Class 17: Agenda and Questions

1 Announcements

- HW7 due tomorrow.
- HW8 (last one!!!) out now.
- You are all done with quizzes!
- Final exam is Th. Dec. 15, 12:15-3:15pm, in 420-040.
- Practice exam released soon.
- Plan for Week 10:
  - Tuesday: Fun day on pseudorandomness (no quiz, not on HW or exam)
  - Thursday: The research frontier! ($\geq 2$ short research talks)

2 Questions?

Any questions from the minilectures and/or the quiz? (Stopping times, Martingale stopping theorem)

3 Wald’s equation

In this exercise we’ll get some practice applying the martingale stopping theorem, to prove Wald’s equation.

**Theorem 1** (Wald’s equation). Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, distributed according to some random variable $X$. Let $T$ be a stopping time for \{X_i\}. If $E[X]$ and $E[T]$ are both bounded, then

$$E \left[ \sum_{i=1}^{T} X_i \right] = E[T] \cdot E[X]. \quad (1)$$

**Group Work**

1. Wald’s equation hopefully seems pretty intuitive. But there is something to prove! Come up with an example of some random variables $X_i$ and $T$ that don’t obey the hypotheses of Theorem 1, so that the (1) does not hold.
2. Let $Z_i = \sum_{j=1}^{i}(X_j - \mathbb{E}[X])$. Prove that $\{Z_i\}$ is a martingale with respect to $\{X_i\}$.

3. Argue that the martingale stopping theorem applies to $\{Z_i\}$ and $T$, where $X, T$ are as in Theorem 1.

4. Use the Martingale stopping theorem to prove Wald’s equation.

5. Consider rolling a fair, six-sided die repeatedly. Let $X$ be the sum of all of the rolls up until the first “6” is rolled, not including that 6. What is $\mathbb{E}X$?

**Group Work: Solutions**

1. There are many examples, but here’s a simple one. Let $X_1 = 0$ with probability 1/2 and 1 with probability 1/2. Let $T = 1 - X_1$. That is, if $X_1 = 0$, then $T = 1$, and if $X_1 = 1$, then $T = 0$. This violates the hypotheses because $T$ is not a stopping time. Indeed, we may find out at time $t = 1$ that the stopping time $T$ was actually 0. To see that this is a counterexample, notice that $\mathbb{E}[T] = \mathbb{E}[X] = 1/2$, while

$$\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = 0.$$ (To see the last thing, notice that in fact this sum is always 0. If $X_1 = 0$, then $T = 1$ and the sum is just $X_1 = 0$. If $X_1 = 1$, then $T = 0$ and the sum is empty.

2. We write

$$\mathbb{E}[Z_t|X_1, \ldots, X_{t-1}] = \sum_{j=1}^{t-1}(X_j - \mathbb{E}X) + \mathbb{E}[X_t - \mathbb{E}X|X_1, \ldots, X_t]$$

$$= \sum_{j=1}^{t-1}(X_j - \mathbb{E}X) = Z_{t-1}.$$ 

3. We use the third condition. By the assumption in Wald’s thm, $\mathbb{E}T < \infty$, so we just need to show that there is some $c$ so that, for all $i$, $\mathbb{E}[|Z_{i+1} - Z_i||X_0, \ldots X_i] < c$. This conditional expectation is just

$$\mathbb{E}|X_{i+1} - \mathbb{E}X| \leq 2\mathbb{E}[X],$$

(using the triangle inequality). And this is again bounded by the assm in Wald’s theorem.
4. Applying the Martingale stopping theorem, we have
\[
0 = \mathbb{E}Z_0 = \mathbb{E}Z_T = \mathbb{E}\left[\sum_{j=1}^{T} (X_j - \mathbb{E}[X])\right] = \mathbb{E}\left[\sum_{j=1}^{T} X_j\right] - \mathbb{E}[T]\mathbb{E}[X]
\]
and rearranging proves (1).

5. Let \(X_i\) be the outcome of the i’th roll, and let \(T\) be the first time we see a six. Then \(T\) is a stopping time for \(X_i\) and \(\mathbb{E}T, \mathbb{E}X\) are both bounded. Thus,
\[
\mathbb{E}\sum_{i=1}^{T} X_i = \mathbb{E}[T]\mathbb{E}[X] = 6 \cdot \frac{7}{2} = 21.
\]
However, what we are after is actually \(\sum_{i=1}^{T-1} X_i\), but by definition the last term is 6, so we have
\[
\sum_{i=1}^{T-1} X_i = 21 - 6 = 15.
\]

4 Ballot Counting

Suppose that there is an election with two candidates, \(A\) and \(B\), and \(n\) voters; say candidate \(A\) is the winner, receiving \(N_A > N_B\) votes. (So \(N_A + N_B = n\)). The ballots are counted in a random order. What is the probably that \(A\) remained ahead for the entire count?

Let \(A_t\) be the number of votes for \(A\) at time \(t\); let \(B_t\) be the number of votes for \(B\) at time \(t\).

Let \(Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}\). That is, we imagine that we’ve already done the count, and then we “uncount” the votes one-by-one.

Group Work

1. Let \(T\) be the smallest \(t\) so that \(Z_t = 0\); if this never occurs, set \(T = n - 1\).

   Explain why \(T\) is a stopping time for \(\{Z_t\}\), and why the Martingale Stopping Theorem applies to it. (Assume for now that \(\{Z_t\}\) is indeed a martingale; you’ll show that soon).

2. Apply the Martingale Stopping Theorem to \(\{Z_t\}\) and \(T\), and use it to compute the
probability that candidate $A$ was ahead throughout the count.

3. Show that $\{Z_t\}$ is a martingale. (Hint: It might help to think of the process that $Z_t$ is tracking as follows. Start with two piles of ballots, one of size $N_A$ and one of size $N_B$. Then choose a uniformly random vote to remove from one of the two piles; that will give you two piles corresponding to $Z_1$. Continue in this way.)

**Group Work: Solutions**

1. Intuitively, $T$ is a stopping time since we don’t need to “look into the future” to compute it: we know at time $t$ whether or not $T = t$. With probability 1, $T < n - 1$, so the second item of the Martingale Stopping Theorem applies.

2. Applying the Martingale Stopping Theorem, we have

$$E[Z_T] = E[Z_0] = \frac{A_n - B_n}{n} = \frac{N_A - N_B}{n}.$$  

On the other hand, there are two possibilities for how $Z_T$ could end up. Either $T < n - 1$, which means that $Z_T = 0$, or else $T = n - 1$, which means that $Z_T = (1 - 0)/1 = 1$. (Notice that if $Z_T = n - 1$, we must have $A_1 = 1$ and $B_1 = 0$, since if $B_1 = 1, A_1 = 0$, we would have had $Z_t = 0$ for some $t < n - 1$, since candidate $B$ got ahead somehow.) Thus, if $Z_T = 1$ (and $T = n - 1$), then candidate $A$ was ahead for the whole count; otherwise $T < n - 1$ and $Z_T = 0$.

Let $p$ be the probability that candidate $A$ was ahead for the whole count. Then the above reasoning shows that

$$E[Z_T] = (1 - p) \cdot 0 + p \cdot 1.$$  

Using the above, this shows

$$p = \frac{N_A - N_B}{n}.$$  

3. To show that $\{Z_t\}$ is a martingale, we have

$$E[Z_{t+1}] = \frac{EA_{n-t-1}}{n - t - 1} - \frac{EB_{n-t-1}}{n - t - 1}.$$  

Consider each of these terms separately. By the intuition in the hint, the expectation $E A_{n-t-1}$ is the probability that we chose our “removed” ballot from pile $A$ (that would be $A_{n-t}/(n-t)$ times $A_{n-t-1}$; plus the probability that we “removed” the ballot from pile $B$ ($B_{n-t}/(n-t)$) times $A_{n-t}$. We have a similar calculation for the other term. Thus,
\[ \mathbb{E}[Z_{t+1}|Z_1, \ldots, Z_t] = \frac{\mathbb{E}A_{n-t-1}}{n-t-1} - \frac{\mathbb{E}B_{n-t-1}}{n-t-1} \]
\[ = \frac{1}{n-t-1} \left( \frac{A_{n-t}}{n-t} \cdot (A_{n-t} - 1) + \frac{B_{n-t}}{n-t} \cdot A_{n-t} \right) + \]
\[ \frac{1}{n-t-1} \left( \frac{B_{n-t}}{n-t} \cdot (B_{n-t} - 1) + \frac{A_{n-t}}{n-t} \cdot B_{n-t} \right) \]

using the fact that \( B_{n-t} + A_{n-t} = n - t \), this simplifies to

\[ \cdots = \frac{A_{n-t}}{n-t+1} + \frac{B_{n-t}}{n-t+1} - \frac{A_{n-t}}{(n-t-1)(n-t)} - \frac{B_{n-t}}{(n-t-1)(n-t)} \]
\[ = \frac{A_{n-t}}{n-t} + \frac{B_{n-t}}{n-t} \]
\[ = Z_t. \]

This is what we wanted, so \( Z_t \) is indeed a martingale.