CS256/Winter 2009 Lecture #6

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Chapter 1

Invariance: Proof Methods

For assertion q
and SPL program P

show $P \vDash \Box\, q$
(i.e., q is P-invariant)

Proving Invariances

Definitions

Recall:

- the variables of assertion:
  - free (flexible) system variables

    $$V = Y \cup \{\pi\}$$

    where Y are the program variables and π is the control variable
  - quantified (rigid) specification variables

- q' is the primed version of q, obtained by replacing each free occurrence of a system variable $y \in V$ by its primed version y'.

- ρ_τ is the transition relation of τ, expressing the relation holding between a state s and any of its τ-successors $s' \in \tau(s)$.

Verification Conditions

(proof obligations)

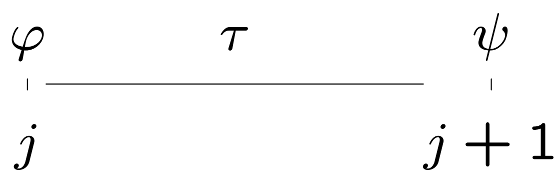
<u>standard verification condition</u>

For assertions φ, ψ and transition τ,

$\{\varphi\}\ \tau\ \{\psi\}$ ("Hoare triple") stands for the state formula

$$\boxed{\rho_\tau \ \wedge\ \varphi \ \rightarrow\ \psi'}$$

"Verification condition (VC) of φ and ψ
relative to transition τ"



Verification Conditions (Con't)

Example:

ρ_τ: $x \geq 0 \;\wedge\; y' = x + y \;\wedge\; x' = x$

φ: $y = 3$ $\qquad$ ψ: $y = x + 3$

Then $\{\varphi\}\ \tau\ \{\psi\}$:

$$\underbrace{x \;\geq\; 0 \;\wedge\; y' = x + y \;\wedge\; x' = x}_{\rho_\tau} \;\wedge\; \underbrace{y = 3}_{\varphi} \;\rightarrow\; \underbrace{y' = x' + 3}_{\psi'}$$

Verification Conditions (Con't)

- for $\tau \in \mathcal{T}$ in P

  $\{\varphi\}\tau\{\psi\}$: $\rho_\tau \wedge \varphi \rightarrow \psi'$

  "τ <u>leads from</u> φ to ψ in P"

- for $\mathcal{T}$ in P

  $\{\varphi\}\mathcal{T}\{\psi\}$: $\{\varphi\}\tau\{\psi\}$ for every $\tau \in \mathcal{T}$

  "$\mathcal{T}$ <u>leads from</u> φ to ψ in P"

`Claim` (Verification Condition)

If $\{\varphi\}\tau\{\psi\}$ is P-state valid,
then every τ-successor of a φ-state is a ψ-state.

Verification Conditions (Con't)

Special Cases

- while, conditional $\qquad \rho_\tau:\ \rho_\tau^{\mathrm{T}} \vee \rho_\tau^{\mathrm{F}}$

$$\{\varphi\}\tau^{\mathrm{T}}\{\psi\}:\quad \rho_\tau^{\mathrm{T}} \ \wedge\ \varphi \ \rightarrow\ \psi'$$

$$\{\varphi\}\tau^{\mathrm{F}}\{\psi\}:\quad \rho_\tau^{\mathrm{F}} \ \wedge\ \varphi \ \rightarrow\ \psi'$$

$$\{\varphi\}\tau\{\psi\}\ :\ \{\varphi\}\tau^{\mathrm{T}}\{\psi\}\ \wedge\ \{\varphi\}\tau^{\mathrm{F}}\{\psi\}$$

- idle

$$\{\varphi\}\tau_I\{\varphi\}:\quad \rho_{\tau_I} \ \wedge\ \varphi \ \rightarrow\ \varphi'$$

always valid, since

$$\rho_{\tau_I} \rightarrow v' = v \qquad \text{for all } v \in V,$$

so $\varphi' = \varphi$.

Verification Conditions (Con't)

Substituted Form of Verification Condition

Transition relation can be written as

$$\rho_\tau: \; C_\tau \;\wedge\; (\overline{V}' = \overline{E})$$

where

C_τ: enabling condition
$\overline{V}'$: primed variable list
$\overline{E}$: expression list

- The substituted form of verification condition $\{\varphi\}\tau\{\psi\}$:

$$\boxed{C_\tau \;\wedge\; \varphi \;\rightarrow\; \psi[\overline{E}/\overline{V}]}$$

where
$\psi[\overline{E}/\overline{V}]$: replace each variable $v \in \overline{V}$ in ψ by the corresponding $e \in \overline{E}$

Note: No primed variables!

> The substituted form of a verification condition is P-state valid iff the standard form is

Verification Conditions (Con't)

`Example:`

$$\rho_\tau : \ x \geq 0 \ \wedge \ y' = x + y \ \wedge \ x' = x$$

$$\varphi : \ y = 3 \qquad \psi : \ y = x + 3$$

Standard

$$\underbrace{x \geq 0 \ \wedge \ y' = x + y \ \wedge \ x' = x}_{\rho_\tau} \ \wedge \ \underbrace{y = 3}_{\varphi} \ \rightarrow \ \underbrace{y' = x' + 3}_{\psi'}$$

Substituted

$$\underbrace{x \geq 0}_{C_\tau} \ \wedge \ \underbrace{y = 3}_{\varphi} \ \rightarrow \underbrace{x + y = x + 3}_{\psi[\overline{E}/\overline{V}]}$$

Verification Conditions (Con't)

Example:

$$\varphi\colon\ x = y \qquad \psi\colon\ x = y + 1$$

$$\rho_\tau\colon\ \underbrace{x \ \geq\ 0}_{C_\tau} \wedge \underbrace{(x', y')}_{\overline{V}'} = \underbrace{(x+1, y)}_{\overline{E}}$$

The substituted form of $\{\varphi\}\tau\{\psi\}$ is

$$\underbrace{x \geq 0}_{C_\tau} \wedge \underbrace{x = y}_{\varphi} \rightarrow$$

$$\underbrace{(x = y + 1)[(x+1, y)/(x, y)]}_{\psi[\overline{E}/\overline{V}]}$$

or equivalently

$$x \geq 0 \ \wedge\ x = y \ \ \rightarrow\ \ x + 1 = y + 1$$

Simplifying Control Expressions

$move(L_1, L_2)$: $\quad L_1 \subseteq \pi \;\wedge\; \pi' = (\pi - L_1) \cup L_2$

e.g., for $L_1 = \{\ell_1\}, L_2 = \{\ell_2\}$
$move(\ell_1, \ell_2)$: $\quad \ell_1 \in \pi \;\wedge\; \pi' = (\pi - \{\ell_1\}) \cup \{\ell_2\}$

Consequences implied by $move(L_1, L_2)$:

- for every $[\ell] \in L_1$
  $at\_\ell = \text{T}$ (i.e., $[\ell] \in \pi$)

- for every $[\ell] \in L_2$
  $at'\_\ell = \text{T}$ (i.e., $[\ell] \in \pi'$)

- for every $[\ell] \in L_1 - L_2$
  $at\_\ell = \text{T}$ (i.e., $[\ell] \in \pi$) and
  $at'\_\ell = \text{F}$ (i.e., $[\ell] \notin \pi'$)

- for every $\ell \notin L_1 \cup L_2$
  $at'\_\ell = at\_\ell$ (i.e., $[\ell] \in \pi, \pi'$ or $[\ell] \notin \pi, \pi'$)

Proving invariance properties: $P \models \Box\, q$

We want to show that for every computation of P

$\sigma : s_0, s_1, s_2, \ldots$

assertion q holds in every state s_j, $j \geq 0$, i.e., $s_j \Vdash q$.

Recall:

A sequence $\sigma : s_0, s_1, s_2, \ldots$ is a computation if the following hold (from Chapter 0):

1. Initiality: $s_0 \Vdash \Theta$

2. Consecution: For each $j \geq 0$,
   s_{j+1} is a τ-successor of s_j for some $\tau \in \mathcal{T}$
   $(s_{j+1} \in \tau(s_j))$

3, 4. Fairness conditions are respected.

Note: Truth of *safety* properties over programs *does not* depend on fairness conditions.

Proving invariance properties (Con't)

This definition suggests a way to prove invariance properties $\Box\, q$:

1. Base case:
   Prove that q holds initially

   $$\Theta \rightarrow q$$

   i.e., q holds at s_0.

2. Inductive step:
   prove that q is preserved by all transitions

   $$\underbrace{q \wedge \rho_\tau \rightarrow q'}_{\{q\}\tau\{q\}} \quad \text{for all } \tau \in \mathcal{T}$$

   i.e., if q holds at s_j, then it holds at every τ-successor s_{j+1}.

Rule B-INV (basic invariance)

Show $P \models \square\, q$ (i.e. q is <u>P-invariant</u>)

For assertion q,

$$\begin{array}{ll} \text{B1.} & P \Vdash \Theta \rightarrow q \\ \text{B2.} & P \Vdash \{q\}\; \mathcal{T}\; \{q\} \\ \hline & P \models \square\, q \end{array}$$

where B2 stands for

$$P \Vdash \{q\}\; \tau\; \{q\} \text{ for every } \tau \in \mathcal{T}$$

- The rule states that if we can prove the P-state validity of $\Theta \rightarrow q$ and $\{q\}\mathcal{T}\{q\}$ then we can conclude that $\square\, q$ is P-valid.
- Thus the proof of a temporal property is reduced to the proof of $1 + |\mathcal{T}|$ first-order verification conditions.

Example 1: REQUEST-RELEASE

$$\textbf{local} \quad x\colon \ \textbf{integer} \quad \textbf{where } x = 1$$

$$\begin{bmatrix} \ell_0: & \textbf{request } x \\ \ell_1: & \textbf{critical} \\ \ell_2: & \textbf{release } x \\ \ell_3: & \end{bmatrix}$$

Θ: $x = 1 \ \wedge \ \pi = \{\ell_0\}$

$\mathcal{T}$: $\{\tau_I, \tau_{\ell_0}, \tau_{\ell_1}, \tau_{\ell_2}\}$

Prove

$$P \models \Box \underbrace{x \geq 0}_{q}$$

using B-INV.

Example 1: request-release (Con't)

B1: $\underbrace{x = 1 \wedge \pi = \{\ell_0\}}_{\Theta} \rightarrow \underbrace{x \geq 0}_{q}$

holds since $x = 1 \rightarrow x \geq 0$

B2:

τ_{ℓ_0}: $\underbrace{x \geq 0}_{q} \wedge \underbrace{move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1}_{\rho_{\tau_{\ell_0}}} \rightarrow \underbrace{x' \geq 0}_{q'}$

holds since $x > 0 \rightarrow x - 1 \geq 0$

τ_{ℓ_1}: $\underbrace{x \geq 0}_{q} \wedge \underbrace{move(\ell_1, \ell_2) \wedge x' = x}_{\rho_{\tau_{\ell_1}}} \rightarrow \underbrace{x' \geq 0}_{q'}$

holds since $x \geq 0 \rightarrow x \geq 0$

τ_{ℓ_2}: $\underbrace{x \geq 0}_{q} \wedge \underbrace{move(\ell_2, \ell_3) \wedge x' = x + 1}_{\rho_{\tau_{\ell_2}}} \rightarrow \underbrace{x' \geq 0}_{q'}$

holds since $x \geq 0 \rightarrow x + 1 \geq 0$

Example 1: request-release (Con't)

$$\textbf{local } x\text{: } \textbf{integer} \;\; \textbf{where } x = 1$$

$$\left[\begin{array}{ll} \ell_0 : & \textbf{request } x \\ \ell_1 : & \textbf{critical} \\ \ell_2 : & \textbf{release } x \\ \ell_3 : & \end{array}\right]$$

We proved

$$P \models \Box\, x \geq 0$$

using B-INV.

Now we want to prove

$$P \models \Box \underbrace{(at\_\ell_1 \rightarrow \; x = 0)}_{q}$$

Example 1: request-release (Con't)

Attempted proof:

B1: $\underbrace{x = 1 \land \pi = \{\ell_0\}}_{\Theta} \rightarrow (\underbrace{at\_\ell_1 \rightarrow x = 0}_{q})$

holds since $\pi = \{\ell_0\} \rightarrow at\_\ell_1 = \text{F}$

B2: $\{q\}\tau_{\ell_0}\{q\}$

$$\underbrace{at\_\ell_1 \rightarrow x = 0}_{q} \land \underbrace{move(\ell_0, \ell_1) \land x > 0 \land x' = x - 1}_{\rho_{\tau_{\ell_0}}}$$

$$\rightarrow \underbrace{at'\_\ell_1 \rightarrow x' = 0}_{q'}$$

We have $move(\ell_0, \ell_1) \rightarrow at\_\ell_1 = \text{F}, \; at'\_\ell_1 = \text{T}$

BUT

$$(\text{F} \rightarrow x = 0) \land x > 0 \land x' = x - 1 \rightarrow (\text{T} \rightarrow x' = 0)$$

Cannot prove: $\boxed{\text{not state-valid}}$

What is the problem?

We need a stronger rule.

Strategies for invariance proofs

Rule B-INV **(basic invariance)**

For assertion q,

$$\begin{array}{ll} \text{B1.} & P \ \Vdash \ \Theta \ \rightarrow \ q \\ \text{B2.} & P \ \Vdash \ \{q\} \ \mathcal{T} \ \{q\} \\ \hline & P \ \models \ \Box \ q \end{array}$$

- q is <u>inductive</u> if B1 and B2 are (state) valid
- By rule B-INV,
  <u>every inductive assertion q is P-invariant</u>
- <u>The converse is not true</u>

`Example:` In REQUEST-RELEASE

$$at\_\ell_1 \ \rightarrow \ x = 0$$

is P-invariant, but not inductive

Rule B-INV(Con't)

The problem is:

"The invariant is not inductive"

i.e., it is not strong enough to be preserved by all transitions.

Another way to look at it is to observe that

$$\{q\}\ \tau_{\ell_0}\ \{q\}$$

is not state valid, but it is P-state valid,
i.e., it is true in all P-accessible states,
since in all P-accessible states

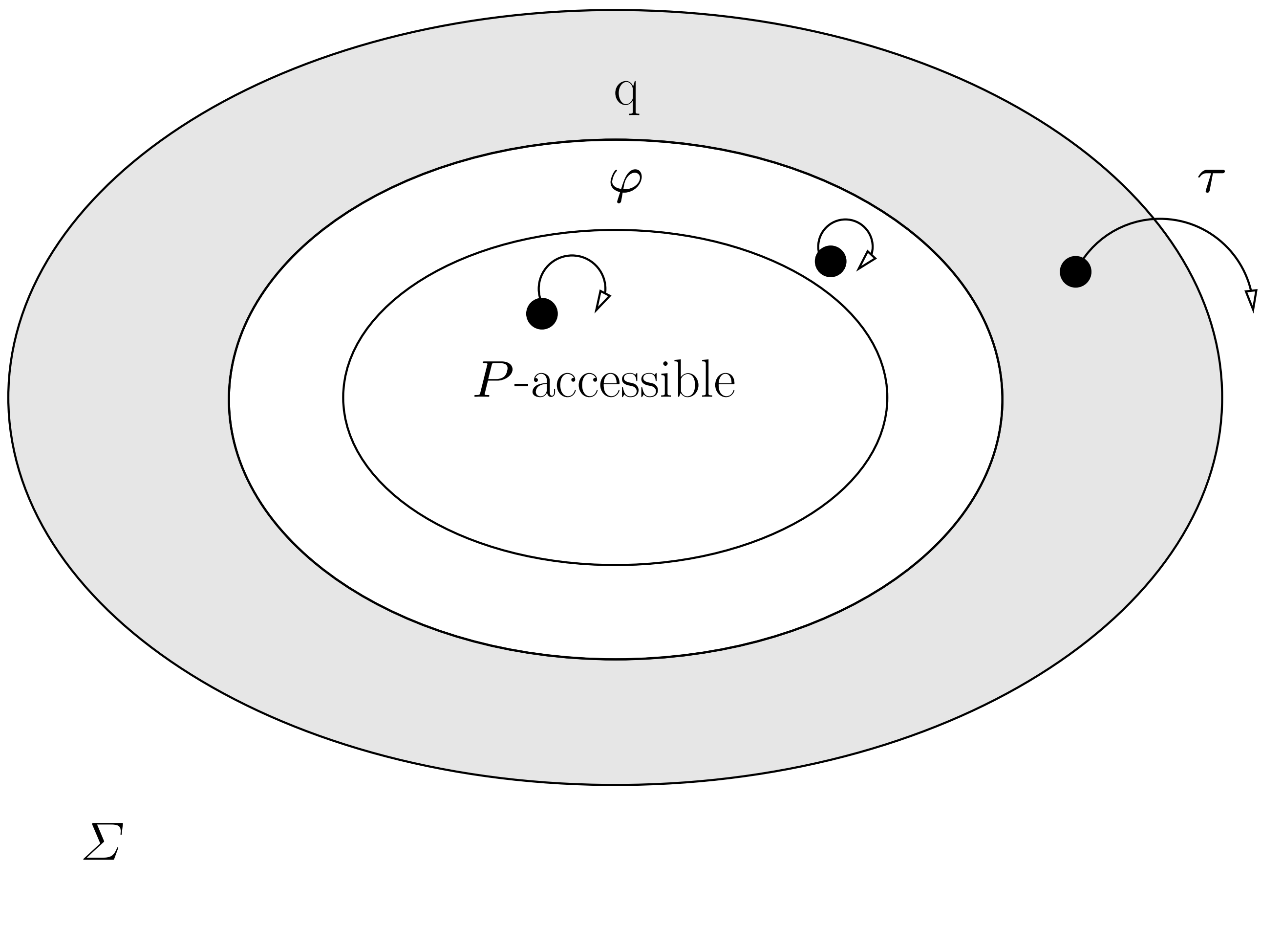
$x = 1$ when at location ℓ_0.

This suggests two strategies to overcome this problem:

- strengthening
- incremental proof

Strategy 1: Strengthening

Find a stronger assertion φ that is inductive and implies the assertion q we want to prove.



In Chapter 2 it will be shown that there always exists such an assertion φ.

Strategy 1: Strengthening (Con't)

`Example:`

To show

$$\Box(\underbrace{at\_\ell_1 \rightarrow x = 0}_{q})$$

strengthen q to

$$\varphi: (at\_\ell_1 \rightarrow x = 0) \wedge (at\_\ell_0 \rightarrow x = 1)$$

and show

$$\Box\, \underbrace{(at\_\ell_1 \rightarrow x = 0) \wedge (at\_\ell_0 \rightarrow x = 1)}_{\varphi}$$

by rule B-INV.

Strategy 1: Strengthening (Con't)

The strengthening strategy relies on the following rule, MON-I, which, combined with B-INV leads to the general invariance rule INV.

Rule MON-I (Monotonicity)

For assertions q_1, q_2,

$$\frac{P \models \Box\, q_1 \qquad P \Vdash q_1 \rightarrow q_2}{P \models \Box\, q_2}$$

Strategy 1: Strengthening (Con't)

<u>Rule INV</u> (general invariance)

For assertions q, φ

$$
\begin{array}{ll}
\text{I1.} & P \Vdash \varphi \;\rightarrow\; q \\
\text{I2.} & P \Vdash \Theta \;\rightarrow\; \varphi \\
\text{I3.} & P \Vdash \{\varphi\}\; \mathcal{T}\; \{\varphi\} \\
\hline
 & P \;\vDash\; \Box\, q
\end{array}
$$

Soundness: If we manage to prove $\Box q$ using the INV rule for some program P, is q really an invariant for the program?

We can prove that this is indeed the case. So INV rule is *sound.*

Completeness: What if q is an invariant for a program P but there is **no** way of proving it under the INV rule?

We can prove that this never happens. There always exists an appropriate φ. In other words INV rule is *complete.*

Strategy 1: Strengthening (Con't)

Motivation:

$$P \models \Box\, \varphi \qquad \text{(by I2 and I3)}$$

$$P \Vdash \varphi \rightarrow q \qquad \text{(by I1)}$$

Therefore,

$$P \models \Box\, q \qquad \text{(by MON-I)}$$

i.e., this rule requires that $\Box\,\varphi$ holds and φ implies $\boldsymbol{q}$, then $\Box\,\boldsymbol{q}$ can be concluded to hold by monotonicity.

Control Invariants

Some control invariants that can always be used (without mentioning them)

- CONFLICT:
  for labels ℓ_i, ℓ_j that are in conflict
  (i.e., not $\sim_L$, not parallel):

  $$\square \neg(at\_\ell_i \wedge at\_\ell_j)$$

- SOMEWHERE:
  for the set of labels $\mathcal{L}_i$ in a
  top-level process:

  $$\square \bigvee_{\ell \in \mathcal{L}_i} at\_\ell$$

- EQUAL:
  for labels l, m, s.t. $l \sim_L m$:

  $$\square(at\_\ell \leftrightarrow at\_m)$$

Control Invariants (Con't)

- PARALLEL:

  for substatement $[S_1||S_2]$:

  $$\square(in\_S_1 \leftrightarrow in\_S_2)$$

  i.e, if control is in S_1 it must also be in S_2 and vice versa.

`Example:`

Using the invariant CONFLICT,

$$move(\ell_2, \ell_3) \quad \text{implies} \quad \begin{array}{r} l_0 \notin \pi,\ l_1 \notin \pi,\ l_3 \notin \pi \\ l_0 \notin \pi',\ l_1 \notin \pi',\ l_2 \notin \pi' \end{array}$$

Strategy 1: Strengthening (Con't)

`Example:`

We proposed the strengthened invariant

$$\varphi : \ (at\_\ell_0 \ \rightarrow \ x = 1) \ \wedge \ (at\_\ell_1 \ \rightarrow \ x = 0)$$

Consider $\{\varphi\}\ \tau_{\ell_0}\ \{\varphi\}$:

$$\underbrace{(at\_\ell_0 \rightarrow x = 1) \ \wedge \ (at\_\ell_1 \rightarrow x = 0)}_{\varphi} \ \wedge$$

$$\underbrace{move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1}_{\rho_{\tau_{\ell_0}}}$$

$$\rightarrow \ \underbrace{(at'\_\ell_0 \rightarrow x' = 1) \ \wedge \ (at'\_\ell_1 \rightarrow x' = 0)}_{\varphi'}$$

$move(\ell_0, \ell_1)$ implies $\ell_0 \in \pi, \ell_1 \notin \pi, \ell_1 \in \pi', \ell_0 \notin \pi'$

Therefore

$$(\text{T} \rightarrow x = 1) \wedge (\text{F} \rightarrow \ldots) \wedge \ldots \wedge x' = x - 1 \wedge \ldots$$

$$\rightarrow \ (\text{F} \rightarrow \ldots) \wedge (\text{T} \rightarrow x' = 0)$$

holds.

Strategy 1: Strengthening (Con't)

`Example (Con't):`

Consider $\{\varphi\}\ \tau_{\ell_2}\ \{\varphi\}$:

$$\underbrace{(at\_\ell_0 \rightarrow x = 1)\ \wedge\ (at\_\ell_1 \rightarrow x = 0)}_{\varphi}\ \wedge$$

$$\underbrace{move(\ell_2, \ell_3) \wedge x' = x + 1}_{\rho_{\tau_{\ell_2}}}$$

$$\rightarrow\ \underbrace{(at'\_\ell_0 \rightarrow x' = 1)\ \wedge\ (at'\_\ell_1 \rightarrow x' = 0)}_{\varphi'}$$

$move(\ell_2, \ell_3)$ implies $\ell_3 \in \pi'$
and by CONFLICT invariants $\ell_0, \ell_1 \notin \pi'$.

Therefore

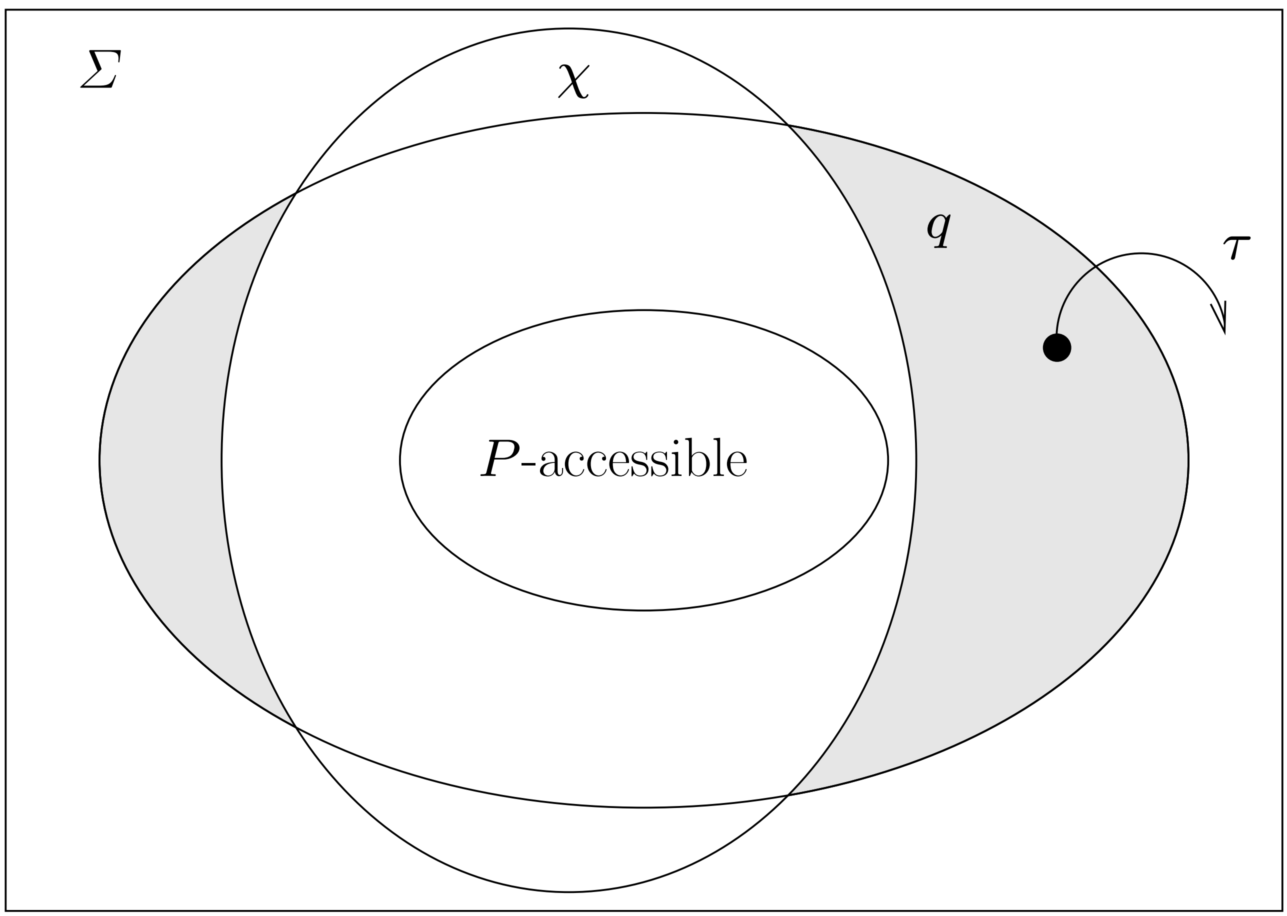
$$\ldots \wedge \ldots\ \rightarrow\ (\mathrm{F} \rightarrow x' = 1) \wedge (\mathrm{F} \rightarrow x' = 0)$$

holds.

$\{\varphi\}\ \tau_{\ell_2}\ \{\varphi\}$ is not state-valid,
but it is P-state valid. Why?

Strategy 2: Incremental proof

Use previously proven invariances χ to exclude parts of the state space from consideration.



Strategy 2: Incremental proof (Con't)

`Example:`

To show

$$\Box(\underbrace{at\_\ell_1 \;\rightarrow\; x = 0}_{q})$$

prove first (separately) by rule B-INV

$$\Box\,\underbrace{(at\_\ell_0 \;\rightarrow\; x = 1)}_{\chi},$$

then show

$$\Box(\underbrace{at\_\ell_1 \;\rightarrow\; x = 0}_{q})$$

by rule B-INV, but add the conjunct

$$at\_\ell_0 \;\rightarrow\; x = 1$$

to the antecedent of all verification conditions.

(`Example` continues...)

Strategy 2: Incremental proof (Con't)

`Example:` (cont'd)

e.g., to show $\{\chi \wedge q\}\tau_{\ell_0}\{q\}$, prove

$$\underbrace{at\_\ell_0 \rightarrow x = 1}_{\chi} \;\wedge\; \underbrace{at\_\ell_1 \rightarrow x = 0}_{q} \;\wedge$$

$$\underbrace{move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1}_{\rho_{\tau_{\ell_0}}}$$

$$\rightarrow \;\; \underbrace{at'\_\ell_1 \rightarrow x' = 0}_{q'}$$

Strategy 2: Incremental proof (Con't)

In an incremental proof we use previously proven properties to eliminate parts of the state space (non P-accessible states) from consideration, relying on the following rules:

Rule SV-PSV**:** from state validities to P-state validities

For assertions q_1, q_2 and χ,

$$\frac{P \vDash \Box\,\chi \qquad P \Vdash \chi \wedge q_1 \;\rightarrow\; q_2}{P \vDash \Box(q_1 \rightarrow q_2)}$$

Rule I-CON**:** Conjunction

For assertions q_1 and q_2,

$$\frac{P \vDash \Box\,q_1 \qquad P \vDash \Box\,q_2}{P \vDash \Box(q_1 \wedge q_2)}$$

Strategy 2: Incremental proof (Con't)

`Example:` Program MUX-SEM
(mutual exclusion by semaphores)

$$\textbf{local } y\text{: } \textbf{integer where } y = 1$$

$$P_1 :: \left[\begin{array}{l} \ell_0\text{: } \textbf{loop forever do} \\ \left[\begin{array}{l} \ell_1 : \textbf{noncritical} \\ \ell_2 : \textbf{request } y \\ \ell_3 : \textbf{critical} \\ \ell_4 : \textbf{release } y \end{array} \right] \end{array} \right] \; || \; P_2 :: \left[\begin{array}{l} m_0\text{: } \textbf{loop forever do} \\ \left[\begin{array}{l} m_1\text{: } \textbf{noncritical} \\ m_2\text{: } \textbf{request } y \\ m_3\text{: } \textbf{critical} \\ m_4\text{: } \textbf{release } y \end{array} \right] \end{array} \right]$$

Prove mutual exclusion

$$\Box \underbrace{\neg(at\_\ell_3 \;\wedge\; at\_m_3)}_{q}$$

Program MUX-SEM (Con't)

3 steps: $\square(\underbrace{y \ \geq 0}_{\varphi_1})$

$$\square(\underbrace{at\_\ell_{3,4} + at\_m_{3,4} + y = 1}_{\varphi_2})$$

$$\square\, \underbrace{\neg(at\_\ell_3 \ \wedge \ at\_m_3)}_{p}$$

where $\text{F} = 0, \text{T} = 1$.

Let π_ℓ: $\quad \pi \cap \{\ell_0, \ldots, \ell_4\}$
π_m: $\quad \pi \cap \{m_0, \ldots, m_4\}$

By control invariants (CONFLICT, SOMEWHERE and PARALLEL)

$$|\pi_\ell| = |\pi_m| = 1$$

Program MUX-SEM (Con't)

Step 1: $\Box(\underbrace{y \geq 0}_{\varphi_1})$

by rule B-INV

B1. $\underbrace{\pi = \{\ell_0, m_0\} \ \wedge \ y = 1}_{\Theta} \ \rightarrow \ \underbrace{y \geq 0}_{\varphi_1}$

B2. $\rho_\tau \ \wedge \ y \geq 0 \ \rightarrow \ y' \geq 0$

check only ℓ_2, ℓ_4, m_2, m_4
("y-modifiable transitions")

Program MUX-SEM (Con't)

$$\ell_2\colon\ \underbrace{move(\ell_2,\ell_3)\ \wedge\ y>0\ \wedge\ y'=y-1}_{\rho_\tau}\ \wedge\ \underbrace{y\geq 0}_{\varphi}\ \rightarrow\ \underbrace{y'\geq 0}_{\varphi'}$$

holds since $y>0\ \rightarrow\ y-1\geq 0$

$$\ell_4\colon\ \underbrace{move(\ell_4,\ell_0)\ \wedge\ y'=y+1}_{\rho_\tau}\ \wedge\ \underbrace{y\geq 0}_{\varphi}\rightarrow\ \underbrace{y'\geq 0}_{\varphi'}$$

holds since $y\geq 0\ \rightarrow\ y+1\geq 0$.

Similarly for $m_2,\ m_4$.

Program MUX-SEM (Con't)

Step 2:

$$\square(\underbrace{at\_\ell_{3,4} + at\_m_{3,4} + y = 1}_{\varphi_2})$$

by rule B-INV

B1. $\underbrace{\pi = \{\ell_0, m_0\} \ \wedge \ y = 1}_{\Theta} \ \rightarrow$

$$\underbrace{\underbrace{at\_\ell_{3,4}}_{0} + \underbrace{at\_m_{3,4}}_{0} + \underbrace{y}_{1} = 1}_{\varphi_2}$$

Program MUX-SEM (Con't)

B2. $\rho_\tau \wedge \varphi_2 \rightarrow \varphi_2'$

$$\rho_{\ell_0} \wedge 0 + at\_m_{3,4} + y = 1 \rightarrow 0 + at\_m_{3,4} + y = 1$$

$$\rho_{\ell_1} \wedge 0 + at\_m_{3,4} + y = 1 \rightarrow 0 + at\_m_{3,4} + y = 1$$

$$\rho_{\ell_2} \wedge 0 + at\_m_{3,4} + y = 1 \rightarrow 1 + at\_m_{3,4} + (y - 1) = 1$$

$$\rho_{\ell_3} \wedge 1 + at\_m_{3,4} + y = 1 \rightarrow 1 + at\_m_{3,4} + y = 1$$

$$\rho_{\ell_4} \wedge 1 + at\_m_{3,4} + y = 1 \rightarrow \underbrace{0}_{at'\_\ell_{3,4}} + \underbrace{at\_m_{3,4}}_{at'\_m_{3,4}} + \underbrace{(y+1)}_{y'} = 1$$

Program MUX-SEM (Con't)

Step 3: Show $P \models \Box \underbrace{\neg(at\_\ell_3 \ \land \ at\_m_3)}_{q}$

- By I-CON

$$\frac{P \models \Box\, \varphi_1,\ P \models \Box\, \varphi_2}{P \models \Box(\varphi_1 \ \land \ \varphi_2)}$$

- By MON-I

$$\frac{\begin{array}{l} P \models \Box(\varphi_1 \ \land \ \varphi_2) \\ P \models \underbrace{y \geq 0}_{\varphi_1} \ \land \ \underbrace{at\_\ell_{3,4} + at\_m_{3,4} + y = 1}_{\varphi_2} \\ \qquad\qquad \rightarrow \ \underbrace{\neg(at\_\ell_3 \ \land \ at\_m_3)}_{q} \end{array}}{P \models \Box \underbrace{\neg(at\_\ell_3 \ \land \ at\_m_3)}_{q}}$$