Class 1: Agenda and in-class Questions

1 Welcome!
Welcome to CS265/CME309!

1.1 Introductions and Course Logistics
- Course staff / student introductions
- Short presentation on course logistics

2 Polynomial Identity Testing

2.1 Group work
- Find a small group to work with.
- If you have questions during group work, please flag one of the teaching team down!

Group Work
Important: as you make progress on the question(s), one person in each room should record your progress on http://PollEv.com/cs265.

(10-20 minutes, depending on how fast folks go through these)

1. First, introduce yourselves to each other. What year/program are you in? What class/activity/etc are you most excited about for this quarter?

2. Quietly read the following definition.

A multivariate polynomial $f(x_1, \ldots, x_m)$ is \textit{identically zero} if all its coefficients are zero. For example, the polynomial $f(x_1, x_2) = (x_1 + 1)(x_2 + 1) - 1 - x_1 x_2 - (x_1 + x_2)$ is identically zero because when you expand it out, all of the terms cancel.

Now, work on the following questions with your group.

3. Which of the following two polynomials are identically zero?

\[
f(x, y) = (x - y)^2 + 2xy + (x + y)^3 - y(3x^2 + y(3x + y + 1)) - (x + 1)x^2
\]
4. Suppose I were to give you a polynomial $f(x_1, x_2, \ldots, x_n)$ of total degree$^a$ $n$ and with $n$ variables and ask you if it is identically zero or not. How long, asymptotically, would it take you in the worst case if you were to do this in the straightforward way, by expanding out every term?

(a) $O(n)$

(b) polynomial in $n$

(c) $2^\Omega(n)$

5. At the end of this group work, we are going to challenge you with two degree-8, 8-variate polynomials $f$ and $g$, and ask you which is identically zero. You’re going to have one minute to answer. Think now about an efficient way to answer this challenge.

As part of your strategy, you may use a basic calculator (eg, https://www.google.com/search?q=calculator), but dumping the expression into WolframAlpha for it to plot or simplify (or something like that) is cheating.

**Hint:** Remember that this is a class on randomized algorithms. Can you think of a randomized strategy?

**Hint:** You might take inspiration from the univariate case. Here’s the graph of a polynomial that is not identically zero. What is true about the values $g(x)$ for most choices of $x$?

6. Would your strategy still work if Mary and Greg know it ahead of time? That is, if we know your strategy — but not necessarily the outcome of any randomness in the strategy — could we come up with polynomials $f, g$ that would foil your strategy? If your strategy would fail if we know it ahead of time, try to come up with another strategy that would (probably) succeed!

7. If you’ve finished all of the above, try to come up with an efficient deterministic strategy that would succeed at this task.

$^a$the total degree of a monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is $\sum_{i=1}^n a_i$. 

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Group Work: Solutions

4. \( f \) is identically zero, and \( g \) is not.

5. Naively, this takes time \( 2^{\Omega(n)} \). That’s because there are \( \binom{2n}{n} = 2^{\Omega(n)} \) possible monomials, so I need to compute at least that many coefficients to check.

6. There are lots of strategies that would work! Some good ideas include:
   - Plug in 10000 (or some other big number) for all of the variables and see if you get zero. Since nonzero polynomials go to \( \pm \infty \) as the inputs \( x_i \to \infty \), and since Mary and Greg surely wouldn’t choose a polynomial with super tiny coefficients, \( f(10000, 10000, 10000, \ldots, 10000) \) will probably be non-zero if \( f \) is.
   - Compute just the constant term, or the coefficient on \( x_1^n \). Both of those are pretty easy to compute just by looking at the function. If they’re not zero, then say the polynomial is not zero. Again, if Mary and Greg don’t know that you are going to be looking at the constant term (or whatever) if they choose a non-zero polynomial it probably won’t have a zero constant term.
   - Choose a random evaluation point \( (Z_1, Z_2, \ldots, Z_8) \) (say, according to a Gaussian). If \( f(Z_1, \ldots, Z_8) \) is not zero, say that \( f \) is not identically zero. Otherwise, guess that \( f \) is identically zero.

   How likely is this to be correct? If \( f \) is identically zero, we will always be correct. If \( f \) is not identically zero, then we claim that we’ll find a nonzero value of it with probability 1. Intuitively, the set of zeros of any degree-8 polynomial is a very small set—it has measure zero in \( \mathbb{R}^8 \)—so no matter what nonzero polynomial is chosen, we will almost always hit a nonzero value.

   All of these are good ideas (and there are more good ideas!) However, the first two are not robust if I know them ahead of time—I could cook up some “bad” examples for them. The last one is robust!

8. This is actually an open question! (Sorry :) )

2.2 Challenge and Discussion!

- Challenge time!
- A bit of lecture on Polynomial Identity Testing

3 What is a randomized algorithm?

- A bit of lecture.
4 Wrap Up

Before next time:

- Watch the two short videos for Class 2 on YouTube (link on course website).
- Do the associated short quiz and check your answers.