Class 7: Agenda and Questions

1 Announcements

- HW3 due Friday

2 Warm-Up

Group Work

Let $G = (V, E)$ be a weighted, undirected graph, on n vertices with edge weights w_{uv} on the edge $\{u, v\} \in E$. Let $d : V \times V \to \mathbb{R}$ be the associated graph metric.

Explain how to efficiently find and apply a map $f : V \to \mathbb{R}^k$, for $k = O(\log^2 n)$, so that

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \leq O(\log n) \frac{\sum_{\{u,v\}\in E} d(u, v)}{\sum_{\{u,v\}\in\binom{V}{2}} d(u, v)}$$

holds with high probability. Above, $\binom{V}{2}$ refers to the set of all unordered pairs $\{u, v\}$ for $u, v \in V$ and $u \neq v$.

Group Work: Solutions

Let $f : V \to \mathbb{R}^k$ be the map given by Bourgain's embedding. Then for all u, v, we have (for some constant b)

$$\frac{k}{b \log n} d(u, v) \leq \|f(u) - f(v)\|_1 \leq k d(u, v),$$

and so

$$\frac{\sum_{\{u,v\}\in E} d(u, v)}{\sum_{\{u,v\}\in\binom{V}{2}} d(u, v)} \geq \frac{\sum_{\{u,v\}\in E} \frac{1}{k} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \frac{b \log n}{k} \|f(u) - f(v)\|_1} = \frac{1}{b \log n} \frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1}.$$

Multiplying both sides by $b \log n$ establishes the statement.

3 Lecture Recap and Questions?

Any questions from the mini-lectures or pre-class-quiz? (Metric Embeddings; Bourgain's Embedding)

4 Sparsest Cuts

[Some slides; summary is below]

For a graph $G = (V, E)$, define

$$\phi(G, S) = \frac{|E(S, \bar{S})|}{|S||\bar{S}|},$$

and

$$\phi(G) = \min_{S \subset V, S \neq \emptyset, V} \phi(G, S),$$

where above $\bar{S} := V \setminus S$ denotes the complement of S, and $E(S, \bar{S})$ denotes the set of edges that have one endpoint in S and one endpoint in $\bar{S}$.

Intuitively, if $\phi(G, S)$ is small, then S is pretty "disconnected" from $\bar{S}$. Notice that the denominator, $|S||\bar{S}|$, is the number of edges that would be between S and $\bar{S}$ in the complete graph, so $\phi(G, S)$ is the fraction of possible edges between S and $\bar{S}$ that actually exist in G.

Finding S to minimize $\phi(G, S)$ is useful, for example, in clustering applications. However, it's also NP-hard. Today we'll see a randomized algorithm to find an S so that $\phi(G, S)$ is *approximately* minimized. More precisely, we'll find S so that $\phi(S, G) \leq O(\log n)\phi(G)$.

Question: How is this definition of $\phi(G)$ different than simply asking for the minimum cut? When might you prefer a sparsest cut to a min cut? (Recall we saw a randomized algorithm for the minimum cut back in Week 1...)

4.1 Connection to metrics

Group Work

In this group work, you will show that

$$\phi(G) = \min_f \frac{\sum_{\{u,v\} \in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \|f(u) - f(v)\|_1}, \tag{1}$$

where the minimum is over all functions $f : V \to \mathbb{R}^k$ for some k, so that f takes on at least two distinct values. (This last bit is needed so that the denominator doesn't vanish).

1. Show that
$$\phi(G) = \min_{f: V \to \{0,1\}} \frac{\sum_{\{u,v\} \in E} |f(u) - f(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)|},$$
where the minimum is over all functions $f : V \to \{0, 1\}$ so that f takes on both values 0 and 1. (The difference between this and the expression above is that f maps to $\{0, 1\}$ instead of $\mathbb{R}^k$ for some k).

Hint: *Consider mapping functions f to sets S by the relationship $S = \{u : f(u) = 1\}$.*

2. Think about why the above extends to show that

$$\phi(G) = \inf_{f:V\to\mathbb{R}} \frac{\sum_{\{u,v\}\in E} |f(u) - f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u) - f(v)|},$$

where now the minimum is over $f : V \to \mathbb{R}$ instead of $f : V \to \{0, 1\}$.

(Don't worry about a formal proof here, just kind of convince yourself intuitively that this is true).

Hint: *Using part (a), it suffices to show that the infimum over all $f : V \to \mathbb{R}$ is actually attained by some f that maps vertices in V to $\{0, 1\}$. To see this, consider the following steps:*

- *Suppose that $f : V \to \mathbb{R}$ takes on three distinct values, $a < b < c$. Consider a new function $f_x : V \to \mathbb{R}$, so that $f_x(u) = x$ if $f(u) = b$, and $f_x(u) = f(u)$ otherwise. That is, $f_x(u)$ just replaces the value b with x. Show that either*

$$R(f_a) \le R(f) \qquad \textit{or} \qquad R(f_c) \le R(f),$$

where

$$R(f) = \frac{\sum_{\{u,v\}\in E} |f(u) - f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u) - f(v)|}.$$

(That is, by sliding the middle value b towards either a or c, you can decrease this quantity.)

Sub-hint: *when you vary $x \in [a, c]$, you can get rid of the absolute values in $R(f_x)$. Looking at a small example might be helpful.*

- *Argue that the above logic implies that there's an f that attains the infimum that takes on only two values.*
- *Argue that those two values may as well be 0 and 1.*

3. Think about why the above extends to show that

$$\phi(G) = \min_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1},$$

where the minimum is over all functions $f : V \to \mathbb{R}^k$ for any k.

Hint: *You may want to use the inequality that $\frac{\sum_i a_i}{\sum_i b_i} \ge \min_i \frac{a_i}{b_i}$ for $a_i, b_i > 0$.*

Group Work: Solutions

1. Using the connection in the hint, the numerator is exactly $|E(S, \bar{S})|$, and the denominator is the number of edges between S and $\bar{S}$ in the complete graph, which is $|S||\bar{S}|$.

2. **Note: this proof is a bit involved; there is an easier proof, but this one involves the least machinery and also is somewhat algorithmic, which will be useful later. I didn't expect students to get all of the details of this proof in group work, I only wanted you to get some basic intuition.**

   For convenience, let
$$R(f) = \frac{\sum_{\{u,v\}\in E} |f(u) - f(v)|}{\sum_{\{u,v\}\in \binom{V}{2}} |f(u) - f(v)|}.$$

   Notice that both the numerator and the denominator of $R(f_{b'})$ are linear in b', for $b' \in [a, c]$. This is because if both $f(u), f(v) = b$, then $|f_{b'}(u) - f_{b'}(v)| = |f(u) - f(v)| = 0$; if neither are equal to b, then the expression does not change; and if only one is equal to b (say WLOG that $f(u) = b$), then the other one is either $\leq a$ or $\geq c$ (say WLOG $\leq a$), meaning that $|f_{b'}(u) = f_{b'}(v)| = |b' - f(v)| = b' - f(v)$ is linear in b'.

   Further, the denominator of $R(f_{b'})$ doesn't vanish, since there's at least one nonzero term in it (e.g., the term $|c - a|$). But then $R(f_{b'})$ is the ratio of linear functions in b', and the denominator never vanishes. It's not too hard to see (e.g., with some calculus) that $R(f_{b'})$ is thus is either increasing or decreasing (or constant), and in particular it attains a minimum at one of the endpoints a or c of the relevant interval.

   We could have done this for any f so that there are ≥ 3 distinct values in the range. By doing this repeatedly, we see that for any f with ≥ 3 distinct values, there is some f^* with only two values (say, a and b) so that $R(f^*) \leq R(f)$. But notice that $R(f^*)$ doesn't change if we change the values of a and b to 0 and 1 respectively. (That is, replace $f^*(x)$ with $\frac{f^*(x)-a}{b-a}$).

   This implies that $\inf_{f:V\to\{0,1\}} R(f) \leq \inf_{f:V\to\mathbb{R}} R(f)$, and since there are only a finite number of functions $f : V \to \{0, 1\}$, the infimum is actually a minimum.

3. We have shown that $\phi(G) = \min_{f:V\to\mathbb{R}} R(f)$. We clearly have
$$\phi(G) = \min_{f:V\to\mathbb{R}} R(f) \geq \min_{f:V\to\mathbb{R}^k} R(f),$$

   since the set we are minimizing over on the right. On the other hand, for any

$f : V \to \mathbb{R}^k$, we can write $f = (f_1, \dots, f_k)$, and so

$$\begin{aligned}
R(f) &= \frac{\sum_{\{u,v\} \in E} \sum_i |f_i(u) - f_i(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} \sum_i |f_i(u) - f_i(v)} \\
&= \frac{\sum_i \sum_{\{u,v\} \in E} |f_i(u) - f_i(v)|}{\sum_i \sum_{\{u,v\} \in \binom{V}{2}} |f_i(u) - f_i(v)|} \\
&\geq \min_i \frac{\sum_{\{u,v\} \in E} |f_i(u) - f_i(v)|}{\sum_{\{u,v\} \in \binom{V}{2}} |f_i(u) - f_i(v)|} \\
&= \min_i R(f_i) \\
&\geq \min_{g : V \to \mathbb{R}} R(g) \\
&= \phi(G).
\end{aligned}$$

Since the above reasoning held for any $f : V \to \mathbb{R}^k$, we conclude

$$\min_{f : V \to \mathbb{R}^k} R(f) \geq \phi(G).$$

4.2 A randomized algorithm

Group Work

1. Based on the result that we got in the first group work, we might think of the following approach:

   Find $f : V \to \mathbb{R}^k$ to minimize

   $$R(f) := \frac{\sum_{\{u,v\} \in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\} \in \binom{V}{2}} \|f(u) - f(v)\|_1}$$

   Unfortunately, this doesn't turn out to be an easy optimization problem to solve. Instead, we'll consider the optimization problem:

   Find values $d_{u,v} \in \mathbb{R}$ for all $u \neq v \in V$ to minimize

   $$Q(d) := \sum_{\{u,v\} \in E} d_{u,v}$$

   subject to:

   - $d_{u,v} = d_{v,u} \geq 0$ for all u, v
   - $d_{u,v} + d_{v,w} \geq d_{u,w}$ for all u, v, w

- $\sum_{\{u,v\}\in\binom{V}{2}} d_{u,v} = 1$

It turns out that this problem *can* be solved efficiently, using linear programming. (If you don't know what that is, it's okay, all that matters now is that we can find $\vec{d}$ to minimize this efficiently).

(There's no question for this part, just understand the optimization problem.)

2. Suppose that d^* is the minimizer of the problem above.

   Explain why $Q(d^*) \le \phi(G)$.

3. Find a randomized algorithm to approximate $\phi(G)$. More precisely, give a randomized algorithm that finds $f : V \to \mathbb{R}^k$ so that, with high probability,

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \le O(\log n)\phi(G).$$

   ***Hint**: Your warm-up exercise might be relevant.*

   ***Hint**: If it comes up, you may assume that Bourgain's embedding works just fine on* pseudo-metrics, *which are functions $d(u,v)$ that obey all of the axioms of metrics except that maybe $d(u,v) = 0$ for $u \neq v$.*

4. Given f as in the previous part, explain how to efficiently find a set $S \subset V$ so that

$$\phi(G,S) \le O(\log n)\phi(G).$$

   ***Hint**: Our proof in the first group-work was somewhat algorithmic...*

Group Work: Solutions

1. Notice that because of the final constraint, and the fact that the ℓ_1 norm satisfies $\|c(f(u) - f(v))\|_1 = c\|f(u) - f(v)\|_1$,

$$R(f) = Q(d_f),$$

   where

$$d_f(u,v) = \frac{\|f(u) - f(v)\|_1}{\sum_{u',v'\in\binom{V}{2}} \|f(u') - f(v')\|_1}.$$

   But $Q(d^*)$ is the minimum over *all* (pseudo-)metrics (aka, distances d that satisfy $d(u,v) = d(v,u) \ge 0$ and also satisfy the triangle inequality), so in particular d_f is in the domain that we are minimizing over. Thus, $Q(d^*) \le Q(d_f) = R(f)$.

Since this holds for any f,

$$Q(d^*) \le \min_f R(f) = \phi(G)$$

using the previous group work.

2. Apply Bourgain's embedding to the metric d^* to get some embedding f. The warm-up exercise exactly implies that

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in \binom{V}{2}} \|f(u) - f(v)\|_1} \le O(\log n) Q(d^*) \le O(\log n)\phi(G).$$

3. Given $f : V \to \mathbb{R}^k$, we saw that we can just find the coordinate f_i of f with the smallest $R(f_i)$ value and that will have $R(f_i) \le R(f)$. From there, if f takes on more than two values, we can "push" any intermediate value to one of its two neighbors. Repeating this leaves us with f taking on only two values, and then we can renormalize f to take on values that are only 0 and 1. Then we let $S \leftarrow \mathrm{Supp}(f)$.