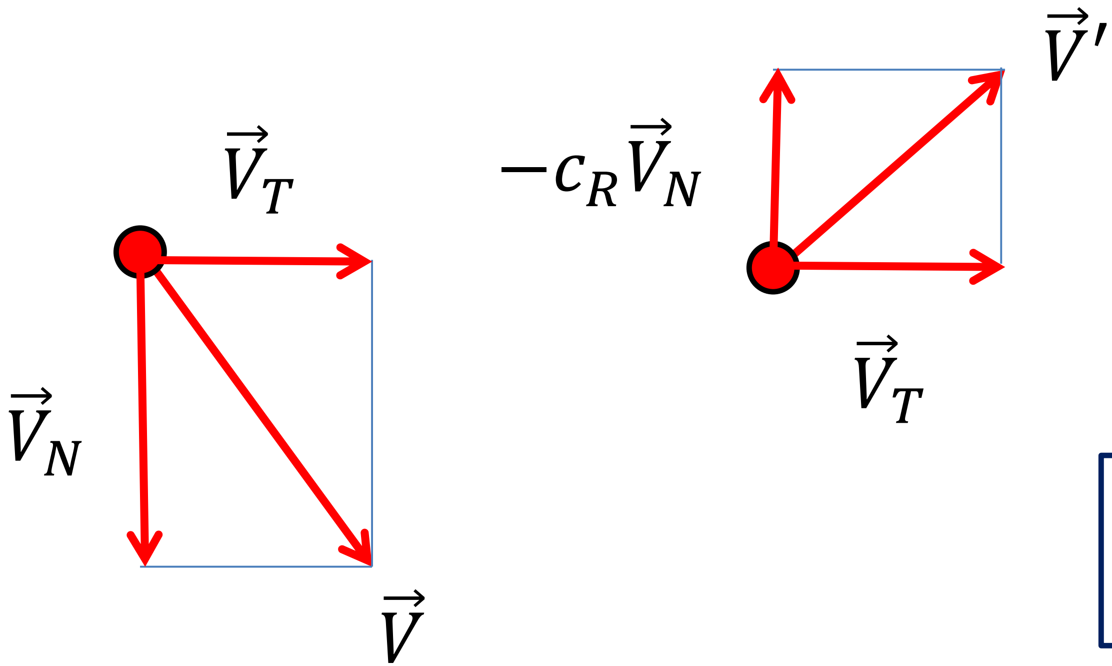
More Collision Detection

- Need to detect that it's colliding with the wall, and not separating
- Make sure it is heading into the wall with: $\vec{V} \cdot \vec{N} < 0$



Collision Response

- Adjust the normal velocity of the particle to account for the collision
- Leave the tangential velocity unchanged
- Probably also want to adjust the position of the particle to move it to the surface of the object (if it is inside)



$$\vec{V}' = \vec{V}_T - c_R\vec{V}_N$$

Friction

- Let j_n be the collision impulse in the normal direction
- The new tangential velocity is $\vec{V}_T' = \vec{V}_T - \frac{\mu|\vec{j}_n|\vec{V}_T}{m|\vec{V}_T|}$, where μ is the coefficient of kinetic friction:

$$\vec{V}' = max\left(0, \left(1 - \frac{\mu|\vec{j}_n|}{m|V_T|}\right)\right)\vec{V}_T - c_R\vec{V}_N$$

- The clamping ensures that friction slows a particle down without changing its direction
- Static friction can be modeled by first applying this formula with the (typically larger) static friction coefficient
  – If the max clamps to 0, static friction stopped the object from moving
  – If not, then recompute the formula with the smaller kinetic friction coefficient

Question #2

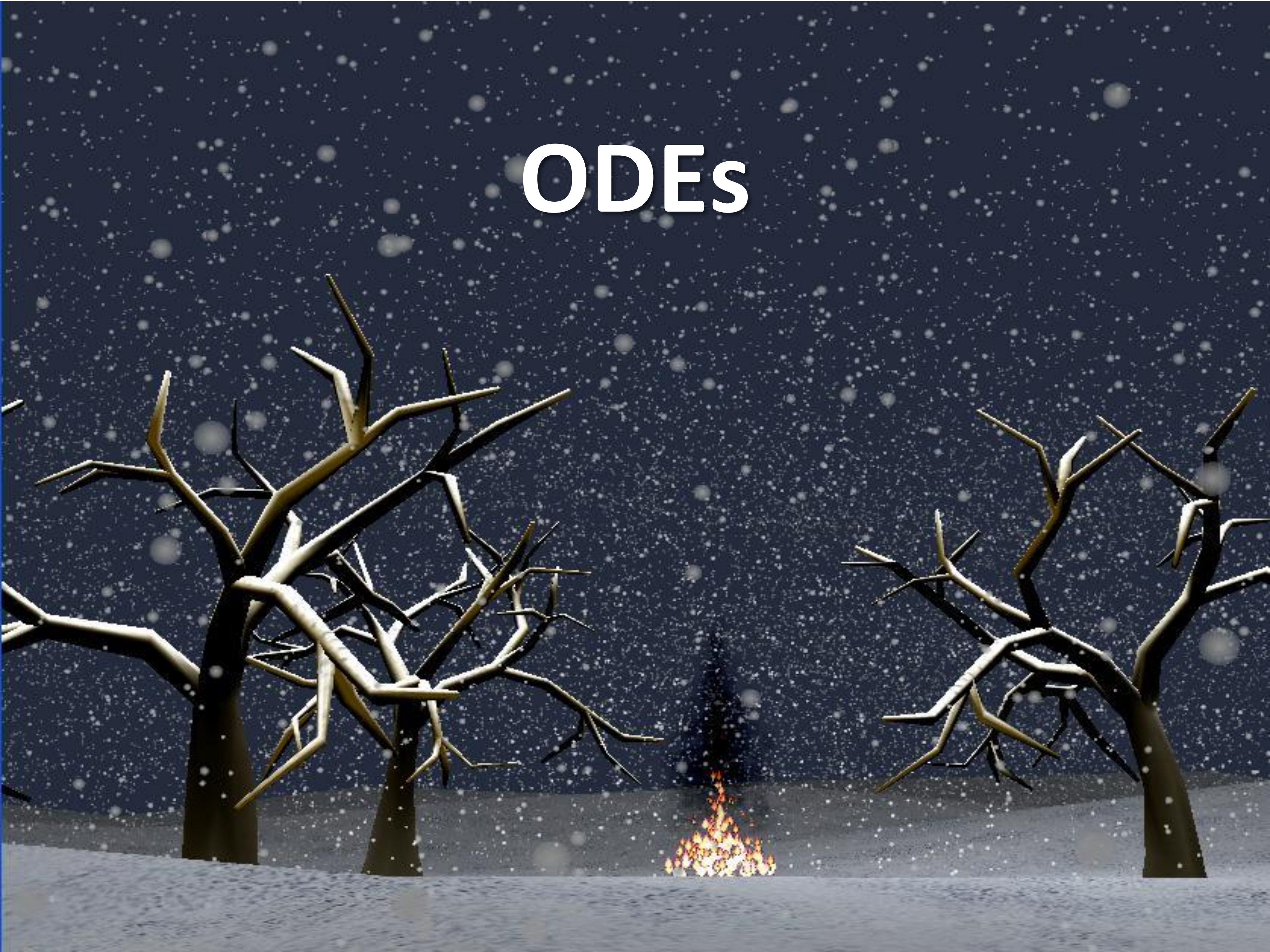
LONG FORM:

- Briefly discuss implementing collisions
- Answer short form question below

SHORT FORM:

- Can you think of a good use for collisions in <u>your</u> video game? Briefly explain

ODEs



Ordinary Differential Equations (ODEs)

- An ODE is an equation containing a function of <u>one independent variable</u> t and its derivatives:

$$f(t, y, y', y'', \dots) = 0$$

- <u>First order</u> ODE's have at most one derivative:

$$f(t, y, y') = 0$$

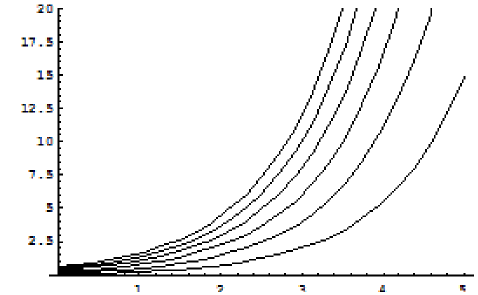
- If we can isolate the derivative term, we call it an <u>explicit</u> ODE (otherwise its implicit):

$$y' = f(t, y)$$

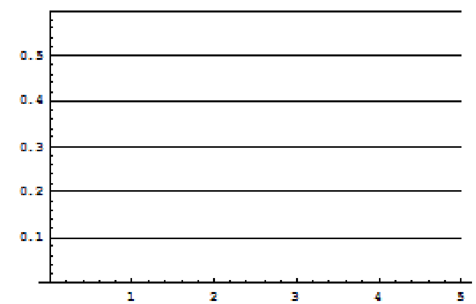
Well-Posed vs. Ill-Posed ODEs

- Model problem
- linear ODE $y' = \lambda y$
  - solution is $y = y_o e^{\lambda(t-t_0)}$
  - 3 kinds of solutions
    - $\lambda > 0$ ill-posed (a)
    - $\lambda = 0$ mildly ill-posed (b)
    - $\lambda < 0$ well-posed (c)
- Ill-posed problems can not (should not) be solved (with any reasonable assurances) on the computer

(a)



(b)



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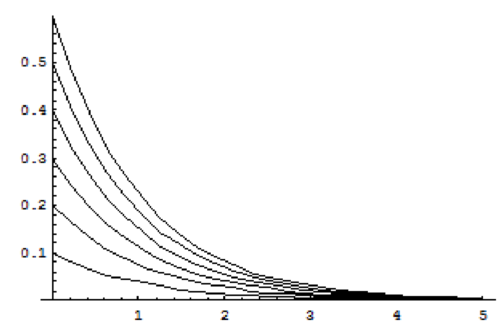
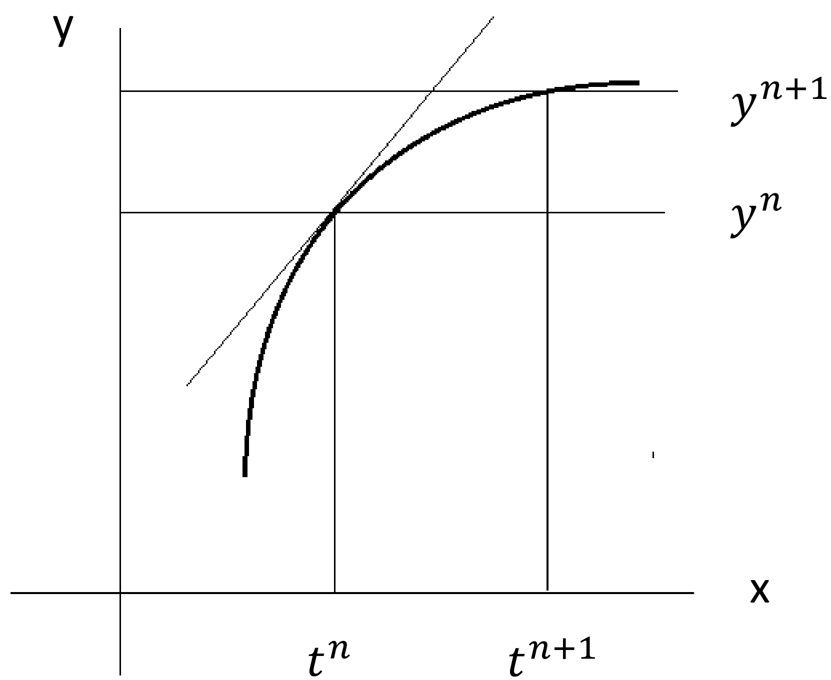


Well-Posed vs. Ill-Posed ODEs

- Scalar ODE $y' = f(t, y)$
  - Derivatives $\frac{df}{dy} = \lambda$ must be negative (or ≤ 0 for mild well-posedness) for all values of t and y we are concerned with
- Systems of ODEs $\vec{y}' = f(t, \vec{y})$
  - All eigenvalues of the Jacobian matrix $\mathrm{J} = \frac{d\vec{f}}{d\vec{y}}$ must be negative (or ≤ 0) for all t and $\vec{y}$ we are concerned with
- Poor choices of the forces in F=ma can lead to ill-posed problems!

Numerical Methods for ODEs

Numerical Approximation of Derivatives



$$y' \approx (y^{n+1} - y^{n})/(t^{n+1} - t^{n})$$

$$or$$

$$y' \approx (y^{n+1} - y^{n})/\Delta t$$

where $\Delta t = t^{n+1} - t^{n}$

Time discretization

$$(y^{n+1} - y^n)/\Delta t = f(t^n, y^n)$$

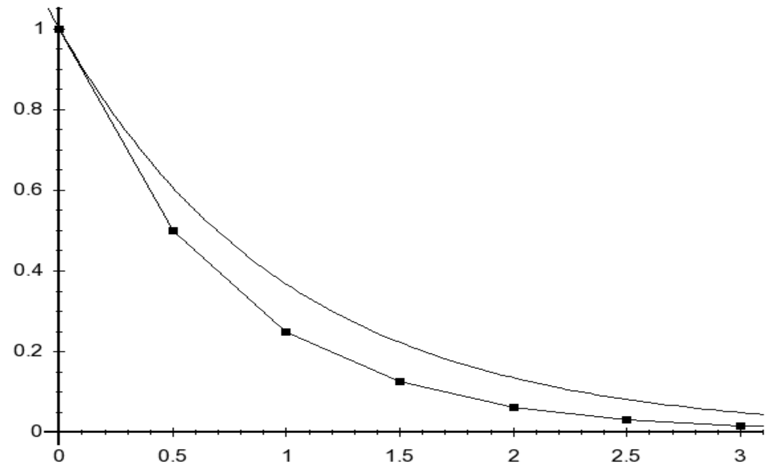
or... $y^{n+1} = y^n + \Delta t\, f(t^n, y^n)$

- This method is called **Forward Euler**
- Start at some initial time t^0 with initial value y^0
- Recursively compute the values for the next time step using the values from the current time step
- Δt can be either fixed or adaptively varied for better accuracy and stability

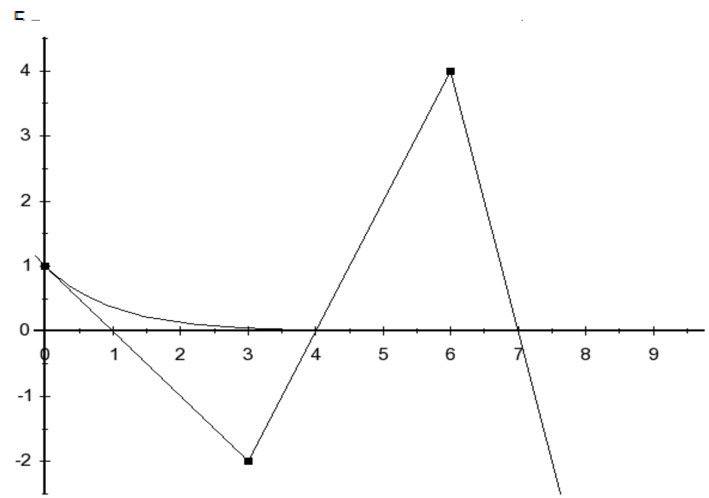
Example

Forward Euler on $y' = -y$ for $y^0 = 1$, $t^0 = 0$

$\Delta t = .5$ is stable



$\Delta t = 3$ is unstable

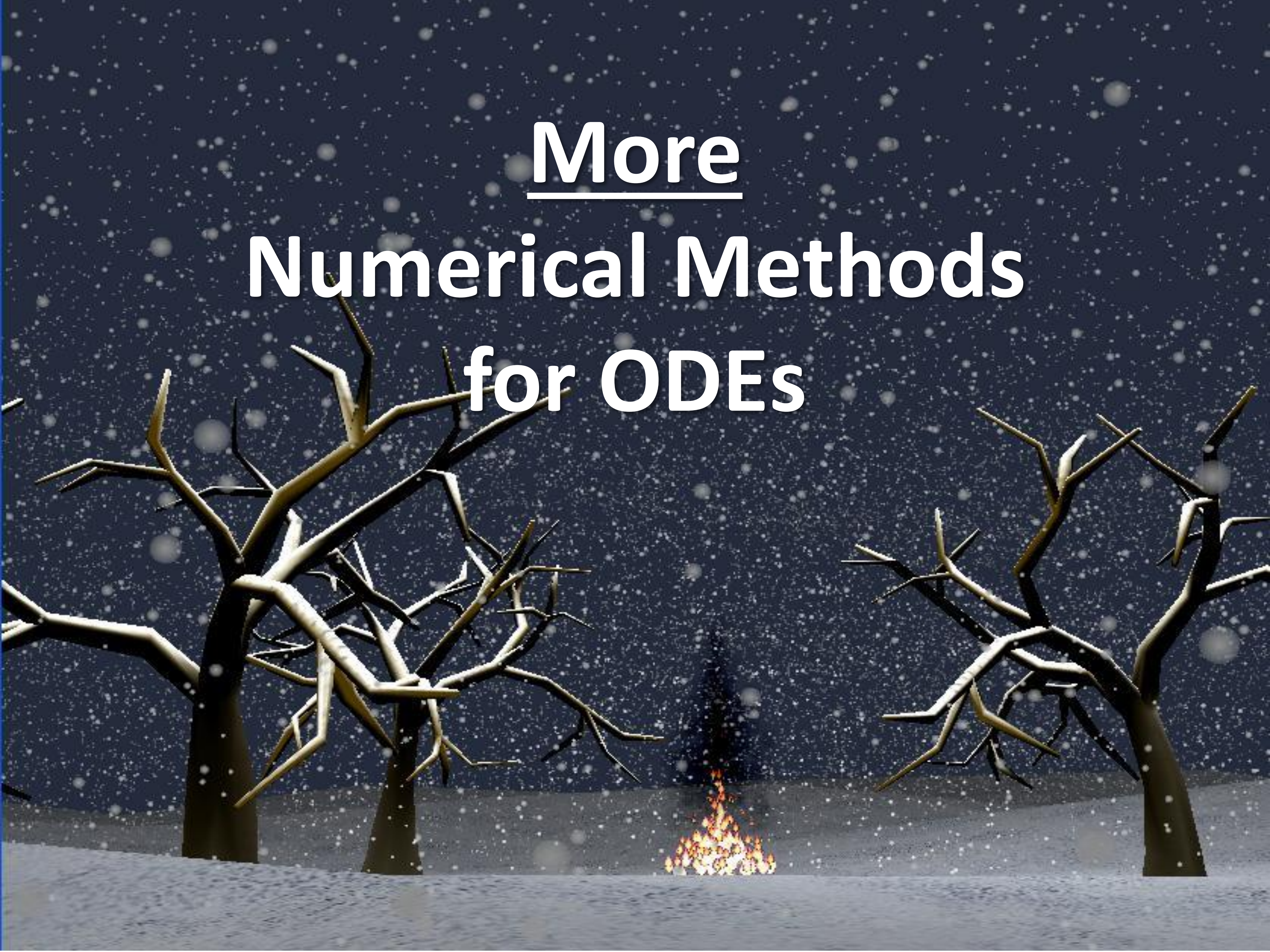


Forward Euler: Stability

- Consider model equation $y' = \lambda y$ with $\lambda < 0$
  - Recall the analytic solution is exponential decay: $y(t) = y_o e^{\lambda(t-t_o)}$
- Forward Euler's method applied to the model equation is $y^{n+1} = y^n + \Delta t \lambda y^n = (1 + \Delta t \lambda) y^n$
- So $y^n = (1 + \Delta t \lambda)^n y^0$, and the solution decays when $|1 + \Delta t \lambda| < 1$
- Thus, $-2 < \Delta t \lambda < 0$ is needed for stability
  - We have $\Delta t \lambda < 0$ trivially, since $\lambda < 0$
- Time step restriction is $\Delta t < 2/|\lambda|$

Forward Euler: Accuracy

- $O(\Delta t^2)$ error in each time step (shown via Taylor series)
- $O\left(\frac{1}{\Delta t}\right)$ time steps to get to an O(1) final time
- $O(\Delta t^2) \times O\left(\frac{1}{\Delta t}\right) = O(\Delta t)$ total error
- 1st order accurate


More
Numerical Methods
for ODEs

Runge-Kutta Schemes

- Runge-Kutta (R.K.) builds on Forward Euler (F.E.)
- Achieves better accuracy by predicting solutions using F.E.
  - and then uses averaging to get new solutions
- Different prediction and averaging schemes give rise to different R.K. schemes
- 1st order (accurate) R.K. is same as F.E.

2nd Order (Accurate) Runge Kutta

- Take two successive F.E. steps:

$$\frac{y^{n+1} - y^{n}}{\Delta t} = f(t^{n}, \mathrm{y}^{\mathrm{n}}) \text{ and } \frac{y^{n+2} - y^{n+1}}{\Delta t} = f(t^{n+1}, \mathrm{y}^{\mathrm{n+1}})$$

- Average the initial and final states:

$$y^{n+1} = \frac{1}{2} y^{n} + \frac{1}{2} y^{n+2}$$

- If the solution is well behaved for each F.E. step, then since linear interpolation is well behaved, the result is well behaved

3rd Order (Accurate) Runge Kutta

- Take two successive F.E. steps:

$$\frac{y^{n+1}-y^{n}}{\Delta t}=f(t^{n},\mathrm{y}^{\mathrm{n}}) \text{ and } \frac{y^{n+2}-\mathrm{y}^{\mathrm{n}+1}}{\Delta t}=f(t^{n+1},\mathrm{y}^{\mathrm{n}+1})$$

- Average the initial and final states:

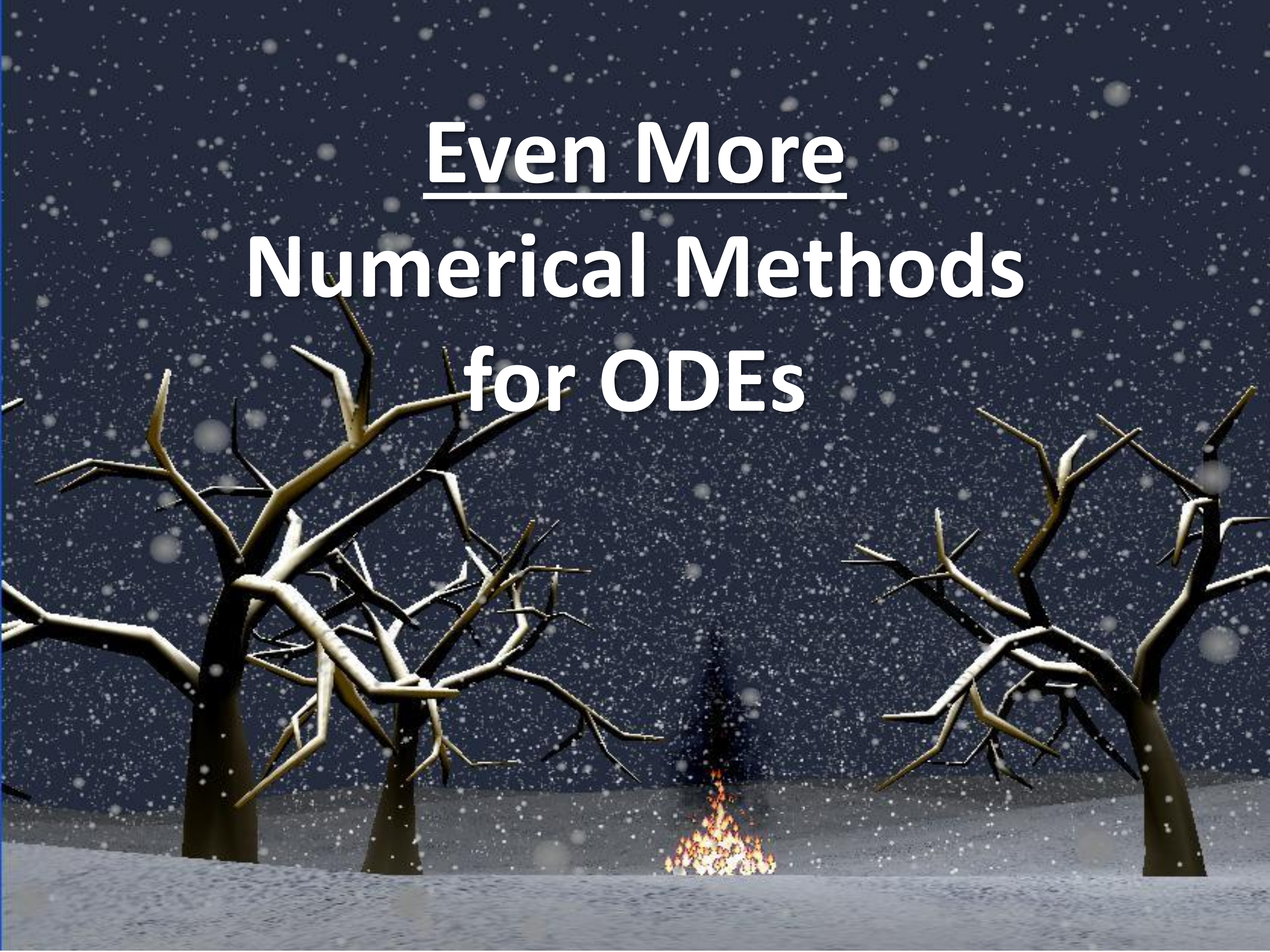
$$y^{n+1/2}=\frac{3}{4}y^{n}+\frac{1}{4}y^{n+2}$$

- Take another F.E. step:

$$\frac{y^{n+3/2}-y^{n+1/2}}{\Delta t}=f(t^{n+1/2},y^{n+1/2})$$

- Then average again: $y^{n+1}=\frac{1}{3}y^{n}+\frac{2}{3}y^{n+3/2}$
- 3rd order R.K is not only more accurate but has some better stability properties

Even More Numerical Methods for ODEs



Backward Euler

$$y^{n+1} = y^n + \Delta t \, f(t^{n+1}, y^{n+1})$$

- Equation is implicit in y^{n+1}, so generally need to solve a nonlinear equation to find y^{n+1}
- Newton iteration…. linearize, solve, linearize, solve, etc.
- Some applications (that allow for larger errors) only use one linearize and solve cycle
- Sometimes f is already linear in y
- Accuracy – 1st order (same as forward Euler)

Backward Euler: Stability

- Consider model equation $y' = \lambda y$ with $\lambda < 0$
- Backward Euler applied to the model equation is $y^{n+1} = y^n + \Delta t \lambda y^{n+1} = (1 - \Delta t \lambda)^{-1} y^n$
- So $y^n = (1 - \Delta t \lambda)^{-n} y^0$ and the solution decays when $|1 - \Delta t \lambda| > 1$
  - Always true!
- Unconditionally stable - works for all Δt
- No time step restriction…

Backward Euler vs. Forward Euler

- Backward Euler (B.E.) is unconditionally stable
  - i.e. one can take very large time steps, whereas Forward Euler (F.E.) requires smaller time steps
- B.E. might excessively damp out the solution, whereas F.E. might blow up (i.e., NaNs)
- Each B.E. time step may be much harder to solve than a F.E. time step
  - B.E. is more theoretically challenging and uses more CPU time
- Not always clear which is better…

Trapezoidal Rule

$$y^{n+1} = y^n + \Delta t \frac{f(t^n, y^n) + f(t^{n+1}, y^{n+1})}{2}$$

- 2nd order accurate
- Unconditionally stable
- Need to solve for y^{n+1} just like Backward Euler
- One can take very large time steps since it is stable
- Sometimes bad oscillatory behavior if Δt is too big

<u>The Best</u>
Numerical Methods….

Back to our problem…

$$\dot{x} = v$$
$$\dot{v} = F/m$$

- We can solve for velocity at one accuracy level lower than for positions (a multivalue method)
- Treating it as a standard system is overkill
- E.g., standard **constant acceleration** equations
  - $\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2}\vec{a}^n$ *piecewise quadratic* position
  - $\vec{v}^{n+1} = \vec{v}^n + \Delta t \vec{a}^n$ *piecewise linear* velocity
  - $\vec{a}^{n+1} = \vec{a}^n$ *piecewise constant* acceleration (constant from time n to just before time n+1)

Newmark Methods

$$\vec{x}^{n+1} = \vec{x}^{n} + \Delta t \vec{v}^{n} + \frac{\Delta t^2}{2}\left[(1-2\beta)\vec{a}^{n} + 2\beta\vec{a}^{n+1}\right]$$

$$\vec{v}^{n+1} = \vec{v}^{n} + \Delta t\left[(1-\gamma)\vec{a}^{n} + \gamma\vec{a}^{n+1}\right]$$

- Most popular multi-value method in *computational mechanics*
- Actually a lot of methods in disguise
- Different choice of β and γ makes a specific method
- β and γ both identically 0 gives the standard constant acceleration case

Newmark Methods

- Second order accurate if and only if $\gamma = 1/2$
- Trapezoidal Rule when $\beta = 1/4$

$$\vec{x}^{n+1} = \vec{x}^n + \Delta t \vec{v}^n + \frac{\Delta t^2}{2} \frac{(a^n + \vec{a}^{n+1})}{2}$$

$$\vec{v}^{n+1} = \vec{v}^n + \Delta t \frac{(a^n + \vec{a}^{n+1})}{2}$$

  - Substitute the acceleration terms from the second equation into the first, to see that the first equation is equivalent to

$$\vec{x}^{n+1} = \vec{x}^n + \Delta t \frac{(v^n + \vec{v}^{n+1})}{2}$$

A Newmark Method…

1. $\vec{v}^{n+1/2} = \vec{v}^{n} + \frac{\Delta t}{2}\vec{a}(t^{n}, \vec{x}^{n}, \vec{v}^{n+1/2})$
2. Modify $\vec{v}^{n+1/2}$ in some cases, e.g. collisions
3. $\vec{x}^{n+1} = \vec{x}^{n} + \Delta t \vec{v}^{n+1/2}$
4. $\vec{v}^{n+1} = \vec{v}^{n+1/2} + \frac{\Delta t}{2}\vec{a}(t^{n+1}, \vec{x}^{n+1}, \vec{v}^{n+1})$
5. Modify $\vec{v}^{n+1}$ in some cases, e.g. collisions

Implicit Solve

- Steps 1 and 4 are implicit in $\vec{v}^{n+1/2}$ and $\vec{v}^{n+1}$ respectively
- Typically the equations are linear in v, so we only need to solve a single matrix system
- The matrix is generally symmetric positive definite (SPD) and we can use fast solvers such as conjugate gradients (CG) for solving the system

- Note that in the first step we are using $\vec{x}^{n}$ instead of $\vec{x}^{n+1/2}$
  - The equations are typically highly nonlinear in x

Question #3

LONG FORM:
- Tell me everything about…. Nvm, have a nice day!

SHORT FORM:
- ☺