Making a Series of Decisions (Revisited)

• Consider this game:

Play?

yes

0.5

$12

0.5

-no

$0

• Would you play 100 times?

• Binomial distribution

In life, you don’t know how many times you’ll play, but if you never play the first time, you can’t get to 100
Making a Series of Decisions

- Consider this game:

  - Play?
    - yes: 0.5 \( \times \) $12
    - no: 0.5 \( \times \) $0

- Would you play 100 times?

- Expected values for each outcome \( (x \cdot p(x)) \)
  - Note: there is far more mass above 0 than below
  - Sum over the graph to get expected value
    - \( E[X] = $100 \)
Recall, Normal Random Variable

- $X$ is a **Normal Random Variable**: $X \sim N(\mu, \sigma^2)$
  - Probability Density Function (PDF):
    \[
    f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{where} \quad -\infty < x < \infty
    \]
  - $E[X] = \mu$
  - $Var(X) = \sigma^2$
  - Also called “Gaussian”
  - Note: $f(x)$ is symmetric about $\mu$
  - Common for natural phenomena: heights, weights, etc.
  - Often results from the sum of multiple variables
The Central Limit Theorem (CLT)

- Consider a series of random variables $X_1, X_2, ...$
  - $X_i$ are all independent
  - $X_i$ have same distribution with $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$
  - Consider the mean of all $X_i$
    - Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
  - Central Limit Theorem: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$
  - Recall, for Binomial $E[X_i] = np$, $\text{Var}(X_i) = np(1 - p)$
    - In example, $np = 100(0.5) = 50$, $np(1 - p) = 100(0.5)(0.5) = 25$
    - To lose money, would need at least 55 losses
    - There’s only an 18.4% chance that would happen
Central Limit Theorem in Real World

- CLT is why many things in “real world” appear Normally distributed
  - Many quantities are sum of independent variables
  - Exams scores
    - Sum of individual problems
  - Election polling
    - Ask 100 people if they will vote for candidate X ($p_1 = \# \text{ “yes”}/100$)
    - Repeat this process with different groups to get $p_1, \ldots, p_n$
    - Will have a normal distribution over $p_i$
Confidence Intervals

- Fact that sample means are normally distributed allows us to compute a “confidence interval”
  - In election example, determine how likely is it that estimate for true $p$ is “close” to our measurement of $p$
  - Rule of thumb for “large” $n$: $n > 30$, but larger is better ($> 100$)
It’s play time!
Sum of Dice

• You will roll 10 dice
  ▪ $X = \text{total value of all 10 dice}$
  ▪ Win if: $X \leq 25$ or $X \geq 45$
  ▪ Roll!

• What is the probability that you would win?

• How can we figure this out?
  ▪ Play game many times, see how often you win
  ▪ Use a computer to simulate it!
  ▪ Or, use Central Limit Theorem to approximate it
  ▪ Probability of winning is about 7.8%
Reasoning About Choice of Another

• Problem
  • n people are on an island, k (> 0) have blue eyes
    o No one knows their own eye color (no mirrors)
    o Each person knows everyone else’s eye color
    o Everyone on the island is a perfect logician
  • If someone determines they have blue eyes, they must leave the island at the coming dawn
  • If someone doesn’t know their own eye color, they sleep past dawn
  • Outsider comes to island and announces to everyone that at least one person on the island has blue eyes

• What happens? Discuss.
Reasoning About Choice of Another

• Solution
  ▪ On the k-th dawn after the announcement, all the blue-eyed people on the island leave
  ▪ Why?
    o Say k = 1, that person realizes they have blue eyes since they know no one else does. They leave island on first dawn.
    o Say k = 2, each blue-eyed person sees one other person with blue eyes. If k = 1, that person would leave at first dawn. But since no one leaves at first dawn, the people with blue eyes realize k > 1. Since they only see one other blue-eyed person, they realize they must have blue eyes in order for k > 1.
    o Say k = 3, same argument as above, but no one leaves on first or second dawn, so k > 2. Since each blue-eyed person only sees two other blue-eyed people, they realize they must have blue eyes for k > 2, and leave at next dawn.
So What Did Announcement Provide?

• If $k > 1$, didn’t everyone already know there was a blue-eyed person on the island?
  - The announcement creates “common knowledge”
    - Everyone now knows that everyone else shares this same knowledge.
    - Common knowledge allows us to reason about each other’s actions
    - In this case, it synchronizes when islanders can start “counting” knowing that everyone else is counting from the same point

• Reasoning using common knowledge forms one of the main bases for Game Theory