1 Announcements

- HW7 due tomorrow.
- HW8 (last one!!!) out now.
- You are all done with quizzes!
- Final exam is Tuesday, December 12, 3:30-6:30pm.
- Practice exam released soon.
- Plan for Week 10:
  - Tuesday: Fun day on pseudorandomness (no quiz, not on HW or exam)
  - Thursday: The research frontier!

2 Questions?

Any questions from the minilectures and/or the quiz? (Stopping times, Martingale stopping theorem)

3 Wald’s equation

In this exercise we’ll get some practice applying the martingale stopping theorem, to prove Wald’s equation.

**Theorem 1** (Wald’s equation). Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, distributed according to some random variable $X$. Let $T$ be a stopping time for \{\{X_i\}\}. If $\mathbb{E}[X]$ and $\mathbb{E}[T]$ are both bounded, then

$$
\mathbb{E} \left[ \sum_{i=1}^{T} X_i \right] = \mathbb{E}[T] \cdot \mathbb{E}[X].
$$

**Group Work**

1. Wald’s equation hopefully seems pretty intuitive. But there is something to prove! Come up with an example of some random variables $X_i$ and $T$ that don’t obey the hypotheses of Theorem 1, so that the (1) does not hold.
Note: To make this more challenging, try to violate as few of the hypotheses as possible.

2. Let \( Z_i = \sum_{j=1}^{i} (X_j - \mathbb{E}[X]) \). Prove that \( \{Z_i\} \) is a martingale with respect to \( \{X_i\} \).

3. Argue that the martingale stopping theorem applies to \( \{Z_i\} \) and \( T \), where \( X, T \) are as in Theorem 1.

4. Use the Martingale stopping theorem to prove Wald’s equation.

5. Consider rolling a fair, six-sided die repeatedly. Let \( X \) be the sum of all of the rolls up until the first “6” is rolled, not including that 6. What is \( \mathbb{E}X \)?

4 Ballot Counting

Suppose that there is an election with two candidates, \( A \) and \( B \), and \( n \) voters; say candidate \( A \) is the winner, receiving \( N_A > N_B \) votes. (So \( N_A + N_B = n \)). The ballots are counted in a random order. What is the probably that \( A \) remained ahead for the entire count?

Let \( A_t \) be the number of votes for \( A \) at time \( t \); let \( B_t \) be the number of votes for \( B \) at time \( t \).

Let \( Z_t = \frac{A_{n-t} - B_{n-t}}{n-t} \). That is, we imagine that we’ve already done the count, and then we “uncount” the votes one-by-one.

Group Work

1. Let \( T \) be the smallest \( t \) so that \( Z_t = 0 \); if this never occurs, set \( T = n - 1 \).
   Explain why \( T \) is a stopping time for \( \{Z_t\} \), and why the Martingale Stopping Theorem applies to it. (Assume for now that \( \{Z_t\} \) is indeed a martingale; you’ll show that soon).

2. Apply the Martingale Stopping Theorem to \( \{Z_t\} \) and \( T \), and use it to compute the probability that candidate \( A \) was ahead throughout the count.

3. Show that \( \{Z_t\} \) is a martingale. (Hint: It might help to think of the process that \( Z_t \) is tracking as follows. Start with two piles of ballots, one of size \( N_A \) and one of size \( N_B \). Then choose a uniformly random vote to remove from one of the two piles; that will give you two piles corresponding to \( Z_1 \). Continue in this way.)