CS361B: Homework 1

Due date: April 17, 2014 at 12:15PM

- Please make your answers clear and concise. In most cases, you should not need more than the equivalent of a page with size 12 font typed. You're more likely to get points subtracted if your proof is difficult to read or understand. To improve legibility, type your solutions if you can.
- Each solution should start on a new page.
- Searching for answers on the Web or consulting solutions is considered a violation of the honor code.
- Every time you are asked to design an algorithm, please provide proof of correctness and compute/prove the approximation factor. Also, explain why your algorithm is polynomial-time. Provide exact running-time analysis where required.

Problem 1-1. **(25pts)** Given optimum dual potentials p^*, one can find optimum min-cost circulation f^* by

1. Saturating all edges with negative reduced cost
2. Removing all edges with positive reduced cost
3. Canceling excesses and deficits using only the remaining zero reduced cost edges (using max-flow as a subroutine).

Provide a formal proof that the result of this computation is indeed a circulation of minimum cost.

Problem 1-2. **(25pts)** Consider trying to solve the min-cost circulation problem on a graph where only two edges have non-zero cost. In particular, one edge, e_-, has a negative cost and the other, e_+, has a positive cost.

1. Suppose $|cost(e_+)| > |cost(e_-)|$. Show we can solve the min-cost circulation problem by doing one iteration of a max flow algorithm.
2. Suppose $|cost(e_+)| < |cost(e_-)|$. Show we can solve the min-cost circulation problem by doing at most two iterations of a max flow algorithm.

In both cases, prove the correctness of your algorithm.

Problem 1-3. **(25pts)** A directed graph $G(V, E)$ is *Eulerian* if there is a directed cycle visiting each edge exactly once. For any vertex $v \in V$, let $\delta^+(v)$ denote the number of edges leaving v and let $\delta^-(v)$ denote the number of edges entering v.

1. (simple) Show that G is Eulerian iff it is connected and for all $v \in V$, we have $\delta^+(v) = \delta^-(v)$.
2. Suppose that for any edge $e = (v, w) \in E$, we can add another copy of e at cost $c_f(e)$. We can add as many copies of e as we wish, but we pay $c_f(e)$ for each copy. Show how to compute the minimum cost set of additions to make G Eulerian by formulating a min cost flow problem.

Problem 1-4. **(25pts)** Let $G(V, E, u, c)$ be an instance of the min cost circulation problem in which all capacities $u(\cdot)$ are integers, but the costs $c(\cdot)$ could be arbitrary reals.

1. Give an example of such a graph in which at least one optimal solution assigns fractional flow on some edges.
2. Consider such a flow f. We will show how to convert f into a flow that is integer on all edges. First, show that the residual graph has a cycle Γ such that each edge along this cycle has fractional residual capacity.

3. What can you say about cost of the cycle Γ you found in part (b)?

4. Design an efficient strongly polynomial time algorithm to convert f into an integer flow. What is the running time of your algorithm (it should not depend on the value of costs or capacities)? [Forgetting the flow and re-solving the problem from scratch is not an acceptable answer.]