CS256/Winter 2009 Lecture #8

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Finding Inductive Assertions
Top-Down Approach

Assertion propagation

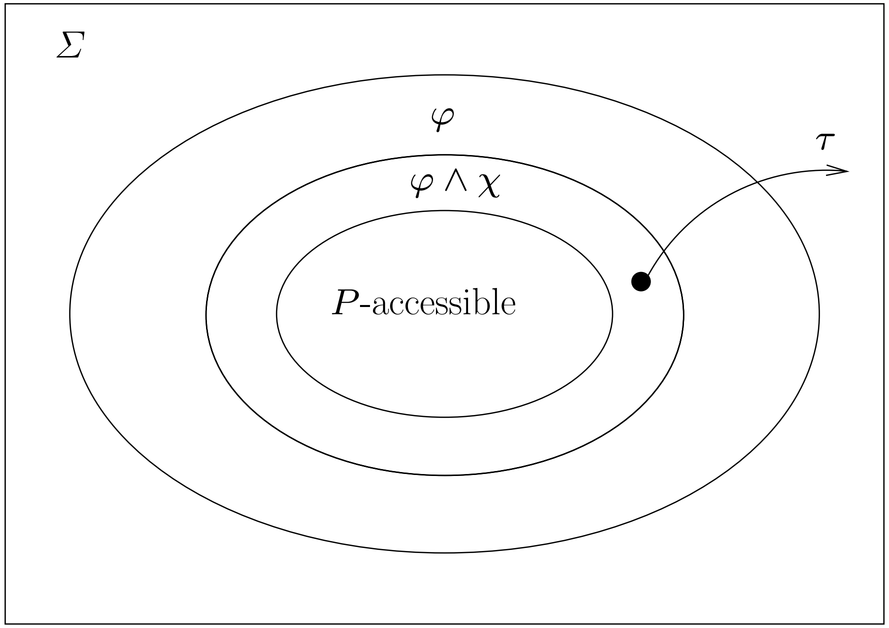
we have previously proven $\square\, \chi$
and we want to prove $\square\, \varphi$
but

$$\{\chi \wedge \varphi\}\tau\{\varphi\}$$

is not state-valid for some $\tau \in \mathcal{T}$.

What is the problem?
(assuming that φ is indeed an invariant)

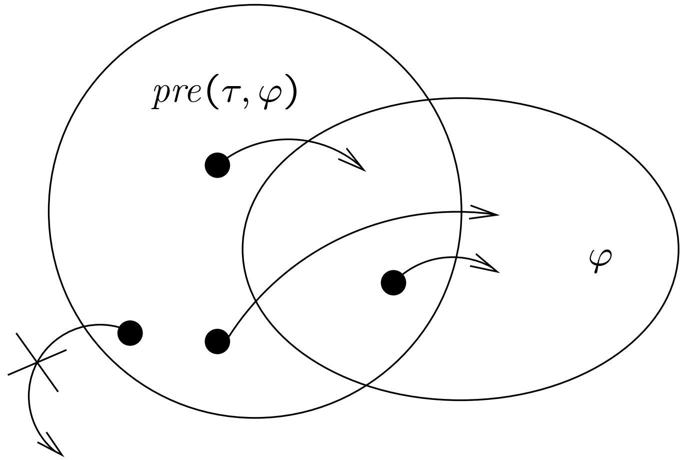
Top-Down Approach (Con'd)



Solution: Take the largest set of states that will result in a φ-state when τ is taken. How?

Precondition of φ w.r.t. τ

$pre(\tau, \varphi) : \forall V' . \rho_\tau \rightarrow \varphi'$



a state s satisfies $pre(\tau, \varphi)$
iff
all τ-successors of s satisfy φ.

Note:
s trivially satisfies $pre(\tau, \varphi)$ if it does not have any τ-successors (i.e., τ is not enabled in s).

Precondition of φ w.r.t. τ (Con'd)

`Example:`

$V : \{x\}$ integer

$\rho_\tau : x > 0 \wedge x' = x - 1$

$\varphi : x \geq 2$

$pre(\tau, \varphi)$:

$$\forall x' . \underbrace{x > 0 \wedge x' = x - 1}_{\rho_\tau} \rightarrow \underbrace{x' \geq 2}_{\varphi'}$$

$$x > 0 \rightarrow x - 1 \geq 2$$

$$x \leq 0 \vee x \geq 3$$

j —— τ —— $j+1$

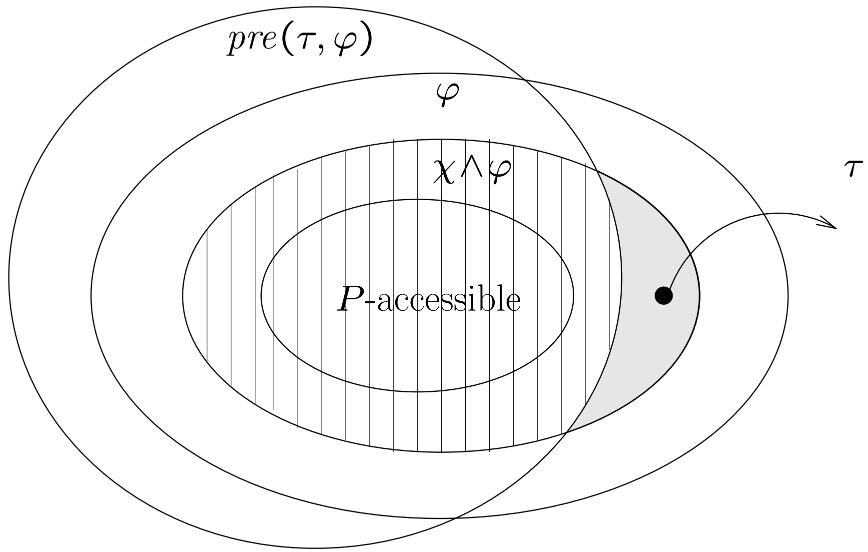
$x \leq 0 \vee x \geq 3$ —— $x \geq 2$

Properties of *pre*(τ, φ)

By the definition of $pre(\tau, \varphi)$,

$$\{\chi \wedge \varphi \wedge pre(\tau, \varphi)\} \ \tau \ \{\varphi\}$$

is guaranteed to be state-valid.



But we have to justify adding the conjunct $pre(\tau, \varphi)$ to the antecedent.
This can be done in two ways:

1. Incremental: prove $\Box \, pre(\tau, \varphi)$
2. Strengthening: prove $\Box(\varphi \wedge pre(\tau, \varphi))$

Properties of $\boldsymbol{pre}(\tau, \varphi)$ (Con'd)

Claim: If φ is P-invariant then so is $pre(\tau, \varphi)$ for every $\tau \in \mathcal{T}$.

Proof:
Suppose φ is P-invariant, but $pre(\tau, \varphi)$ is not P-invariant.

Then there exists a P-accessible state s such that

$$s \not\models pre(\tau, \varphi).$$

But then, by the definition of $pre(\tau, \varphi)$, there exists a τ-successor s' of s such that $s' \not\models \varphi$.

Since s is P-accessible, s' is also P-accessible, contradicting that φ is a P-invariant.

Properties of $\boldsymbol{pre}(\tau, \varphi)$ (Con'd)

Definition: A transition τ is said to be self-disabling if for every state s, τ is disabled in all τ-successors of s.

Claim: For every assertion φ and self-disabling transition τ

$$\{\varphi \wedge pre(\tau, \varphi)\}\ \tau\ \{\varphi \wedge pre(\tau, \varphi)\}$$

is state-valid.

Proof:
Assume $s \models \varphi \wedge pre(\tau, \varphi)$.

Then by definition of $pre(\tau, \varphi)$, for every s', τ-successor of s,
$s' \models \varphi$.

Since τ is self-disabling, τ is disabled in all τ-successors s' of s, and so trivially
$s' \models pre(\tau, \varphi)$

Thus for all τ-successors s' of s,
$s' \models \varphi \wedge pre(\tau, \varphi)$.

Heuristic

If the verification condition

$$\{\chi \wedge \varphi\}\tau\{\varphi\}$$

is not state-valid:

Find $pre(\tau, \varphi)$ and then
• Strengthening approach:
strengthen φ by adding the conjunct $pre(\tau, \varphi)$
prove $\square(\varphi \;\wedge\; pre(\tau,\; \varphi))$
or,
• Incremental approach:
prove $\square\; pre(\tau, \varphi)$
and add $pre(\tau, \varphi)$ to χ.

Note:
$pre(\tau, \varphi)$ is not guaranteed to be an inductive invariant, so the premises of INV have to be checked again.

`Example:`

$$\textbf{local } x\textbf{: integer where } x = 1$$

$$\left[\begin{array}{l} \ell_0 : \textbf{request } x \\ \ell_1 : \textbf{critical} \\ \ell_2 : \textbf{release } x \end{array}\right]$$

We want to prove

$$\boxed{\square\, \underbrace{(at\_\ell_1 \rightarrow x = 0)}_{\varphi}}$$

Problem:
$\{at\_\ell_1 \rightarrow x = 0\}\ \tau_{\ell_0}\ \{at\_\ell_1 \rightarrow x = 0\}$
is not state-valid.

If we use the above heuristic we get

$$pre(\tau_{\ell_0}, \varphi) = \forall x', \pi' .\ \underbrace{(move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1)}_{\rho_{\ell_0}} \rightarrow \underbrace{(at'\_\ell_1 \rightarrow x' = 0)}_{\varphi'}$$

Example (Con'd):

$$pre(\tau_{\ell_0}, \varphi) = \\ \forall x', \pi' \,.\, \underbrace{(move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1)}_{\rho_{\ell_0}} \\ \rightarrow \underbrace{(at'\_\ell_1 \rightarrow x' = 0)}_{\varphi'}$$

Since

$$move(\ell_0, \ell_1) \;\rightarrow at\_\ell_0 = \mathrm{T}, at'\_\ell_1 = \mathrm{T}$$

$$x' = x - 1 \wedge x' = 0 \;\rightarrow\; x = 1$$

it simplifies to

$$pre(\tau_{\ell_0}, \varphi):\; at\_\ell_0 \wedge x > 0 \;\rightarrow\; x = 1$$

Strengthened assertion

$\varphi \wedge pre(\tau_{\ell_0}, \varphi)$: $(at\_\ell_1 \rightarrow x = 0) \wedge (at\_\ell_0 \rightarrow x = 1)$

what we "guessed" before

Show that $\varphi \wedge pre(\tau_{\ell_0}, \varphi)$ is inductive
("strengthening approach")

Substituted form of $pre(\tau, \varphi)$

Many transition relations have the form

$$\rho_\tau:\;\; C_\tau \;\wedge\; \overline{V}' = \overline{E}$$

where C_τ is the enabled condition of τ.

And so

$$pre(\tau, \varphi):\; \forall \overline{V}' \,.\, C_\tau \;\wedge\; \overline{V}' = \overline{E} \;\rightarrow\; \varphi'$$

can be simplified to

$$\forall \overline{V}' \,.\, C_\tau \;\rightarrow\; \varphi[\overline{E}/\overline{V}]$$

replacing all primed variables by its
corresponding expression,
thus the quantifier can be eliminated to obtain

$$pre(\tau, \varphi):\; C_\tau \rightarrow\; \varphi[\overline{E}/\overline{V}]$$

Example: Program mux-pet1(Fig. 2.25)

(Peterson's Algorithm for mutual exclusion)

$$
\begin{array}{llll}
\textbf{local} & y_1, y_2\text{:} & \textbf{boolean} & \textbf{where } y_1 = \text{F}, y_2 = \text{F} \\
 & s \quad : & \textbf{integer} & \textbf{where } s = 1
\end{array}
$$

$$
\ell_0 : \ \textbf{loop forever do}
$$

$$
P_1 :: \left[\begin{array}{ll}
\ell_1 : & \textbf{noncritical} \\
\ell_2 : & (y_1,\ s) := (\text{T},\ 1) \\
\ell_3 : & \textbf{await } (\neg y_2) \vee (s \neq 1) \\
\ell_4 : & \textbf{critical} \\
\ell_5 : & y_1 := \text{F}
\end{array}\right]
$$

$$||$$

$$
m_0 : \ \textbf{loop forever do}
$$

$$
P_2 :: \left[\begin{array}{ll}
m_1 : & \textbf{noncritical} \\
m_2 : & (y_2,\ s) := (\text{T},\ 2) \\
m_3 : & \textbf{await } (\neg y_1) \vee (s \neq 2) \\
m_4 : & \textbf{critical} \\
m_5 : & y_2 := \text{F}
\end{array}\right]
$$

Example: Program mux-pet1 (Fig. 2.25) (Con'd)

We want to prove mutual exclusion:

$$
\boxed{\Box\, \underbrace{\neg(at\_\ell_4 \ \wedge \ at\_m_4)}_{\psi}}
$$

Bottom-up invariants:

$$
\begin{array}{ll}
\varphi_0\text{:} & s = 1 \ \vee \ s = 2 \\
\varphi_1\text{:} & y_1 \ \leftrightarrow \ at\_\ell_{3..5} \\
\varphi_2\text{:} & y_2 \ \leftrightarrow \ at\_m_{3..5}
\end{array}
$$

Problem: the verification conditions

$$
\begin{array}{l}
\{\varphi_0 \ \wedge \ \varphi_1 \ \wedge \ \varphi_2 \ \wedge \ \psi\} \ \ell_3 \ \{\psi\} \\
\{\varphi_0 \ \wedge \ \varphi_1 \ \wedge \ \varphi_2 \ \wedge \ \psi\} \ \ m_3 \ \{\psi\}
\end{array}
$$

are not state-valid

Example: Program mux-pet1 (Fig. 2.25) (Con'd)

$$pre(\tau_{\ell_3}, \psi):\ \forall \pi':\ \underbrace{move(\ell_3, \ell_4)\ \wedge\ (\neg y_2\ \vee\ s \neq 1)}_{\rho_{\ell_3}}\ \rightarrow\ \underbrace{\neg(at'\_\ell_4\ \wedge\ at'\_m_4)}_{\psi'}$$

since

$move(\ell_3, \ell_4)$ implies $at'\_\ell_4 = \text{T}$, $at'\_m_4 = at\_m_4$

$pre(\tau_{\ell_3}, \psi)$ simplifies to:

$$at\_\ell_3\ \wedge\ (\neg y_2\ \vee\ s \neq 1)\ \rightarrow\ \neg at\_m_4$$

$$\boxed{\varphi_3:\ at\_\ell_3\ \wedge\ at\_m_4\ \rightarrow\ y_2\ \wedge\ s = 1}$$

$pre(\tau_{m_3}, \psi)$: $\forall \pi' \ldots\ldots$

simplifies to:

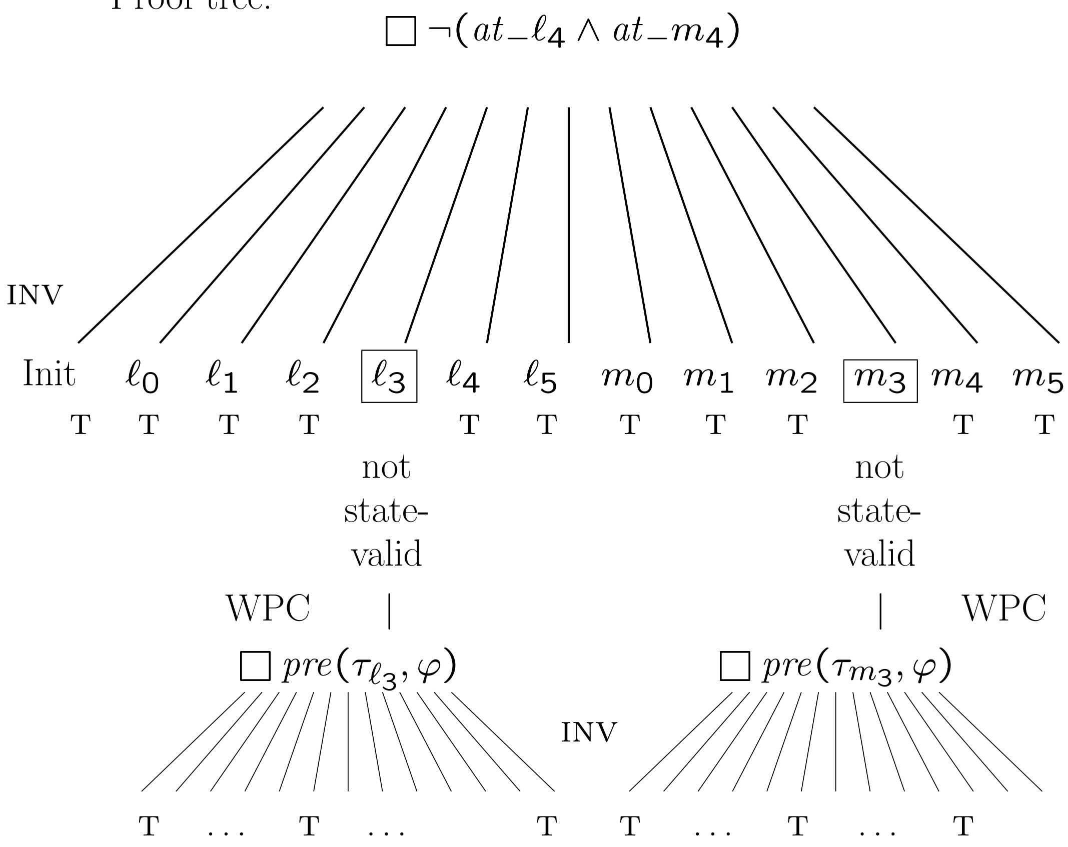
$$\boxed{\varphi_4:\ at\_\ell_4\ \wedge\ at\_m_3\ \rightarrow\ y_1\ \wedge\ s = 2}$$

Show that φ_3: $pre(\tau_{\ell_3}, \psi)$ and φ_4: $pre(\tau_{m_3}, \psi)$ are inductive relative to $\varphi_0 \wedge \varphi_1 \wedge \varphi_2$ ("incremental approach")

Then show that ψ is inductive relative to $\varphi_0 \wedge \ldots \wedge \varphi_4$.

Example: Program mux-pet1 (Fig. 2.25) (Con'd)

Proof tree:



T = state-valid (relative to the bottom-up invariants)

Example: *pre* may never terminate

The transition is

$$\rho_\tau : \; x' = x + y \;\wedge\; y' = y$$

The property is

$$\varphi : \; x \geq 0$$

The VC is

$$\underbrace{x' = x + y \;\wedge\; y' = y}_{\rho_\tau} \;\wedge\; \underbrace{x \geq 0}_{\varphi} \;\rightarrow\; \underbrace{x' \geq 0}_{\varphi'}$$

which is not state valid.

Step 1: The precondition is

$$pre(\tau, x \geq 0) : \forall x', y' \colon x' = x + y \wedge y' = y \;\rightarrow\; x' \geq 0$$

that is $y \geq -x$.

Attempting to prove $\square\, pre(\tau, \varphi)$ state valid, the VC

$$\underbrace{x' = x + y \;\wedge\; y' = y}_{\rho_\tau} \;\wedge\; \underbrace{y \geq -x}_{pre} \;\rightarrow\; \underbrace{y' \geq -x'}_{pre'}$$

is not state-valid.

Step 2: Compute $pre(\tau, \; y \geq -x)$

$$\forall x', y' \colon \; \underbrace{x' = x + y \;\wedge\; y' = y}_{\rho_\tau} \;\rightarrow\; \underbrace{y' \geq -x'}_{pre'}$$

that is $y \geq -\frac{x}{2}$.

In general the precondition

$$pre\left(\tau, \; y \geq -\frac{x}{n}\right) : \; y \geq -\frac{x}{n+1}$$

Taking the limit as n approaches infinity, we obtain

$$y \geq 0$$

which is what we want.

Finite-State Algorithmic Verification

finite-state program P

each $x \in V$ assumes only finitely many values in all P-computations

Therefore,

there are only finitely many distinct P-accessible states.

Example:

MUX-PET1 (Fig 2.25) is finite-state program:

$s = 1, 2$

$y_1 = \text{T}, \text{F} \qquad y_2 = \text{T}, \text{F}$

π can assume at most 36 different values

Example: Program mux-pet1 (Fig. 2.25)

(Peterson's Algorithm for mutual exclusion)

local y_1, y_2: **boolean** **where** $y_1 = \text{F}, y_2 = \text{F}$
s : **integer** **where** $s = 1$

$$P_1 :: \quad \ell_0 : \ \textbf{loop forever do} \begin{bmatrix} \ell_1 : & \textbf{noncritical} \\ \ell_2 : & (y_1, s) := (\text{T}, 1) \\ \ell_3 : & \textbf{await } (\neg y_2) \vee (s \neq 1) \\ \ell_4 : & \textbf{critical} \\ \ell_5 : & y_1 := \text{F} \end{bmatrix}$$

$||$

$$P_2 :: \quad m_0 : \ \textbf{loop forever do} \begin{bmatrix} m_1 : & \textbf{noncritical} \\ m_2 : & (y_2, s) := (\text{T}, 2) \\ m_3 : & \textbf{await } (\neg y_1) \vee (s \neq 2) \\ m_4 : & \textbf{critical} \\ m_5 : & y_2 := \text{F} \end{bmatrix}$$

Algorithm (transition-graph)

For a given finite-state program P
Incrementally construct the state-transition graph G_P, where each node represents a state.

- Initially
  Place as nodes in G_P all initial states (satisfy Θ)

- Repeat until no new nodes or new edges can be added to G_P

  [For some $s \in G_P$, let $s_1, \ldots, s_k$ be its successors
  Add to G_P all new nodes in $\{s_1, \ldots, s_k\}$
  and draw edges connecting s to s_i, $i = 1, \ldots, k$]

Algorithmic Verification of Invariance

For assertion q,
To check validity of $\square\, q$ over finite-state program P:

1. Construct the state-transition graph G_{P}.

2. Check if q holds in each state of the graph.

Example: Program MUX-SEM (Fig 2.26)

Generates finite state-transition graph (Fig 2.27)

Check assertion

$$\varphi: \quad \neg(at\_\ell_3 \ \land \ at\_m_3)$$

in the graph.

φ holds over all accessible states.
Thus, $\square\, \varphi$ for MUX-SEM.

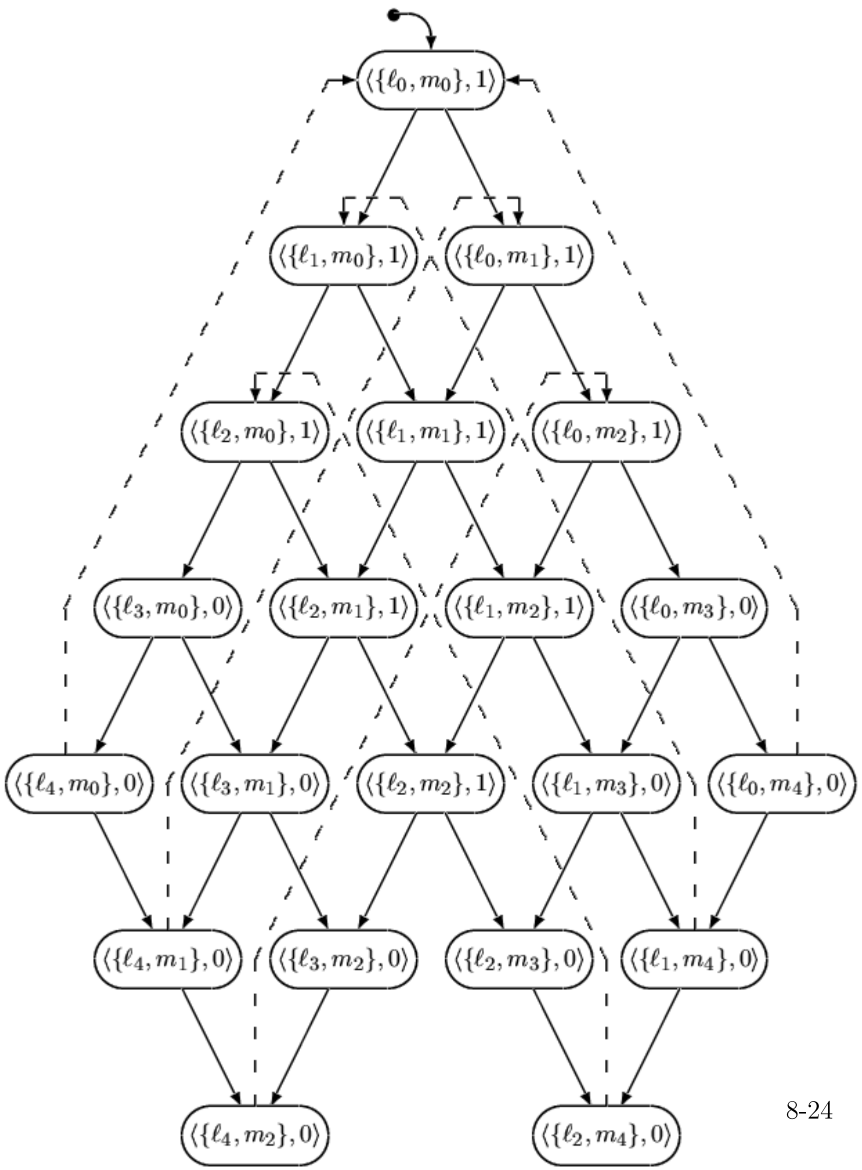
Program MUX-SEM (Fig. 2.26)

(mutual exclusion by semaphores)

local y: **integer where** $y = 1$

$$P_1 :: \left[\begin{array}{l} \ell_0\text{: } \textbf{loop forever do} \\ \quad \left[\begin{array}{l} \ell_1 : \textbf{noncritical} \\ \ell_2 : \textbf{request } y \\ \ell_3 : \textbf{critical} \\ \ell_4 : \textbf{release } y \end{array} \right] \end{array} \right] \; || \; P_2 :: \left[\begin{array}{l} m_0\text{: } \textbf{loop forever do} \\ \quad \left[\begin{array}{l} m_1\text{: } \textbf{noncritical} \\ m_2\text{: } \textbf{request } y \\ m_3\text{: } \textbf{critical} \\ m_4\text{: } \textbf{release } y \end{array} \right] \end{array} \right]$$

Program MUX-SEM state-transition graph (Fig. 2.27)



`Example:` Program MUX-PET1 (Fig 2.25)

State-transition graph G_P (Fig 2.28)

(i, j, v) means $\pi\colon \{\ell_i, m_j\},\ s\colon v$

No y_1, y_2 since

$$y_1 = \text{T} \quad \text{iff} \quad 3 \le i \le 5$$
$$y_2 = \text{T} \quad \text{iff} \quad 3 \le j \le 5$$

Property checked

$$\Box \underbrace{\neg(at\_\ell_4 \ \wedge \ at\_m_4)}_{\psi}$$

Example: Program mux-pet1(Fig. 2.25)

(Peterson's Algorithm for mutual exclusion)

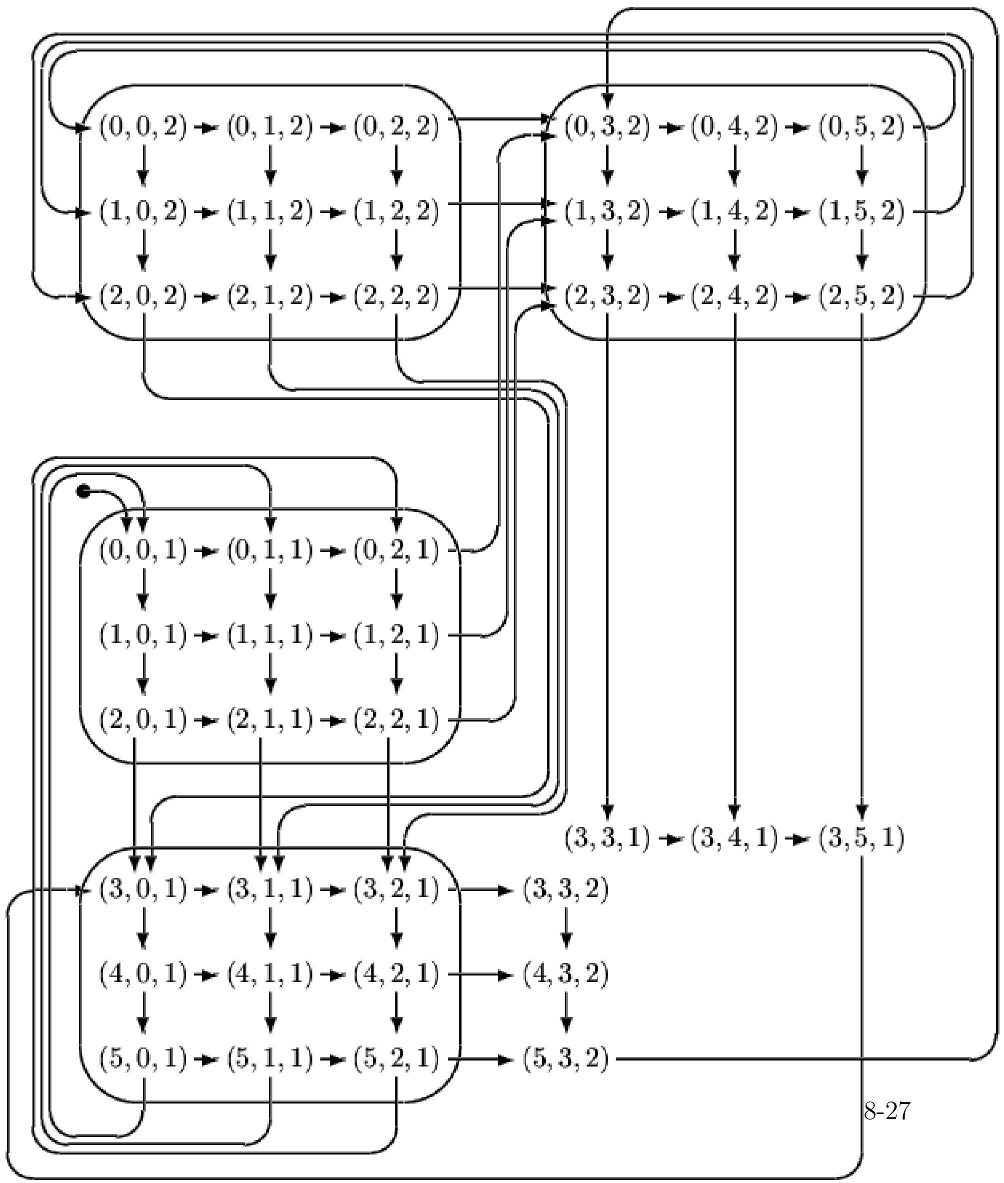
local y_1, y_2: **boolean** **where** $y_1 = \text{F}, y_2 = \text{F}$
s : **integer** **where** $s = 1$

$$P_1 :: \quad \ell_0 : \textbf{loop forever do} \begin{bmatrix} \ell_1 : & \textbf{noncritical} \\ \ell_2 : & (y_1,\ s) := (\text{T},\ 1) \\ \ell_3 : & \textbf{await}\ (\neg y_2) \vee (s \neq 1) \\ \ell_4 : & \textbf{critical} \\ \ell_5 : & y_1 := \text{F} \end{bmatrix}$$

$||$

$$P_2 :: \quad m_0 : \textbf{loop forever do} \begin{bmatrix} m_1 : & \textbf{noncritical} \\ m_2 : & (y_2,\ s) := (\text{T},\ 2) \\ m_3 : & \textbf{await}\ (\neg y_1) \vee (s \neq 2) \\ m_4 : & \textbf{critical} \\ m_5 : & y_2 := \text{F} \end{bmatrix}$$

MUX-PET1 State-transition graph (Fig 2.28)



Completeness of rule INV

Rule INV (general invariance)

For assertions φ, q,

$$
\begin{array}{ll}
\text{I1.} & \Vdash \varphi \rightarrow q \\
\text{I2.} & \Vdash \Theta \rightarrow \varphi \\
\text{I3.} & \Vdash \{\varphi\}\ \mathcal{T}\ \{\varphi\} \\
\hline
 & \models \Box q
\end{array}
$$

Theorem (Relative completeness of rule INV)

For every assertion q such that

$\Box q$ is P-valid

there exists an assertion φ such that I1 – I3 are provable from state validities

We actually show

"completeness relative to first-order reasoning"

taking all state-valid assertions as axioms

Outline of proof

Given FTS P with system variables (program + control variables)

$$\overline{y} = (y_1, \ldots, y_m)$$

- Assume $\Box\, q$ is P-valid, i.e., (†) q holds over every P-accessible state
- Construct (to be shown) accessibility assertion
  $acc_P(\overline{y})$
  such that for any state s,
  (*) s is P-accessible state iff $s \Vdash acc_P$
- Take $\varphi = acc_P$

We have to show :

1. acc_P satisfies I1 – I3
2. acc_P can be "constructed"

1. acc_P satisfies I1 – I3

- Premise I1: $\underbrace{acc_P}_{\varphi} \rightarrow q$

$$s \Vdash acc_P \overset{(*)}{\Rightarrow} s \text{ is } P\text{-accessible state}$$

$$\overset{(\dagger)}{\Rightarrow} s \Vdash q$$

Thus

$$\underbrace{acc_P}_{\varphi} \rightarrow q$$

is state-valid

- Premise I2: $\Theta \rightarrow \underbrace{acc_P}_{\varphi}$

$$s \Vdash \Theta \Rightarrow s \text{ is } P\text{-accessible}$$

$$\overset{(*)}{\Rightarrow} s \Vdash \underbrace{acc_P}_{\varphi}$$

Thus

$$\Theta \rightarrow \underbrace{acc_P}_{\varphi}$$

is state-valid

- Premise I3: for every $\tau \in \mathcal{T}$,

$$\rho_\tau \wedge acc_P \rightarrow acc'_P$$

where $acc'_P = acc_P(\overline{y}')$.

Take s' to be a $\overline{y}$-variant of s (s agrees with s' on all variables other than $\overline{y}$) and for each y_i take

$$s'[y_i] = s[y'_i]$$

Then

$$\left.\begin{array}{l} s \Vdash \rho_\tau \quad \Rightarrow \quad s' \text{ is a } \tau\text{-successor of } s \\ s \Vdash acc_P \quad \overset{(*)}{\Rightarrow} \quad s \text{ is } P\text{-accessible} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad s' \text{ is } P\text{-accessible}$$

$$\overset{(*)}{\Rightarrow} \quad s' \Vdash acc_P$$

$$\Rightarrow \quad s \Vdash acc'_P$$

`Example:`

V: $\{y\}$ Θ: $y = 0$
$\mathcal{T}$: $\{\tau_I, \tau\}$, where ρ_τ: $y' = y + 2$
For this program: $acc_P(y)$: $y \geq 0 \wedge even(y)$

2. Construction of acc_P

Assume assertion language includes

dynamic array $\underline{a}$ over D

Array $\underline{a}$ is viewed as function,

$$a\colon [1..n] \mapsto D$$

where n is the size of the array

The assumption is <u>not essential</u>

We can use Gödel numbering

$$(k_1, \ldots, k_n) \quad \mapsto \quad n = p_1^{k_1} \cdots p_n^{k_n}$$

where p_i is the ith prime number

Case: single-variable y

Define

$$acc_P(y): \quad (\exists n > 0)\ (\exists a \in [1..n] \mapsto D)\ . \\ init \ \wedge\ last\ \wedge\ evolve$$

where

init: $\Theta(a[1])$

last: $a[n] \ = \ y$

evolve: $\forall i\,.\,1 \leq i < n\,.\ \bigvee_{\tau \in \mathcal{T}} \rho_\tau(a[i], a[i{+}1])$

i.e., there exists an array a, such that

- $a[1]$ is an initial state
- $a[n]$ has value y (last element)
- every two consecutive elements are related by some transition relation

array a represents a prefix

$$s_1, \ldots, s_n$$

of a computation where $a[i]$ stands for

the value of y at state s_i

Claim:

For any value $d \in D$,

$acc_P(d) = \mathrm{T}$

iff

d is a possible value of y in a P-accessible state

$acc_P(d)$ asserts the existence of a computation prefix that leads to a state where $y = d$.

`Example:` Transition system EVEN

V: $\{y\}$ ranges over $\mathbb{Z}$ (the integers)	
Θ: $y = 0$	
ρ_τ: $y' = y + 2$	

$acc_P(y)$:

$$(\exists n > 0)(\exists a \in [1..n] \mapsto \mathbb{Z}) \,. \begin{pmatrix} a[1] = 0 \;\wedge\; a[n] = y \;\wedge \\ \forall i \,.\, 1 \le i < n \,.\, a[i+1] = a[i] + 2 \end{pmatrix}$$

simplifies to

$$(\exists n > 0)(\exists a \in [1..n] \mapsto \mathbb{Z}) \,. \begin{pmatrix} a[n] = y \;\wedge \\ \forall i \,.\, 1 \le i \le n \,.\, a[i] = 2 \cdot (i-1) \end{pmatrix}$$

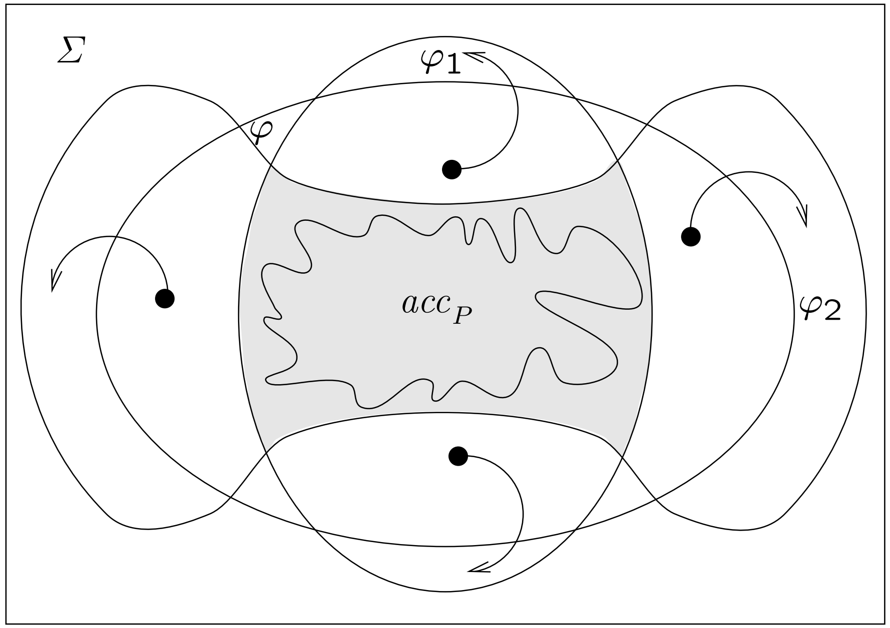
simplifies to

$$\boxed{y \ge 0 \;\wedge\; even(y)}$$

Precisely characterizes the values that y may assume in P-accessible states of EVEN

Discussion

Although the assertion acc_P is inductive and strengthens any P-invariant, it is not very useful in practice.



The shaded area is preserved by all transitions. Its description is much simpler than that of acc_P.

Multivariable $\overline{y} = (y_1, \ldots, y_m)$ case

Use 2-dimensional array a

$\underline{y_1}$		$\underline{y_m}$
$a[1,1]$	. . . .	$a[1,m]$
$a[2,1]$	. . . .	$a[2,m]$
⋮		⋮

Example: Transition system FACT

y,z ranging over $\mathbb{N}$ (the nonnegative integers)

Θ: $y = 1 \;\wedge\; z = 1$

ρ_τ: $y' = y + 1 \;\wedge\; z' = (y+1) \cdot z$

Construction of acc_P:

$(\exists n > 0)(\exists a \in [1..n] \times [1,2] \mapsto \mathbb{N})\,.$

$$\begin{pmatrix} a[1,1] = 1 \;\wedge\; a[1,2] = 1 \;\wedge \\ a[n,1] = y \;\wedge\; a[n,2] = z \\ \bigwedge \\ \forall i\colon\; 1 \leq i < n\colon\; a[i+1,1] = a[i,1] + 1 \;\wedge \\ a[i+1,2] = (a[i,1]+1) \cdot a[i,2] \end{pmatrix}$$

$(\exists n > 0)(\exists a \in [1..n] \times [1,2] \mapsto \mathbb{N})\,.$

$$\left(\begin{array}{c} a[1,1] = 1 \ \wedge \ a[1,2] = 1 \ \wedge \\ a[n,1] = y \ \wedge \ a[n,2] = z \\ \bigwedge \\ \forall i\colon\ 1 \leq i < n\colon\ a[i+1,1] = a[i,1] + 1 \ \wedge \\ a[i+1,2] = (a[i,1]+1) \cdot a[i,2] \end{array}\right)$$

simplifies to

$(\exists n > 0)(\exists a \in [1..n] \times [1,2] \mapsto \mathbb{N})\,.$

$$\left(\begin{array}{c} a[n,1] = y \ \wedge \ a[n,2] = z \\ \wedge \\ \forall i\colon\ 1 \leq i \leq n\colon\ a[i,1] = i \ \wedge \ a[i,2] = i! \end{array}\right)$$

simplifies to

$$\boxed{y \geq 1 \ \wedge \ z = y!}$$

Precisely characterizes the P-accessible states for the transition system FACT