CS265, Fall 2020

Class 10: Agenda, Questions, and Links

1 Warm-Up

Go to http://PollEv.com/cs265 and answer the following questions.

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on http://PollEv.com/cs265.

1. Show that, in any undirected, unweighted graph $G = (V, E)$ with no self-loops, there is a cut with at least $|E|/2$ edges that cross it. (Recall that a *cut* in G is just a partition of the vertices $V = S \cup \bar{S}$, and that an edge $\{u, v\}$ crosses the cut if $u \in S$ and $v \in \bar{S}$ or the other way around).
2. Let φ be a 3-CNF formula. That is, φ is the AND of a bunch of clauses that look like $(x \vee y \vee z)$ (or $(x \vee \bar{y} \vee \bar{z})$, or ..., where $\bar{x}$ means "not x"). For example, maybe

$$\varphi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_4} \vee x_5) \wedge \cdots \wedge (x_{23} \vee \overline{x_1} \vee \overline{x_5}).$$

   Given an *assignment* σ to the variables $x_1, x_2, \ldots$ (eg, $x_1 = TRUE, x_2 = FALSE$, etc), we say that a clause of φ is *satisfied* by σ if that clause evaluates to TRUE.

   Show that any 3-CNF formula φ has an assignment σ so that at least 7/8 of the clauses are satisfied.

Once you are done with those, **go to PollEverywhere and say what you did!** (and/or upvote others' answers). If you have time, think about how you would find such assignments *efficiently.*

2 Announcements

- HW4 due Friday! HW5 out Friday.

3 Questions?

Any questions from the minilectures and/or the quiz? (The probabilistic method; Ramsey numbers; Independent sets)

- Go into small groups and ask each other your questions.

- Go to https://pollev.com/cs265 and ask your questions/comments there, or else upvote others' questions.

4 Derandomization via conditional expectation

In class today, we'll explore a general way to turn an existence proof—like the ones from your warm-up exercise—into an algorithm. This is called "Derandomization via conditional expectation."

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on http://PollEv.com/cs265.

Our goal in this group work is to find an efficient, deterministic algorithm to find a cut $(S, \bar{S})$ so that the number of edges crossing the cut is at least $|E|/2$. In general, finding a cut with the *maximum* number of edges crossing it is NP-hard; but this will at least find a large-ish cut.

Note: There is a straightforward deterministic greedy algorithm to do this. Here, we'll see a way to derive a deterministic algorithm using conditional expectations.

1. Let $G = (V, E)$ be as in warm-up question 1. Suppose the vertices are ordered $V = \{v_1, v_2, \ldots, v_n\}$.

   Suppose that $S \subseteq V$ is chosen uniformly at random (that is, each v_i is included in S indepedently with probability 1/2). Let X be the number of edges crossing the cut $(S, \bar{S})$.

   Convince yourself that $\mathbb{E}[X | v_1 \in S] = |E|/2$.

2. Suppose that you have made some choices for $v_1, v_2, \ldots, v_{t-1}$ (eg, $v_1 \in S, v_2 \notin S, v_3 \in S, \ldots, v_{t-1} \in S$), so that

$$\mathbb{E}[X | \text{ choices for } v_1, \ldots, v_{t-1} \ ] \geq \frac{|E|}{2}.$$

   Show that **either**

$$\mathbb{E}[X \mid \text{choices for } v_1, \ldots, v_{t-1}; \text{ and } v_t \in S] \geq \frac{|E|}{2}$$

   or

$$\mathbb{E}[X \mid \text{choices for } v_1, \ldots, v_{t-1}; \text{ and } v_t \notin S] \geq \frac{|E|}{2}$$

   At this point, please record your progress on PollEverywhere.

3. Again, suppose that you have made choices for $v_1, \ldots, v_{t-1}$ so that

$$\mathbb{E}[X| \text{ choices for } v_1, \ldots, v_{t-1} \ ] \geq \frac{|E|}{2}.$$

Show how to *deterministically, efficiently* make a choice for v_t so that

$$\mathbb{E}[X| \text{ choices for } v_1, \ldots, v_{t-1}; \text{ and } v_t \ ] \geq \frac{|E|}{2}.$$

4. Building on your method above, design an algorithm to make a choice for v_1, and then v_2, and then v_3, and so on, so that eventually you have (efficiently, deterministically) found a set S so that at least $|E|/2$ edges cross the cut $(S, \bar{S})$.

   At this point, please record your progress on PollEverywhere.

[Solutions and discussion of the general paradigm]

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on `http://PollEv.com/cs265`.

1. Let φ be a 3-CNF formula with n variables and m clauses. Give an efficient (polynomial in n, m) deterministic algorithm to find an assignment to φ so that at least a 7/8-fraction of the clauses are satisfied.

   Hint: Use the method you developed in the previous group work, and consider the second warm-up question.

   At this point, please fill out the PollEverywhere to record your progress.