Convolution

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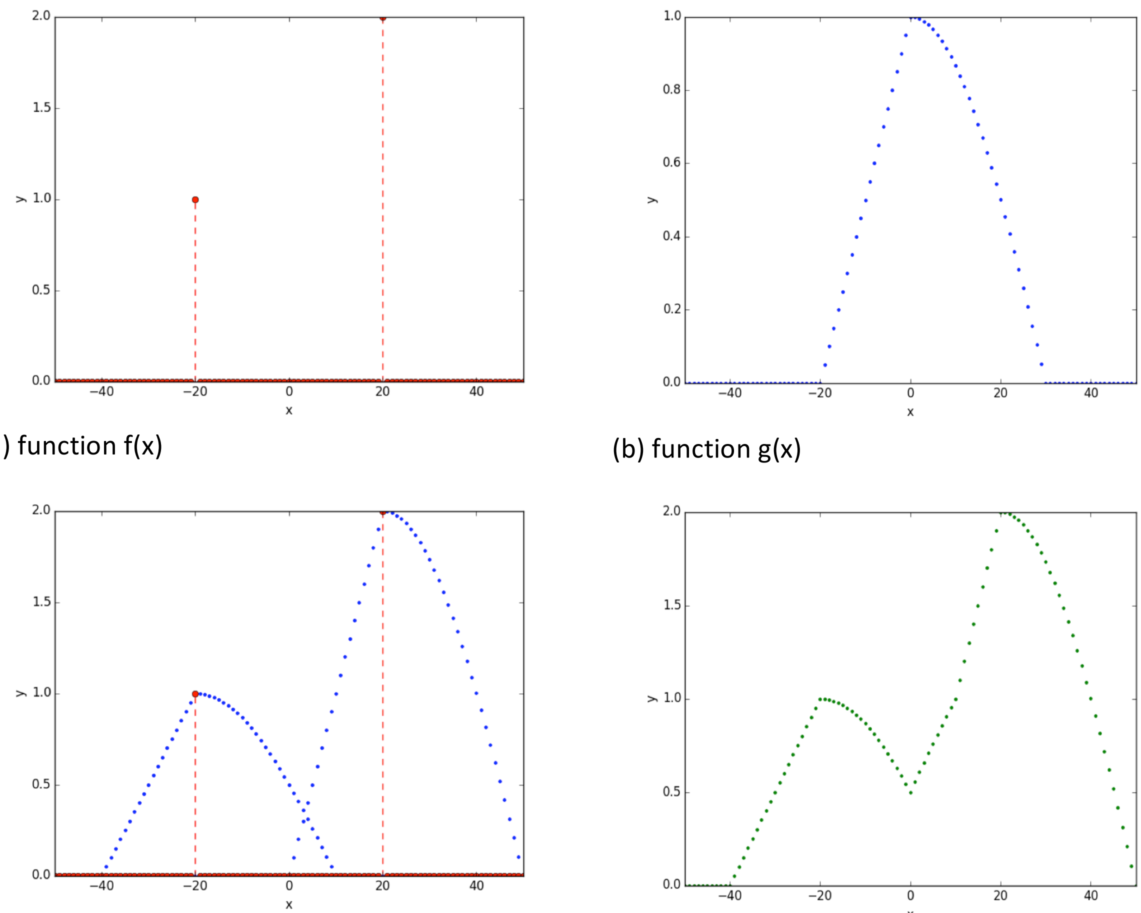
Convolution

Convolution is a mathematical operation that generalizes the idea of a moving average. Fig. 1 shows an example to illustrate how convolution works (for functions defined at discrete, evenly spaced points).



(a) function f(x)

(b) function g(x)

(c) Scaled and shifted versions of g(x)

(d) Result: f(x) convolved with g(x)

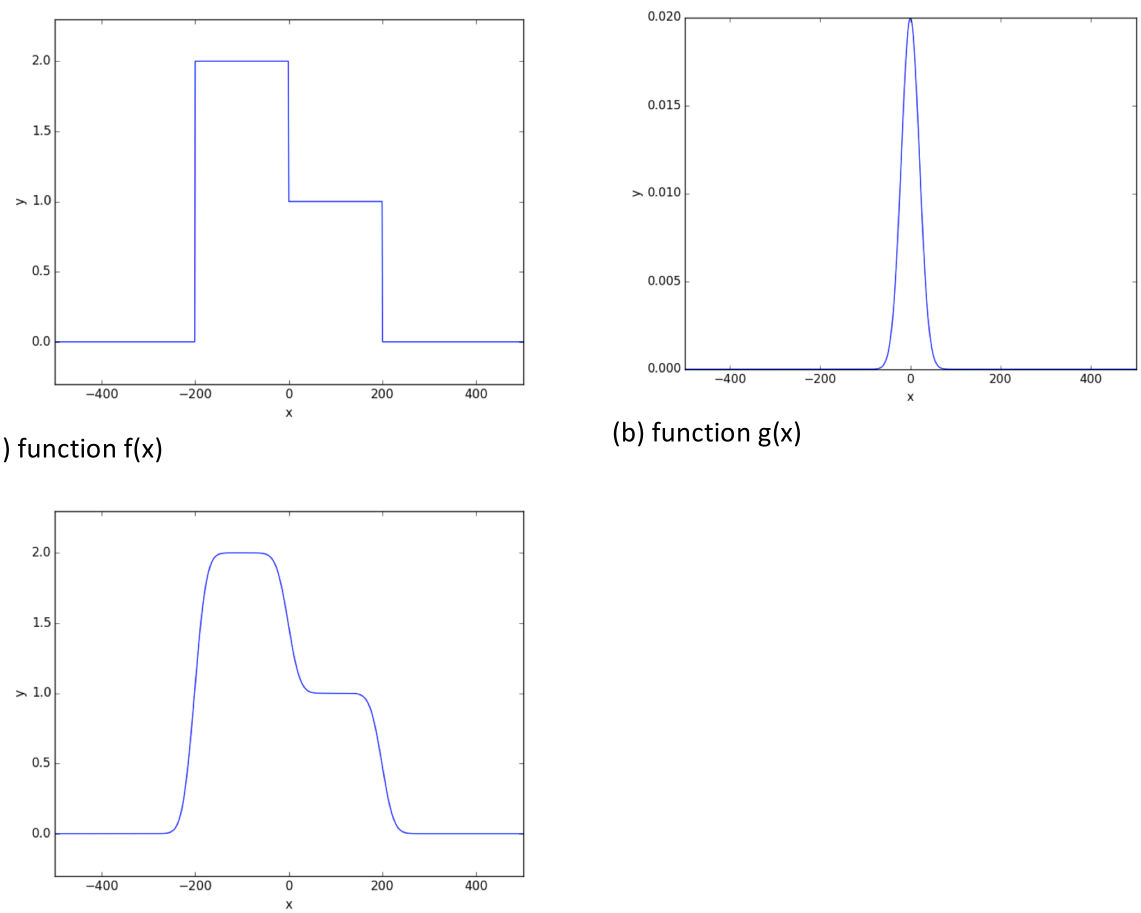
Fig.1: Function g(x) spreads around the two non-zero pixels of function f(x), the convolution between the two functions is the sum of the contributions from both pixels

The function in fig. 1a, f(x), takes on non-zero values at only two points (–20 and +20). To compute the convolution of f(x) and g(x), we center a version of g(x) around each non-zero point of f(x), scaling it by the value of f(x) at that point (Fig 1c). The contributions from the differently scaled and shifted g(x) functions are then summed together at each point to get the final convolution result.   in fig. 1(b) is multiplied by the value of each pixels of f(x) and spread around these pixels. Finally, the contributions from spreading around both pixels are added together to form the convolution between the two functions.

Mathematically, the convolution (denoted by "⊗" here) between two discrete functions f(n) and g(n) is defined as:

$$[f \otimes g](n) = \sum_{m=-\infty}^{\infty} f(m)g(n-m)$$

While so far we have demonstrated how convolution works on discrete functions, the same concept may be readily applied to continuous function, by replacing the summation with integration. Fig. 2



(a) function f(x)

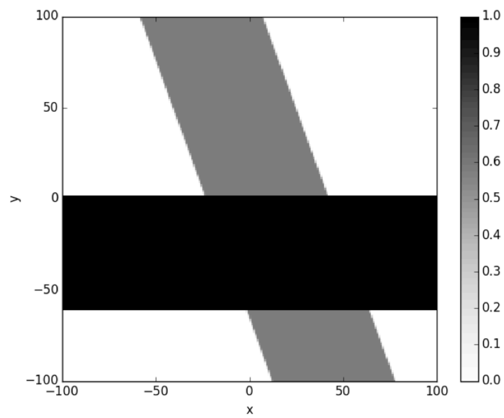
(b) function g(x)

(c) f(x) convolved with g(x)

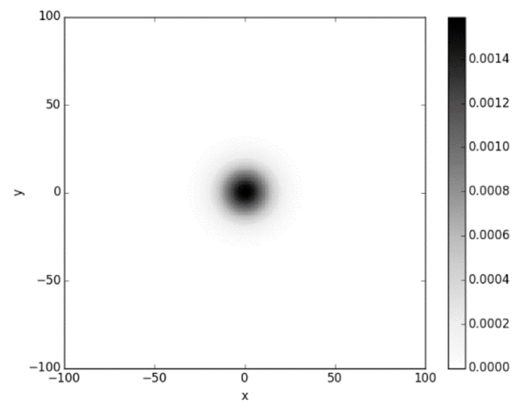
Fig.2: (a) a step function convolved with (b) a Gaussian function resulting in (c) rounded edges

shows such an example with continuous functions, where the convolution of a function with sharp features with a Gaussian function results in rounded features.

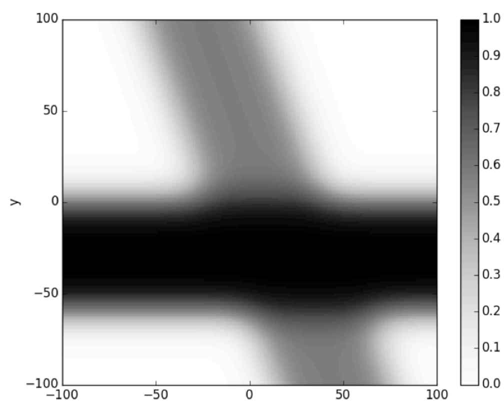
The concept of convolution generalizes to multiple dimensions. For example, in fig. 3a simple 2-D function with sharp step-like features is convolved with a 2-D Gaussian function, producing a blurred version of the original function. (Note that functions of two variables can be represented as images,



(a) function f(x, y)



(b) function g(x, y)



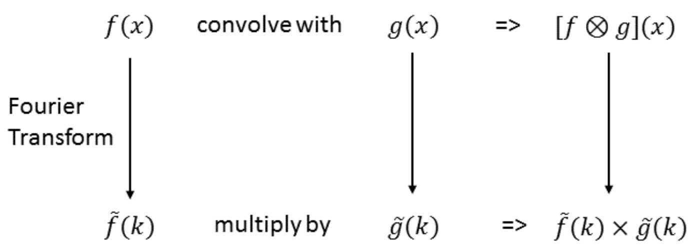
(c) f(x, y) convolved with g(x, y)

Fig.3: The convolution of a function of a 2D function (a) with a 2-D Gaussian function (b) results in a blurred version of the original function (c).

where the value of the function for particular x and y coordinates specifies the brightness of the image pixel at that point).

Convolution Theorem

The Convolution theorem states that convolution in real space is equivalent to multiplication in the Fourier space:



In other words, if we compute the Fourier transforms of f and g, and then multiply the two resulting complex numbers for each frequency, we will get the same result as if we computed the Fourier transform of the convolution of f and g. Thus, one can compute a convolution by performing the Fourier transform of the original functions, multiplying the results, and then performing an inverse Fourier transform. Because Fourier transforms can be performed very efficiently with the FFT algorithm, this is often the fastest way to compute a convolution.