Problem Set 7 CS265, Autumn 2023

Due: Friday December 1 at 11:59PM

Please follow the homework policies on the course website.

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1. **(9 pt.) Fundamental Theorem of Markov Chains: A Special Case**

   Let $X_0, X_1, \ldots$ be a Markov chain over n states (labeled $1, 2, \ldots, n$) with transition matrix $P \in \mathbb{R}^{n\times n}$, i.e., for any $t \geq 0$, $\Pr[X_{t+1} = j | X_t = i] = P_{ij}$. In addition, we assume that $P_{ij} > 0$ for all $i, j \in [n]$, and define $p_{\min} \coloneqq \min_{i,j\in[n]} P_{ij} > 0$. In this problem, we will prove part of the fundamental theorem of Markov chains for this special case. In particular, we will show that there exists a unique stationary distribution π such that for all $i, j \in [n]$,

   $$\lim_{t\to+\infty} \Pr[X_t = j | X_0 = i] = \pi_j.$$

   (a) **(2 pt.)** As a warmup, show that the assumption $P_{ij} > 0$ for all $i, j \in [n]$ implies that the Markov chain is irreducible and aperiodic. Thus, the assumption that we made is not weaker than the one in the original theorem.

   (b) **(2 pt.)** Let $a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ be a row vector that satisfies $\sum_{i=1}^n a_i = 0$. Prove that $\|aP\|_1 \leq (1 - np_{\min}/2)\|a\|_1$.

   [**HINT:** *You can use the following fact: For vectors* $a, b \in \mathbb{R}^n$ *satisfying* $\sum_{i=1}^n a_i = 0$ *and* $\min_{i\in[n]} b_i \geq \epsilon > 0$, $|\sum_{i=1}^n a_i b_i| \leq \sum_{i=1}^n |a_i| b_i - \frac{\epsilon}{2} \sum_{i=1}^n |a_i|$.]

   (c) **(3 pt.)** Prove that there exists an n-dimensional row vector $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}$ such that: (1) $\pi = \pi P$; (2) $\sum_{i=1}^n \pi_i = 1$.

   [**HINT:** *First prove the existence of a non-zero vector* π *satisfying* $\pi = \pi P$, *and then show that the second condition can be satisfied by scaling* π. *For the first step, you may use the following fact without proof: if* λ *is an eigenvalue of a square matrix* A, λ *is also an eigenvalue of* A^T. *Part 1b might be helpful for the second step.*]

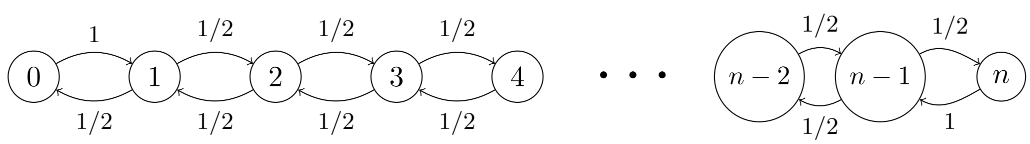
   (d) **(2 pt.)** Let $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ be a row vector that satisfies $\sum_{i=1}^n v_i = 1$. Let π be a vector chosen as in Part 1c. Prove that $\lim_{t\to+\infty} vP^t = \pi$. Then, derive that for all $i, j \in [n]$,

   $$\lim_{t\to+\infty} \Pr[X_t = j | X_0 = i] = \pi_j.$$

   [**HINT:** *Apply Part 1b to* $(v - \pi), (v - \pi)P, (v - \pi)P^2, \ldots$.]

   (e) **(0 pt.)** **[Optional: this won't be graded.]** Extend the proof to the general case, where the Markov chain is irreducible and aperiodic but $P_{ij} > 0$ might not hold.

2. **(11 pt.)** Let $n > 2$, and consider the Markov chain $\{X_t\}$ defined on the states $\{0, 1, \ldots, n\}$ consisting of a random walk with reflecting barriers at 0 and n:



That is, $\{X_t\}$ is defined by the following transition probabilities:

- For $i \in \{1, \ldots, n-1\}$, we have

$$\Pr[X_t = i+1 | X_{t-1} = i] = \Pr[X_t = i-1 | X_{t-1} = i] = \frac{1}{2}.$$

- At 0 and n, we have reflecting barriers:

$$\Pr[X_t = 1 | X_{t-1} = 0] = \Pr[X_t = n-1 | X_{t-1} = n] = 1.$$

(a) **(2 pt.)** Is this chain periodic or aperiodic? Is it irreducible? Justify your answers in one sentence each.

(b) **(5 pt.)** Consider the "lazy" version of $\{X_t\}$ that, at every timestep, flips a fair coin and with probability 1/2 stays in its current state, and with probability 1/2 transitions as prescribed above. Call this lazy version $\{\tilde{X}_t\}$. Define a coupling for $\tilde{X}_t$ that ensures that the two chains in your coupling "never cross without meeting." That is, if you are coupling $\{\tilde{X}_t\}$ and $\{\tilde{Y}_t\}$, you should ensure that if $\tilde{X}_0 \leq \tilde{Y}_0$, then it will hold that $\tilde{X}_t \leq \tilde{Y}_t$ for all t.

(c) **(4 pt.)** Show that $\{\tilde{X}_t\}$ has a unique stationary distribution, and that the mixing time of $\{\tilde{X}_t\}$ is bounded by $O(n^2)$.

[**HINT:** *To bound the mixing time, use the coupling you defined in part (b).*]

[**HINT:** *Recall Lemma 6 from Class 13, which says that if Z_t is a walk on $\{0, 1, 2, \ldots\}$ with a reflecting barrier at 0 (so $\Pr[Z_t = 1 | Z_{t-1} = 0] = 1$, and otherwise $Z_t = Z_{t-1} \pm 1$ with probability 1/2 each), then the expected amount of time before $Z_t = n$, given that $Z_0 \leq n$, is at most n^2.*]