CS265, Fall 2020

Class 15: Agenda, Questions, and Links

1 Announcements

- HW7 is out...and you have TWO WEEKS to do it!!
- Wednesday (Nov 4) is a "buffer" day, which means that we'll just use it for catch-up and questions. This means no videos/quiz for Wednesday Nov 4.
- Please fill out this feedback form! https://forms.gle/uLuQ3ZDxbcyQzpKo6
- PSA: If you are eligable to vote (and haven't voted already), make plans to vote! Hopefully having nothing due on Wednesday 11/4 will make it easier for you to vote by 11/3.

2 Questions?

Any questions from the minilectures and/or the quiz and/or the warm-up? (Coupling, Mixing times)

- Go into small groups and ask each other your questions.
- Go to https://pollev.com/cs265 and ask your questions/comments there, or else upvote others' questions.

3 Shuffling I

In this exercise we'll practice coming up with a coupling.

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on http://PollEv.com/cs265.
Consider the following Markov Chain for shuffling a deck of n cards:

> At each timestep, choose a uniformly random card and move it to the top of the deck. (If you choose the top card, don't do anything).

Let X_t denote the state of the deck after t steps.

1. Convince yourself that this Markov chain is irreducible and aperiodic. What is the stationary distribution of $X_0, X_1, \ldots$?

2. We are going to bound the mixing time of this Markov chain using *couplings.* Come up with a coupling on this Markov chain that you think will "couple" quickly.

   Hint: *You might want to take inspiration from the graph-coloring example we saw, where we tried to make the same choice in both chains.*

3. For the coupling that you came up with, how long is it likely to take for the two chains to couple?

   Hint: *Assuming you came up with the coupling that I think you did, it might be helpful to remember the* coupon collector's problem. *In particular, if you are trying to collect n coupons, the probability that you need more than $2n \log n$ tries to collect them all is $o(1)$.*

4. Come up with a bound on τ_{mix} using your coupling. Can you show that $\tau_{mix} = O(n \log n)$? (You may assume that n is sufficiently large).

   Hint: *Remember that $\Delta(t) \leq \max_{s,s'} \Pr[T_{s,s'} \geq t]$, and use your answer to the previous part.*

   At this point, please fill out the pollEverywhere!

4 Shuffling II

In this exercise, we'll see a different way to bound mixing times, other than coupling.

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on `http://PollEv.com/cs265`.

Consider this **different shuffling scheme** for shuffling a deck of n cards:

> At each timestep, choose the top card and move it to a uniformly random position in the deck. (So with probability $1/n$, you move it to its own position and don't do anything).

Let X_t denote the state of the deck after t steps.

1. Convince yourself that this chain is aperiodic and irreducible, and that the stationary distribution is uniform.

2. Let T be the first time at which the original bottom card of the deck is placed randomly somewhere. (That is, if the deck starts out with the ace of spades on the bottom, then time $T - 1$ is the first time that the ace of spades is on the top).

   Argue that, at any time $t \geq T$, the deck is completely uniform. (That is, the Markov chain has converged exactly to its stationary distribution).

3. What is $\mathbb{E}[T]$?

Hint: *Assuming that the ace of spades is originally on the bottom, write*

$$\begin{aligned} T = &\ \textit{time it takes for the ace of spades to move to the second-from-bottom position} \\ &+ \textit{time it takes for the ace of spades to move to the third-from-bottom position} \\ &+ \cdots \\ &+ \textit{time it takes for the ace of spades to move from the 2nd to the top position} \\ &+ 1, \end{aligned}$$

and use linearity of expectation.

At this point, please fill out the pollEverywhere!

4. Notice that Markov's inequality implies that

$$\Pr[T \geq 2e\mathbb{E}[T]] \leq 1/(2e).$$

Explain why this implies that the mixing time of $X_0, X_1, \ldots$ is at most $2e\mathbb{E}[T]$.

Hint: *Write* $P_s^t = \Pr[t \geq T] \cdot \pi + \Pr[t < T] \cdot \sigma$*, where* π *is the uniform (stationary) distribution and* σ *is some other distribution (the distribution of* P_s^t *conditioned on* $t < T$*). Then use that expression in the definition* $\Delta(t) = \|\pi - P_s^t\|$*.*