CS265, Fall 2020

Class 18: Agenda, Questions, and Links

1 Announcements

- HW7 is out, due Friday (HW8, THE LAST ONE, will be released Friday)
- No quiz due Monday 11/16. You are all done with quizzes!!
- Plan for next week: Monday is still a "normal" class (there are videos, just no quiz). Wednesday (last class) will be special. Mary will give a research talk!
- Please fill out course evaluations on Axess!

2 Questions?

Any questions from the minilectures and/or the quiz and/or the warm-up? (Stopping times, Martingale stopping theorem)

- Go into small groups and ask each other your questions.
- Go to `https://pollev.com/cs265` and ask your questions/comments there, or else upvote others' questions.

3 Wald's equation

In this exercise we'll get some practice applying the martingale stopping theorem, to prove **Wald's equation.**

Theorem 1 (Wald's equation)**.** *Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, distributed according to some random variable X. Let T be a stopping time for $\{X_i\}$. If $\mathbb{E}[X]$ and $\mathbb{E}[T]$ are both bounded, then*

$$\mathbb{E}[\sum_{i=1}^{T} X_i] = \mathbb{E}[T] \cdot \mathbb{E}[X]. \quad (1)$$

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on `http://PollEv.com/cs265`.

1. Wald's equation hopefully seems pretty intuitive. But there is something to prove!

Come up with an example of some random variables X_i and T that don't obey the hypotheses of Theorem 1, so that the (1) does not hold.

Note: To make this more challenging, try to violate as few of the hypotheses as possible.

2. Let $Z_i = \sum_{j=1}^{i} (X_j - \mathbb{E}[X])$. Prove that $\{Z_i\}$ is a martingale with respect to $\{X_i\}$.

   At this point, please fill out the PollEverywhere!

3. Argue that the martingale stopping theorem applies to $\{Z_i\}$.

4. Use the Martingale stopping theorem to prove Wald's equation.

   Hint: *Linearity of expectation.*

   At this point, please fill out the PollEverywhere!

5. Consider the following procedure. Roll one fair six-sided die to get a number $T \in \{1, 2, \ldots, 6\}$. Then flip T fair coins. What is the expected number of heads that you get total?

   Hint: *Apply Wald's equation to the sum* $\sum_{i=1}^{T} X_i$, *where* X_i *is the outcome of the i'th coin flip.*

   At this point, please fill out the PollEverywhere!

4 How long until everyone speaks?

In this group work, we'll see another example of using Wald's equation.

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on http://PollEv.com/cs265**.**

Consider the following setup. There are n people, each with an infinite number of messages that they'd like to say. However, only one person can speak at a time. More formally, time is discrete, and at each timestep, any player can choose to broadcast one of their messages. If they are the only person to speak at that timestep, their message goes through. However, if multiple people try to speak during the same timestep, everyone hears an unintelligable noise that sounds a bit like `FAIL`. (If you try to speak and `FAIL`, then the message you tried to say goes back to the front of your queue).

The players decide to try the following strategy. At each timestep, everyone will try to speak, indepdently, with probability $1/n$.

In this question, you'll figure out the expected amount of time until everybody speaks once.

1. Let N be the number of messages that are **successfully** broadcast before everyone

has had a chance to successfully broadcast at least once. What is $\mathbb{E}[N]$?

***Hint**: Coupon collecting.*

2. Let T be the number of timesteps until everyone has successfully sent at least one message. What is $\mathbb{E}[T]$?

   ***Hint**: Let t_i be the time at which the i'th successful message is sent, and let $r_i = t_i - t_{i-1}$. Argue that $T = \sum_{i=1}^{N} r_i$, and apply Wald's equation.*

   At this point, please fill out the PollEverywhere!