CS361B: HOMEWORK 2

Due date: May 8, 2014 at 12:15PM

- Please make your answers clear and concise. In most cases, you should not need more than the equivalent of a page with size 12 font typed. You're more likely to get points subtracted if your proof is difficult to read or understand. To improve legibility, type your solutions if you can.
- Each solution should start on a new page.
- Searching for answers on the Web or consulting solutions is considered a violation of the honor code.
- Every time you are asked to design an algorithm, please provide proof of correctness and compute/prove the approximation factor. Also, explain why your algorithm is polynomial-time. Provide exact running-time analysis where required.

Problem 1-1. Consider a network $G(V, E)$ with a specially designated sink node $s \in V$. Each node $i \in V - \{s\}$ has a demand r_i which is either 0 or 1. These demands must be routed to the sink. Edge e in the network has non-negative cost $c(e)$ and unit capacity. The goal is to construct a tree to route all demand to s. We define the cost of the tree as the sum of the cost of its edges. Show how to find such a tree with minimum cost. Can your approach be used to solve the problem if each edge in the network has capacity 2 instead of one? Prove that it still works (with minor modifications) or explain where it will fail.

Problem 1-2. In the class we talked about a non strongly-polynomial cost-scaling algorithm where, at each phase, we computed potentials $p(v)$ such that all reduced costs are at least μ, the minimum mean cycle cost, and then canceled all the cycles in the graph induced by negative reduced cost edges (see top of page 32 in the notes).

Apply this algorithm for the case where both costs and capacities are integer, with U being the largest capacity and C being the largest absolute value of edge cost. Lets say that we do not need an optimum solution. Instead, we need a solution whose cost is within a $1 + \epsilon$ factor of the minimum cost. How many phases of the cycle canceling should be sufficient ? Can you show a solution where the number of phases depends on $\log U$ but not on $\log C$? (Assume that arithmetic operations take one time unit, independent of the number of bits in the operands.)

Problem 1-3. Recall the transshipment problem introduced in class. In this problem, each edge has a cost, any node may have a demand or an excess, and the goal is to find a flow of minimum cost that satisfies all the demands from the excesses. That is, for each node with an excess, the net outgoing flow must equal the excess, and for each node with a demand, the net incoming flow must equal the demand. (This problem models the task of finding the cheapest way to transport goods from suppliers (excesses) to customers (demands).)

Does there exist an instance of the transshipment problem with non-negative costs on all the edges such that, if we increase one of the demands (and increase one of the excesses), the value of the minimum-cost feasible solution decreases? Give an example of such an instance, or prove that none exists.

Problem 1-4. Consider the primal and dual linear programs

$$\max\{cx \mid Ax \leq b\}$$

and

$$\min\{yb \mid yA = c, y \geq 0\}.$$

Assume that both programs have the same finite optimal value v^*, and let x^* be an optimal solution to the primal program. The goal of this problem is to show that there is an optimal dual solution $y^* \geq 0$ such that the rows of the matrix A corresponding to the positive components of y^* are linearly independent. [Hint: review the proof in section 6.2 of the notes.]

1.Show that the vector $[\ c \quad v^*\ ]$ is in the cone generated by the rows of the matrix $[\ A \quad b\ ]$.

2.For a vector $y \geq 0$, let A_y be the submatrix of A consisting of the rows i such that $y_i > 0$, and define a vector b_y similarly. Show that there is a vector $y^* \geq 0$ such that $[\ c \quad v^*\ ] = y^*[\ A \quad b\ ]$, and the matrix $[\ A_{y^*} \quad b_{y^*}\ ]$ has linearly independent rows.

3.Show that $A_{y^*}x^* = b_{y^*}$.

4.Prove that the matrix A_{y^*} has linearly independent rows.