Class 3 Exercises

CS250/EE387, Winter 2022

1. In the videos/notes, we saw that with high probability, a random linear code of rate $R \geq 1 - \frac{\log_q(\mathrm{Vol}_q(d-1,n))-1}{n} \approx 1 - H_q(d/n) - o(1)$ has distance at least d with high probability.

   Would this argument have worked if we had started with a completely random code of about that rate? (That is, let $C \subseteq \mathbb{F}_q^n$ be defined by including each element of $\mathbb{F}_q^n$ in C independently with probability q^{Rn}/q^n). If yes, what if anything needs to change about the proof? If not, what goes wrong with our proof?

2. Let $q \geq 3$ and fix some paramater $\alpha \in (1/q, 1-1/q)$. Suppose we draw an element $x \in \{1, 2, \ldots, q\}^n$, independently at random. Give an expression for the (approximate) probability that x has at least αn "3"'s in it. Your answer should be simple, and it should have a q-ary entropy term in it.

3. Your friend is confused by the statement, from the videos/lecture notes, that decoding a random binary linear code from up to half the distance is thought to be hard. They think that there is a polynomial time algorithm. Their reasoning is as follows.

   - Suppose that G is the generator matrix for a code C with distance d. Let $t < \lfloor \frac{d-1}{2} \rfloor$ be the number of errors that might occur.
   - The goal is, given a noisy codeword $y = Gx + e$ for $\mathrm{wt}(e) \leq t$, to find the x.
   - Since $t < \lfloor \frac{d-1}{2} \rfloor$, there is a unique such x, and we have $e = Gx - y$. In particular, x is the solution to the optimization problem
   $$x = \mathrm{argmin}_{x'} \mathrm{wt} Gx' - y.$$
   - Since we are working over $\mathbb{F}_2$, for any vector v we have $\mathrm{wt}(v) = \|v\|_2^2$, where $\|v\|_2 = \sqrt{\sum_i v_i^2}$ is the ℓ_2 norm. Thus, x is the solution to
   $$x = \mathrm{argmin}_{x'} \|Gx' - y\|_2^2.$$
   - But this is just linear regression! Use your favorite efficient technique to solve it. (For example, we could compute the pseudoinverse $G^\dagger = (G^T G)^{-1} G^T$ and compute $G^\dagger y$).

   Unfortunately, your friend has missed something. What's wrong with the above approach?

The following problem is quite long, but most of it is exposition—we will walk through parts (a), (b) and (c) together as a class.

4. In this exercise we'll walk through an attack on the McEliece cryptosystem called "Stern's attack." It's not a devastating attack—by making the numbers big enough you can still protect against it—but it does give a non-trivial way for Eve to figure out what Bob's message is. (Note: If you don't care about crypto, this is still an interesting algorithm for decoding an arbitrary linear code!)

   (a) Recall that the problem Eve wants to solve to break the McEliece cryptosystem is to decode a binary linear code. Let $C \subseteq \mathbb{F}_2^n$ be the binary linear code that Eve has to decode in the McEliece cryptosystem. (So, in the language of the vidoes/notes, a generator matrix for C had a special form, $P \cdot G_0 \cdot S$). Say that C has dimension k, length n, and distance $d \geq 2t+1$. Let G be the generator matrix for C. (Note: in the lecture notes, G was $\hat{G}$...we're losing the hat since the original G won't be relevant for this question.) Eve's job is to find a vector x, given $y = Gx + e$, where $\mathrm{wt}(e) = t$.

   Consider the code C', one dimension larger than C, given by $C' = C + \{0, y\}$. (That is, $C' = C \cup \{c + y : c \in C\}$ — convince yourself that this is indeed a linear code if it's not immediately clear).

   Show that, if Eve can find a weight-t vector in C', then she can find Bob's message x.

   (b) In light of the previous part, we will focus on the problem of finding a low-weight vector in a linear code C'. (This will actually work for *any* linear code.). In part (b) of this problem, there is no question, we're just going to present Stern's algorithm for finding a codeword of C' of weight t.

   Before we get into it, here's a quick overview:

   A. We are going to construct a randomized parity-check matrix H for C'.
   B. We will enumerate over some guesses for (part of) the support of a weight-t codeword c.
   C. We will check to see if we can fill out the rest of the support.
   D. It turns out that A-C will succeed with some small, but not-too-small, probability. We'll repeat A-C a bunch of times until we win.

   Okay, now we'll go through steps A,B,C,D in more detail.

   A. **Construct a randomized parity-check matrix.** We'll also set up some notation. Fix parameters p and ℓ to be determined later. (Think of $p \ll t/2$, and think of $\ell > p$ as being pretty small as well). We are given as input a generator matrix of the code C'; use linear algebra to compute a parity-check matrix.

      i. There are several parity-check matrices of C'. We will choose a random parity-check matrix $H \in \mathbb{F}_2^{n-k}$ as follows. Choose a random set $W \subseteq \{1, \ldots, n\}$ of size $n-k$ and choose H — by doing row operations on the parity-check matrix you already have — so that the $(n-k) \times (n-k)$ given by the columns indexed by W form the identity matrix. (Note: The astute reader will realize that not every set W will allow this! That is, if the columns indexed by W are linearly dependent you will not be able to diagonalize them to get I. Just ignore this[1]...)

      ii. Choose a random subset $Z \subset W$ of size ℓ. Let $Z' \subset \{1, \ldots, n-k\}$ be the set of columns that correspond to Z according to the entries of H. That is, for each $z \in Z$, the z'th column of H is equal to $e_{z'}$ for some $z' \in \{1, \ldots, n-k\}$. Let Z' be the set of all such z'.

      iii. Consider the k elements of $\{1, \ldots, n\} \setminus W$. Partition them randomly into two parts, X and Y. (That is, each of the k elements joins X with probability 1/2 or joins Y with probability 1/2, independently).

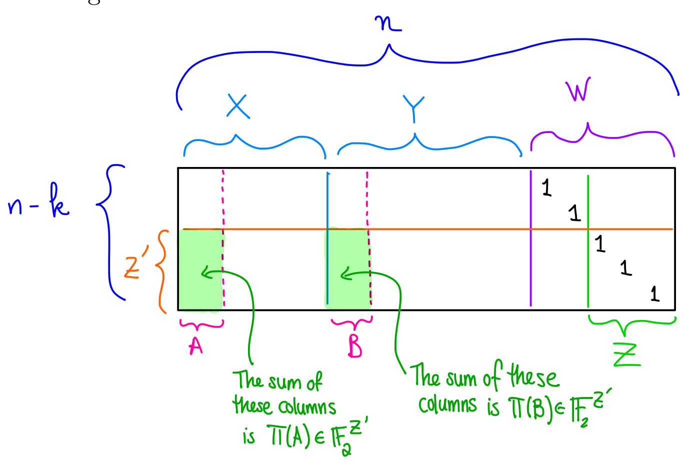
[1] Stern's original algorithm says you should resample W until you can make the identity in those columns, and notes that this doesn't seem to affect the distribution of W very much in practice.

Notation: Let h_i denote the i'th column of H. Given a set $A \subset X$, define $\pi(A) \in \mathbb{F}_2^{Z'}$ by

$$\pi(A) := \left(\sum_{a \in A} h_a \right) \Big|_{Z'}.$$

That is, we look at all the columns indexed by A and add them together, then restrict to the rows in Z'. For $B \subset Y$, we define $\pi(B)$ similarly.

Altogether, the picture looks something like this, except the sets X, Y, Z, W are random and so probably not contiguous.



B. **"Guess" some potential supports.** For each set $A \subseteq X$ of size p, compute $\pi(A)$. For each set $B \subseteq Y$ of size p, compute $\pi(B)$. If you find A, B with $\pi(A) = \pi(B)$, make a note of it.

Aside: this will come back later. The time it takes to do this is roughly:

- $O\left(p\ell\binom{|X|}{p}\right) + O\left(p\ell\binom{|Y|}{p}\right) \approx O\left(p\ell\binom{k/2}{p}\right)$ to enumerate over all A and compute $\pi(A)$, and then (in a separate loop) do the same thing for all of the B's.
- The number of vectors in $\mathbb{F}_2^{Z'}$ is 2^ℓ. So we can keep a hash table with 2^ℓ keys to find collisions. As a back-of-the-envelope calculation, the number of collisions that we expect (using the fact that everything in sight is random, so we hope that $\pi(A)$ and $\pi(B)$ are each approximately uniformly random in $\mathbb{F}_2^{Z'}$) is approximately:

$$\begin{aligned}
\mathbb{E}[\text{number of collisions}] &= \sum_B \left(\sum_A \mathbb{P}[\pi(A) = \pi(B)] \right) \\
&= \mathbb{E}\binom{|Y|}{p} \cdot \binom{|X|}{p} \cdot \frac{1}{2^\ell} \\
&\approx \binom{k/2}{p}^2 \cdot 2^{-\ell}.
\end{aligned}$$

(The above is not strictly legit — e.g., $|X|$ and $|Y|$ are correlated so I shouldn't just apply $\mathbb{E}$ to each of them independently — but it's close enough).

Thus, the amount of time it takes to iterate over all collisions and check the weight of $H(\mathbf{1}_A + \mathbf{1}_B)$ is about $O((n-k) \cdot 2p)$ per collision, or about

$$O\left((n-k)p \cdot \binom{k/2}{p}^2 \cdot 2^{-\ell} \right)$$

total.

C. **For each potential support, try to fill in the rest.** For each collision — that is, for each pair A, B so that $\pi(A) = \pi(B)$ — check to see if $\sum_{a \in A} h_a + \sum_{b \in B} h_b$ has weight exactly $t - 2p$. If it does, we **claim** that you can find a vector c of weight exactly t so that $Hc = 0$. Return this vector c. (And if none of these collisions result in returning something, return `fail`.)

D. **Repeat until you win.** Repeat steps A through C with independent randomness until you return something other than `fail`.

(Again, there is no question in part (b), just make sure you understand the algorithm).

(c) Justify the **claim** above: *If* $\pi(A) = \pi(B)$ *and if* $\sum_{a \in A} h_a + \sum_{b \in B} h_b$ *has weight exactly* $t - 2p$*, then there is a vector* c *so that* $\mathrm{wt}(c) = t$ *and* $Hc = 0$. Observe that such a vector c is indeed what wanted to return.

(d) Explain why the algorithm will succeed (with a given choice of Z, X, Y) if there is a vector c of weight t so that:

I. $c|_X$ and $c|_Y$ both have weight exactly p.

II. $c|_Z$ has weight zero.

(e) The expected running time of the algorithm is thus:

$$O(\text{time for A-C}) \cdot \frac{1}{\Pr[\text{I. and II. occur}]}.$$

This might seem pretty big. After all, in steps B and C we are iterating over all possible A's and B's and collisions. Moreover, the probability that this works seems pretty small, so we are probably repeating the whole thing a lot. However, it turns out that this can result in a non-trivial speed-up over the naive algorithm. To see this, let's fix:

$$n = 300, k = 150, t = 20, p = 3, \ell = 12.$$

i. What order of magnitude is the running time of the naive algorithm to find a weight-t vector c? (The naive algorithm is "iterate over all $c \in C$ and see if it has weight t"). In particular, this running time is on the order of $2^{\text{something}}$. What is that something, for the choice of parameters above?

ii. What is the order of magnitude for the running time of Stern's attack? Just try to come up with a back-of-the-envelope running time, focusing on the value of "something" in $2^{\text{something}}$. **We will walk you through some key components below; you just have to put them together in the right way.** (You may want to use your phone as a calculator or something).

- Iterating over A and computing $\pi(A)$ (and then doing the same for the B's) takes time on the order of:
$$p\ell \binom{k/2}{p} = 3 \cdot 12 \cdot \binom{75}{3} = 2,430,900.$$
- Iterating over all colliding pairs and checking the weight of the resulting vector times time on the order of:
$$(n-k)p\binom{k/2}{p}^2 \cdot 2^{-\ell} = 150 \cdot 3 \cdot \binom{75}{3}^2 \cdot 2^{-12} \approx 500,935,432.$$
- The probability, when choosing a random subset W of size $k = 150$ out of $n = 300$ things, that the $t = 20$ ones in our desired codeword c end up with exactly 14 ones in W and exactly 6 ones outside of W is:
$$\frac{\binom{20}{6} \cdot \binom{300-20}{150-6}}{\binom{300}{150}} \approx 0.03414.$$

- The probability, when choosing the partition X, Y, that the six ones not in W get split with 3 in X and 3 in Y is:
$$\binom{6}{3}/2^6 = 0.3125.$$
- The probability, when choosing a random $Z \subseteq W$ of size $\ell = 12$, that none of the $t = 20$ ones in c end up in Z is:
$$\frac{\binom{150-14}{12}}{\binom{150}{12}} \approx 0.3.$$