CS265, Fall 2023

Class 17: Agenda and Questions

1 Announcements

- HW7 due tomorrow.
- HW8 (last one!!!) out now.
- **You are all done with quizzes!**
- Final exam is Tuesday, December 12, 3:30-6:30pm.
- Practice exam released soon.
- Plan for Week 10:
  - Tuesday: Fun day on pseudorandomness (no quiz, not on HW or exam)
  - Thursday: The research frontier!

2 Questions?

Any questions from the minilectures and/or the quiz? (Stopping times, Martingale stopping theorem)

3 Wald's equation

In this exercise we'll get some practice applying the martingale stopping theorem, to prove **Wald's equation.**

Theorem 1 (Wald's equation)**.** *Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, distributed according to some random variable X. Let T be a stopping time for $\{X_i\}$. If $\mathbb{E}[X]$ and $\mathbb{E}[T]$ are both bounded, then*

$$\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbb{E}[T] \cdot \mathbb{E}[X]. \tag{1}$$

Group Work

1. Wald's equation hopefully seems pretty intuitive. But there is something to prove! Come up with an example of some random variables X_i and T that don't obey the hypotheses of Theorem 1, so that the (1) does not hold.

Note: To make this more challenging, try to violate as few of the hypotheses as possible.

2. Let $Z_i = \sum_{j=1}^{i}(X_j - \mathbb{E}[X])$. Prove that $\{Z_i\}$ is a martingale with respect to $\{X_i\}$.
3. Argue that the martingale stopping theorem applies to $\{Z_i\}$ and T, where X, T are as in Theorem 1.
4. Use the Martingale stopping theorem to prove Wald's equation.
5. Consider rolling a fair, six-sided die repeatly. Let X be the sum of all of the rolls up until the first "6" is rolled, not including that 6. What is $\mathbb{E}X$?

Group Work: Solutions

1. There are many examples, but here's a simple one. Let $X_1 = 0$ with probability 1/2 and 1 with probability 1/2. Let $T = 1 - X_1$. That is, if $X_1 = 0$, then $T = 1$, and if $X_1 = 1$, then $T = 0$. This violates the hypotheses because T is *not* a stopping time. Indeed, we may find out at time $t = 1$ that the stopping time T was actually 0. To see that this is a counterexample, notice that $\mathbb{E}[T] = \mathbb{E}[X] = 1/2$, while

$$\mathbb{E}[\sum_{i=1}^{T} X_i] = 0.$$

(To see the last thing, notice that in fact this sum is always 0. If $X_1 = 0$, then $T = 1$ and the sum is just $X_1 = 0$. If $X_1 = 1$, then $T = 0$ and the sum is empty.

2. We write

$$\begin{aligned}\mathbb{E}[Z_t|X_1,\ldots,X_{t-1}] &= \sum_{j=1}^{t-1}(X_j - \mathbb{E}X) + \mathbb{E}[X_t - \mathbb{E}X|X_1,\ldots,X_t] \\ &= \sum_{j=1}^{t-1}(X_j - \mathbb{E}X) = Z_{t-1}.\end{aligned}$$

3. We use the third condition. By the assumption in Wald's thm, $\mathbb{E}T < \infty$, so we just need to show that there is some c so that, for all i, $\mathbb{E}[|Z_{i+1} - Z_i||X_0,\ldots X_i] < c$. This conditional expectation is just

$$\mathbb{E}|X_{i+1} - \mathbb{E}X| \leq 2\mathbb{E}[X],$$

(using the triangle inequality). And this is again bounded by the assm in Wald's theorem.

4. Applying the Martingale stopping theorem, we have

$$\begin{aligned} 0 &= \mathbb{E}Z_0 \\ &= \mathbb{E}Z_T \\ &= \mathbb{E}[\sum_{j=1}^{T}(X_j - \mathbb{E}[X])] \\ &= \mathbb{E}[\sum_{j=1}^{T} X_j] - \mathbb{E}[T]\mathbb{E}[X] \end{aligned}$$

   and rearranging proves (1).

5. Let X_i be the outcome of the i'th roll, and let T be the first time we see a six. Then T is a stopping time for X_i and $\mathbb{E}T$, $\mathbb{E}X$ are both bounded. Thus,

$$\mathbb{E}\sum_{i=1}^{T} X_i = \mathbb{E}[T]\mathbb{E}[X] = 6 \cdot \frac{7}{2} = 21.$$

   However, what we are after is actually $\sum_{i=1}^{T-1} X_i$, but by definition the last term is 6, so we have

$$\sum_{i=1}^{T-1} X_i = 21 - 6 = 15.$$

4 Ballot Counting

Suppose that there is an election with two candidates, A and B, and n voters; say candidate A is the winner, receiving $N_A > N_B$ votes. (So $N_A + N_B = n$). The ballots are counted in a random order. What is the probably that A remained ahead for the entire count?

Let A_t be the number of votes for A at time t; let B_t be the number of votes for B at time t.

Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$. That is, we imagine that we've already done the count, and then we "uncount" the votes one-by-one.

Group Work

1. Let T be the smallest t so that $Z_t = 0$; if this never occurs, set $T = n - 1$.

   Explain why T is a stopping time for $\{Z_t\}$, and why the Martingale Stopping Theorem applies to it. (Assume for now that $\{Z_t\}$ is indeed a martingale; you'll show that soon).

2. Apply the Martingale Stopping Theorem to $\{Z_t\}$ and T, and use it to compute the

probability that candidate A was ahead throughout the count.

3. Show that $\{Z_t\}$ is a martingale. (Hint: It might help to think of the process that Z_t is tracking as follows. Start with two piles of ballots, one of size N_A and one of size N_B. Then choose a uniformly random vote to remove from one of the two piles; that will give you two piles corresponding to Z_1. Continue in this way.)

Group Work: Solutions

1. Intuitively, T is a stopping time since we don't need to "look into the future" to compute it: we know at time t whether or not $T = t$. With probability 1, $T < n-1$, so the second item of the Martingale Stopping Theorem applies.
2. Applying the Martingale Stopping Theorem, we have

$$\mathbb{E}[Z_T] = \mathbb{E}[Z_0] = \frac{A_n - B_n}{n} = \frac{N_A - N_B}{n}.$$

   On the other hand, there are two possibilities for how Z_T could end up. Either $T < n-1$, which means that $Z_T = 0$, or else $T = n-1$, which means that $Z_T = (1-0)/1 = 1$. (Notice that if $Z_T = n-1$, we must have $A_1 = 1$ and $B_1 = 0$, since if $B_1 = 1, A_1 = 0$, we would have had $Z_t = 0$ for some $t < n-1$, since candidate B got ahead somehow.) Thus, if $Z_T = 1$ (and $T = n-1$), then candidate A was ahead for the whole count; otherwise $T < n-1$ and $Z_T = 0$.

   Let p be the probability that candidate A was ahead for the whole count. Then the above reasoning shows that

$$\mathbb{E}[Z_T] = (1-p)\cdot 0 + p \cdot 1.$$

   Using the above, this shows

$$p = \frac{N_A - N_B}{n}.$$

3. To show that $\{Z_t\}$ is a martingale, we have

$$\mathbb{E}Z_{t+1} = \frac{\mathbb{E}A_{n-t-1}}{n-t-1} - \frac{\mathbb{E}B_{n-t-1}}{n-t-1}.$$

   Consider each of these terms separately. By the intuition in the hint, the expectation $\mathbb{E}A_{n-t-1}$ is the probability that we chose our "removed" ballot from pile A (that would be $A_{n-t}/(n-t)$) times $A_{n-t}-1$; plus the probability that we "removed" the ballot from pile B ($B_{n-t}/(n-t)$) times A_{n-t}. We have a similar calculation for the other term. Thus,

$$
\begin{aligned}
\mathbb{E}[Z_{t+1}|Z_1,\ldots,Z_t] &= \frac{\mathbb{E}A_{n-t-1}}{n-t-1} - \frac{\mathbb{E}B_{n-t-1}}{n-t-1} \\
&= \frac{1}{n-t-1}\left(\frac{A_{n-t}}{n-t}\cdot(A_{n-t}-1) + \frac{B_{n-t}}{n-t}\cdot A_{n-t}\right) + \\
&\qquad \frac{1}{n-t-1}\left(\frac{B_{n-t}}{n-t}\cdot(B_{n-t}-1) + \frac{A_{n-t}}{n-t}\cdot B_{n-t}\right)
\end{aligned}
$$

using the fact that $B_{n-t} + A_{n-t} = n - t$, this simplifies to

$$
\begin{aligned}
\cdots &= \frac{A_{n-t}}{n-t+1} + \frac{B_{n-t}}{n-t+1} - \frac{A_{n-t}}{(n-t-1)(n-t)} - \frac{B_{n-t}}{(n-t-1)(n-t)} \\
&= \frac{A_{n-t}}{n-t} + \frac{B_{n-t}}{n-t} \\
&= Z_t.
\end{aligned}
$$

This is what we wanted, so Z_t is indeed a martingale.