CS265, Fall 2020

Class 8: Agenda, Questions, and Links

1 Announcements

- HW4 out!

2 Questions?

Any questions from the minilectures and/or the quiz? (Compressed sensing; RIP; Gaussian matrices have the RIP whp)

- Go into small groups and ask each other your questions.
- Go to pollev.com/cs265 and ask your questions/comments there, or else upvote others' questions.

3 More matrices with the RIP whp

Group Work

Important: as you make progress on the question(s), one person in each room should record your progress on http://PollEv.com/cs265.

1. Let $\varepsilon \in (0, 1/4)$. Suppose that $A \in \mathbb{R}^{m \times n}$ is a distribution on matrices so that, for some constant c:

$$\forall x \in \mathbb{R}^n, \Pr\left\{ \left| \|Ax\|_2 - \|x\|_2 \right| \geq \varepsilon \|x\|_2 \right\} \leq 2\exp(-cm\varepsilon^2). \tag{1}$$

   (a) Is it the case that A is a good JL transform (aka, for any set $S \subseteq \mathbb{R}^n$ of size N, $\|A(x-y)\|_2 = (1 \pm \varepsilon)\|x-y\|_2$ with high probability), with $m = O(\varepsilon^{-2} \log N)$?

   (b) Is it the case that, with high probability, A has the (k, ε)-RIP with $m = O(\varepsilon^{-2} k \log n)$?

   At this point, please record your progress on PollEverywhere!

2. Let $A \in \left\{ -\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}} \right\}^{m \times n}$ be a matrix where every entry is independent, with probability 1/2 of of $\frac{1}{\sqrt{m}}$ and probability 1/2 of $\frac{-1}{\sqrt{m}}$.

In this question, you'll show that A satisfies (1). (Notice that A might be much nicer to compute with than a random Gaussian matrix, since ± 1's are easier to deal with than real numbers).

(a) Show that, for any vector $x \in \mathbb{R}^n$, $\mathbb{E}\|Ax\|_2^2 = \|x\|_2^2$.

(b) For the next part, you'll use the *Hanson-Wright inequality.* Here's a statement of one form of this inequality:

Theorem. Let $Z_1, \ldots, Z_N$ be $\pm\frac{1}{\sqrt{m}}$-valued independent mean-zero random variables. Let $\Phi \in \mathbb{R}^{N \times N}$ be any matrix. Then for any $t \geq 0$,

$$\Pr\left\{|\vec{Z}^T \Phi \vec{Z} - \mathbb{E}\vec{Z}^T \Phi \vec{Z}| > t\right\} \leq 2\exp\left(-c\min\left(\frac{t^2 m^2}{\|\Phi\|_F^2}, \frac{tm}{\|\Phi\|}\right)\right),$$

where above $\vec{Z} = (Z_1, \ldots, Z_N)$ is the length-N vector with the random variables Z_i in it, $\|\Phi\|_F^2 = \sum_{i,j} \Phi_{i,j}^2$ denotes the Frobenius norm, and $\|\Phi\| = \sup_{v \in \mathbb{R}^N \setminus \{0\}} \frac{\|\Phi v\|_2}{\|v\|_2}$ is the operator norm.

Use the Hanson-Wright inequality (or in other inequality you feel like) to show that (1) holds for the matrix A in the previous part.

Hint: Let $N = nm$, and write $\|Ax\|_2^2$ as $\vec{Z}^T \Phi \vec{Z}$ for some matrix Φ, where the elements of $\vec{Z}$ are the entries of A.

Hint: A further hint for how to do the above: Let a_i be the i'th row of A. Then $\|Ax\|_2^2 = \sum_i a_i^T (xx^T) a_i$ (why?). Consider a matrix Φ that is block-diagonal where each block is equal to the matrix xx^T:

$$\Phi = \begin{pmatrix} xx^T & & & & \\ & xx^T & & & \\ & & xx^T & & \\ & & & \ddots & \\ & & & & xx^T \end{pmatrix}$$

Hint: It might be useful that (a) for a vector x, we have $\|xx^T\|_F^2 = \|x\|_2^4$ and $\|xx^T\| = \|x\|_2^2$, and (b) for a block-diagonal matrix Φ with blocks $\Phi_1, \Phi_2, \ldots$ on the diagonal, $\|\Phi\| = \max_i \|\Phi_i\|$. (These facts are not too hard to derive, but you can take them as given if you like).

At this point, please record your progress on PollEverywhere!

3. If you are done with the above two, here's a few "challenge" questions to think about:

(a) What other distributions on a matrix A can you come up with (other than i.i.d. Gaussians and i.i.d. $\pm 1/\sqrt{m}$ entries) that are (a) natural and (b) seem like they'd satisfy (1)? For example, what about any matrix with i.i.d. mean-zero entries? What about any matrix with i.i.d. mean-zero *bounded* entries? (i.e., the entries should never be larger than 100).

(b) Suppose that A has the RIP. Consider a matrix $A \cdot D$, where D is a diagonal matrix with i.i.d. mean-zero ± 1 entries on the diagonal. Show that AD satisfies (1), up to log factors.

Hint: This is pretty tricky to do quantitatively, but you may be able to come up with some intuition for why it should be true qualitatively.

Hint: Write $\|ADx\|_2^2 = Z^T \Phi Z$ where Z is a vector of independent sign flips, and apply the Hanson-Wright inequality above...if A has the RIP, what can you say about every $k \times k$ block of Φ?

Hint: For a complete solution, check out this paper: `https://arxiv.org/pdf/1009.0744.pdf`