CS265, Fall 2023

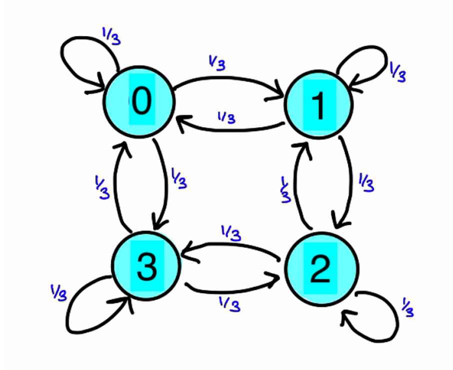
Class 13: Agenda and Questions

1 Questions/Lecture Recap

Any questions from the minilectures and/or the quiz? (Markov chains and a randomized algorithm for 2SAT)

2 Spectral Analysis of Markov Chains

Consider the Markov chain given by:



Here's a quick warm-up (we may do this together):

Group Work

1. What is the transition matrix for this Markov chain?
2. Suppose that you start in state 0. What is the probability that you are in state 2 after one step? Two steps? Three steps? 100 steps? (Don't actually compute this, just say how you would).
3. As $t \to \infty$, what do you think is $\lim_{t\to\infty} \Pr[X_t = 2|X_0 = 0]$?

Next, we'll see how we can use linear algebra to help us out in computing things like $\Pr[X_t = 2|X_0 = 0]$ for general t. We'll focus on this particular example, but as we go, keep in mind what you think the general principle should be.

Group Work

1. Let

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

   where $i = \sqrt{-1}$. (You may recognize F as the 4×4 discrete Fourier matrix, so $F_{jk} = \frac{1}{2}e^{-2\pi ijk/4}$.) Notice that F is a Hermitian matrix, which means that $F^*F = FF^* = I$, where F^* denotes the Hermitian conjugate (e.g., take the transpose and change all of the i's to $-i$'s).

   Convince yourself that

$$P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$$

   ***Hint**: Check that the columns of F are eigenvectors for P.*

   ***Note:** If your linear algebra is rusty and you trust me, just remind yourself what an eigenvector actually is. The main point here is that you should understand this so that you can use it in the next part.*

2. Given the previous part, for the Markov chain defined at the top, how would you figure out the probability of being in state 2 at time 100, if you started at state 0? (This time, use the previous part to get an easier-to-compute-with expression.) Come up with a statement like

$$\Pr[X_t = 2 | X_0 = 0] = \frac{1}{4} \pm O(\_\_\_)$$

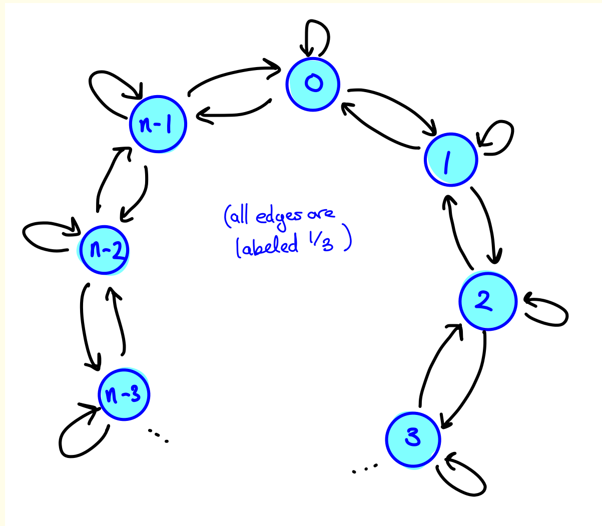
   where the thing in the $O()$ term depends on t. What is the best bound you can get?

Before we move on to larger cycles, let's take a minute to reflect on what just went on. [A bit of lecture about spectral analysis. The point is that if we have a symmetric Markov chain, we can always write the transition matrix as $P = VDV^*$ for a Hermitian matrix V and a diagonal matrix D with real values on the diagonals. Then we can write $P^t = VD^tV^*$, and as long as the second-largest eigenvalue is strictly less than 1, eventually D^t will look like $\mathrm{diag}(1, \mathrm{tiny}, \mathrm{tiny}, \ldots, \mathrm{tiny})$. This means that we can compute transition probabilities after t steps up to very small error terms.]

In this next part, you'll generalize what you saw above to larger cycles.

Group Work

1. Consider the analogous Markov chain to the 4-state one that you saw before, except that it has n states. That is, it looks like this:



   Let $P \in \mathbb{R}^{n \times n}$ be the transition matrix for this Markov chain. Here is a **fact:**

$$P = F_n D F_n^*,$$

   where D is a diagonal matrix whose j'th entry is

$$D_{j,j} = \frac{1 + 2\cos(2\pi j/n)}{3},$$

   where $j = 0, \ldots, n-1$. (Importantly, j is zero-indexed here!) Above, F_n is the $n \times n$ DFT, so

$$(F_n)_{j,k} = \frac{1}{\sqrt{n}} e^{-2\pi ijk/n}.$$

   (There is no question here, just acknowledge it.)

   Note: *As before, you can work this out for yourself if you feel like. As a hint, check that the columns of F are eigenvectors of P with the appropriate eigenvalues. You may find it helpful that $2\cos(x) = e^x + e^{-x}$.*

2. Come up with an expression for $\Pr[X_t = 0 | X_0 = 0]$. You should get a kind of nasty sum involving some cosines, but it shouldn't be *too* nasty.

3. Convince yourself that as $t \to \infty$, $\Pr[X_t = 0 | X_0 = 0] \to 1/n$.

4. Try to think about how *fast* this convergence is. That is, how large does t have to be before $\Pr[X_t = 0 | X_0 = 0] = \frac{1+o(1)}{n}$? (Don't try to come up with a formal proof, just some back-of-the-envelope calculations).

Also, how does this compare to what we saw in the mini-lectures about the walk on the line?

Hint: *You may find the Taylor expansion* $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$ *of* $\cos(x)$ *about zero helpful. In particular, when* x *is small,* $\cos(x) \approx 1 - \frac{x^2}{2}$*. You may also want to use the approximation* $1 - x \approx e^{-x}$ *for small* x *liberally.*