Class 17

Martingale Stopping Theorem; Wald’s equation
Announcements

• HW7 due tomorrow!
• HW8 out now!
• You are now all done with quizzes!!!
• FINAL EXAM will be Thursday Dec 15, 12:15-3:15 in 420-040.
  • We’ll release a practice exam with our cheat sheet soon.
• Check out Ed for an addendum to my flailing at the end of Tuesday’s class.

• Plan for week 10:
  • Tuesday: Fun “bonus” material on pseudorandomness. (No quiz, will not be on HW or exam).
  • Thursday: The research frontier! (At least 2 short research talks).
Recap!

- Stopping times
- Martingale Stopping theorem
- Applications
Stopping times

• $T$ is a **stopping time** for $\{X_t\}$ if the event that $T = i$ is mutually independent of all the random variables $X_j \mid X_0, \ldots, X_i$, for all $j > i$.
  
  • Informally, you should be able to tell that $T$ has occurred at time $T$, without looking into the future.
  
  • Example: The first time $X_t$ hits 100.
  
  • Non-example: The last time $X_t$ hits 100.
Martingale Stopping Theorem

- Let \( \{Z_t\} \) be a martingale w.r.t. \( \{X_t\} \).
- Let \( T \) be a stopping time for \( \{X_t\} \).
- Then \( \mathbb{E}[Z_T] = \mathbb{E}[Z_0] \) if at least one of the following hold:
  - There is a \( c \) so that \( |Z_i| < c \) for all \( i \)
  - There is a \( c \) so that \( T < c \) with probability 1
  - There is a \( c \) so that \( \mathbb{E}[|Z_{i+1} - Z_i| \mid X_0, \ldots, X_i] < c \) for all \( i \), and \( \mathbb{E}[T] < \infty \)
Why do we care?

• Hitting time of random walks!

• Set up a martingale $Z_t$ so that $E[Z_T]$ has something to do with something you care about.
  - E.g., $E[Z_T] = Pr[Z_T = b] \cdot b - Pr[Z_T = a] \cdot a$

• You know what $E[Z_0]$ is.
  - E.g., $E[Z_0] = 0$.

• Solve FTW.
  - E.g., $Pr[Z_T = b] = \frac{a}{a+b}$
Questions?
Stopping times, martingale stopping thm, quiz?
Q1. Is it a stopping time?

• $Z_t$ is the sum of $t$ rolls of a fair six-sided die.

• $T_1 = \min\{t: Z_t \geq 12\}$

• $T_2 = \min\{t: Z_{t+2} \geq 12\}$

• $T_3 = \min\{t: Z_t - \frac{7t}{2} \geq 12\}$
Q2.

• Same $Z_t$ as before.
• $Y_t = Z_t - \frac{7t}{2}$

• To which of the stopping times does the MST apply?
  • $T_1$ and the $Y_t$’s: Yes, since $T_1 < 13$ with probability 1.
  • $T_1$ and the $Z_t$’s: No, since the $Z_t$’s are not a martingale.
  • $T_3$ and the $Y_t$’s: No, since the conditions of the MST aren’t met.
    • Intuitively, $Y_t$ can wander off to $-\infty$ without ever hitting 12.
Q3. Hitting times

- $Q_0 = 0$
- $Q_t = \begin{cases} Q_{t-1} + 3 & \Pr \frac{1}{2} \\ Q_{t-1} - 3 & \Pr \frac{1}{2} \end{cases}$

- $\mathbb{E}[\min\{t : |Q_t| = 30\}] = 100$
Plan for today

• Wald’s Equation
• (If time) Ballot counting thm
Wald’s Equation

**Theorem.**
- Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, $X_i \sim X$.
- Let $T$ be a stopping time for $\{X_i\}$.
- Suppose that $\mathbb{E}[T], \mathbb{E}[X] < \infty$.
- Then
\[
\mathbb{E} \left[ \sum_{i=1}^{T} X_i \right] = \mathbb{E}[T] \cdot \mathbb{E}[X]
\]
Group Work

1. Find an example where Thm 1 fails if the hypotheses aren’t met.
   • Try to violate as few of the hypothesis as you can!
2. Let \( Z_i = \sum_{j=1}^{i} (X_j - E[X]) \). Prove that \( \{Z_t\} \) is a martingale wrt \( \{X_t\} \).
3. Show that the martingale stopping thm applies to \( \{Z_t\} \) and \( T \).
4. Use the martingale stopping thm to prove Wald’s equation.
5. Consider rolling a fair six-sided die repeatedly. Let \( X \) be the the sum of all of the rolls up until the first ”6” is rolled. (Not including that first “6”). What is \( E[X] \)?
1. Examples where Wald’s eqn doesn’t hold
1. An example where the conclusion does not hold.

- Let $X_1, X_2, \ldots$, be i.i.d. $\{0,1\}$ random variables, mean $1/2$.
  - Actually we only need $X_1$
- Let $T$ be $1 - X_1$. (Note that $T$ is not a stopping time!)

- Then $\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = 0$.
  - If $X_1 = 1$ then we sum zero things
  - If $X_1 = 0$ then we sum one thing which is equal to 0
- But $\mathbb{E}[T] \cdot \mathbb{E}[X] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
2. $Z_i$ is a martingale

- $\mathbb{E}[Z_i|X_1,\ldots,X_{i-1}] = Z_{i-1}$

- $\mathbb{E}[|Z_i|] < \infty$

Let $Z_i = \sum_{j=i}^{\infty} (X_j - \mathbb{E}[X])$
3. The Martingale Stopping Thm applies

**Theorem 1** (Martingale Stopping Theorem). Letting \( \{Z_i\} \) denote a martingale with respect to \( \{X_t\} \), and \( T \) a stopping time for \( \{X_t\} \), then \( \mathbb{E}[Z_T] = \mathbb{E}[Z_0] \) if at least one of the following conditions hold:

1. If there exists a constant \( c \) such that for all \( i \), \( |Z_i| < c \).
2. If there exists a constant \( c \) such that with probability 1, \( T < c \).
3. If \( \mathbb{E}[T] < \infty \), and there exists a constant \( c \) such that for all \( i \), \( \mathbb{E}[|Z_{i+1} - Z_i| | X_0, \ldots, X_i] < c \).
3. The Martingale Stopping Thm applies

• Apply the third condition:

\[ \mathbb{E}[T] < \infty \quad \text{(by assm)} \]

\[ \mathbb{E} \left[ \left| Z_{i+1} - Z_i \right| \mid X_{i}, \ldots, X_i \right] = \mathbb{E} \left[ X_{i+1} - \mathbb{E}[X_{i+1}] \right] \leq 2 \mathbb{E}[X] \text{ is bounded. (by assm)} \]
4. Apply the martingale stopping theorem
4. Apply the martingale stopping theorem

\[ \mathbb{E} [Z_T] = \mathbb{E} [Z_1] = 0. \]

\[
0 = \mathbb{E} [Z_T] = \mathbb{E} \left[ \sum_{j=1}^{T} (X_j - \mathbb{E}[X]) \right] \\
= \mathbb{E} \left[ (\sum_{j=1}^{T} X_j) - T \cdot \mathbb{E}[X] \right] \\
= \mathbb{E} \left[ \sum_{j=1}^{T} X_j \right] - \mathbb{E}[T] \cdot \mathbb{E}[X] \\
\Rightarrow \mathbb{E} \left[ \sum_{j=1}^{T} X_j \right] = \mathbb{E}[T] \cdot \mathbb{E}[X]. \]
5. Application of Wald’s equation

- $X = \text{sum of die-rolls up until you get a six. (Not including that six).}$

- $E[X] =$
Ballot counting

• Election with two candidates, A and B, and n voters.
• A will win, receiving $N_A > N_B$ votes. (so $N_A + N_B = n$).
• Ballots are counted in a random order.
• What is the probability that A remains ahead the whole time?
Ballot counting

• Let $A_t$ be number of votes for A at time $t$
• Let $B_t$ be number of votes for B at time $t$
• Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$

• Election with two candidates, A and B, and $n$ voters.
• A will win, receiving $N_A > N_B$ votes. (so $N_A + N_B = n$).
• Ballots are counted in a random order.
• What is the probability that A remains ahead the whole time?
Group Work

1. Let $T$ be smallest $t$ so that $Z_t = 0$; if this never occurs, $T = n - 1$. Explain why MST applies to $\{Z_t\}$ and $T$.
   - Assume for now that $\{Z_t\}$ is a martingale.

2. Apply MST to $\{Z_t\}$ and $T$, and use it to compute the probability that $A$ was ahead the whole time.

3. Show that $\{Z_t\}$ is indeed a martingale.
1. Let $T$ be smallest $t$ so that $Z_t = 0$; if this never occurs, $T = n - 1$. Explain why MST applies to $\{Z_t\}$ and $T$.

**Theorem 1** (Martingale Stopping Theorem). Letting $\{Z_t\}$ denote a martingale with respect to $\{X_t\}$, and $T$ a stopping time for $\{X_t\}$, then $E[Z_T] = E[Z_0]$ if at least one of the following conditions hold:

1. If there exists a constant $c$ such that for all $i$, $|Z_i| < c$.
2. If there exists a constant $c$ such that with probability 1, $T < c$.
3. If $E[T] < \infty$, and there exists a constant $c$ such that for all $i$, $E[|Z_{i+1} - Z_i| | X_0, \ldots, X_i] < c$. 
2. Apply MST to compute the prob. A was ahead the whole time.

- $E[Z_T] = E[Z_0] = \frac{A_{n-t} - B_{n-t}}{n - t}$

- OTOH, $E[Z_T] =$
3. Show $Z_t$ is indeed a martingale

- Start two piles of ballots. Take one from a random pile at each step.
Recap

• The Martingale Stopping Theorem is useful!