1 Announcements

- HW7 due tomorrow.
- HW8 (last one!!!) out now.
- You are all done with quizzes!
- Final exam is Th. Dec. 15, 12:15-3:15pm, in 420-040.
- Practice exam released soon.
- Plan for Week 10:
  - Tuesday: Fun day on pseudorandomness (no quiz, not on HW or exam)
  - Thursday: The research frontier! ($\geq 2$ short research talks)

2 Questions?

Any questions from the minilectures and/or the quiz? (Stopping times, Martingale stopping theorem)

3 Wald’s equation

In this exercise we’ll get some practice applying the martingale stopping theorem, to prove Wald’s equation.

**Theorem 1** (Wald’s equation). Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, distributed according to some random variable $X$. Let $T$ be a stopping time for $\{X_i\}$. If $\mathbb{E}[X]$ and $\mathbb{E}[T]$ are both bounded, then

$$
\mathbb{E} \left[ \sum_{i=1}^{T} X_i \right] = \mathbb{E}[T] \cdot \mathbb{E}[X].
$$

Group Work

1. Wald’s equation hopefully seems pretty intuitive. But there is something to prove! Come up with an example of some random variables $X_i$ and $T$ that don’t obey the hypotheses of Theorem 1, so that the (1) does not hold.
Note: To make this more challenging, try to violate as few of the hypotheses as possible.

2. Let $Z_i = \sum_{j=1}^{i} (X_j - \mathbb{E}[X])$. Prove that $\{Z_i\}$ is a martingale with respect to $\{X_i\}$.

3. Argue that the martingale stopping theorem applies to $\{Z_i\}$ and $T$, where $X, T$ are as in Theorem 1.

4. Use the Martingale stopping theorem to prove Wald’s equation.

5. Consider rolling a fair, six-sided die repeatedly. Let $X$ be the sum of all of the rolls up until the first “6” is rolled, not including that 6. What is $\mathbb{E}X$?

4 Ballot Counting

Suppose that there is an election with two candidates, $A$ and $B$, and $n$ voters; say candidate $A$ is the winner, receiving $N_A > N_B$ votes. (So $N_A + N_B = n$). The ballots are counted in a random order. What is the probably that $A$ remained ahead for the entire count?

Let $A_t$ be the number of votes for $A$ at time $t$; let $B_t$ be the number of votes for $B$ at time $t$.

Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$. That is, we imagine that we’ve already done the count, and then we “uncount” the votes one-by-one.

Group Work

1. Let $T$ be the smallest $t$ so that $Z_t = 0$; if this never occurs, set $T = n - 1$.

   Explain why $T$ is a stopping time for $\{Z_t\}$, and why the Martingale Stopping Theorem applies to it. (Assume for now that $\{Z_t\}$ is indeed a martingale; you’ll show that soon).

2. Apply the Martingale Stopping Theorem to $\{Z_t\}$ and $T$, and use it to compute the probability that candidate $A$ was ahead throughout the count.

3. Show that $\{Z_t\}$ is a martingale. (Hint: It might help to think of the process that $Z_t$ is tracking as follows. Start with two piles of ballots, one of size $N_A$ and one of size $N_B$. Then choose a uniformly random vote to remove from one of the two piles; that will give you two piles corresponding to $Z_1$. Continue in this way.)